Polynomial Sets Generated By $e^t \phi_1(x_1 t) \phi_2(x_2 t) \phi_3(x_3 t)$

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Abstract: The present paper deal with three variables polynomial sets generated by functions of the form $e^t \phi_1(x_1 t) \phi_2(x_2 t) \phi_3(x_3 t)$. Its special case analogous to Laguerre polynomials have been discussed.

Keywords: Laguerre polynomials of three variables, hypergeometric function.

1 Introduction

Laguerre polynomials $L^{(\alpha)}_n(x)$ possess the generating relation [See Rainville [6] pp-130], $L^{(\alpha)}_n(x)$ is well known Laguerre Polynomials of one variable

$$e^t _0F_1(-;1+\alpha;-xt) = \sum_{n=0}^{\infty} \frac{L^{(\alpha)}_n(x)t^n}{(1+\alpha)_n}$$  \hspace{1cm} (1)

and $L^{(\alpha,\alpha_2)}_n(x_1, x_2)$ is Laguerre Polynomials of two variables due to S.F. Ragab [4] and Chatterjea [1] gave generating function of Laguerre polynomials of two variable $L^{(\alpha,\beta)}_n(x,y)$in the form

$$e^t _0F_1(-;\alpha+1;-xt)_{0F_1(-;\beta+1;-yt)} = \sum_{n=0}^{\infty} \frac{n!L^{(\alpha,\beta)}_n(x,y)t^n}{(\alpha+1)_n(\beta+1)_n}$$  \hspace{1cm} (2)

One arrives at properties hold by $L^{(\alpha)}_n(x)$ (See Rainville [6], pp. 132-133). Motivated by (2) an attempt has been made to study three variables polynomials similar to one given in (2) and generated by functions of the form $e^t \phi_1(x_1 t) \phi_2(x_2 t) \phi_3(x_3 t)$.

2 Three-variable polynomial sets analogous to (2)

Let us consider the generating relation of the type

$$e^t \phi_1(x_1 t) \phi_2(x_2 t) \phi_3(x_3 t) = \sum_{n=0}^{\infty} \sigma_n(x_1, x_2, x_3)t^n$$  \hspace{1cm} (3)

Let

$$F(t,x_1,x_2,x_3) = e^t \phi_1(x_1 t) \phi_2(x_2 t) \phi_3(x_3 t)$$  \hspace{1cm} (4)

Then

$$\frac{\partial F}{\partial x_1} = te^t \phi'_1 \phi_2 \phi_3$$  \hspace{1cm} (5)

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Then (4) yields
\[ \frac{\partial F}{\partial t} = e^t \phi_1 \phi_2 \phi_3 + x_1 e^t \phi_1 \phi_2 \phi_3 + x_2 e^t \phi_1 \phi_2 \phi_3 + x_3 e^t \phi_1 \phi_2 \phi_3 \] (8)

Eliminating \( \phi_1, \phi_1', \phi_2, \phi_2', \phi_3 \) and \( \phi_3' \) from the five equations (4), (5), (6), (7) and (8), we obtain
\[ \left( x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) F - t \frac{\partial F}{\partial t} = -tF \] (9)

Since
\[ F = e^t \phi_1(x_1t) \phi_2(x_2t) \phi_3(x_3t) = \sum_{n=0}^{\infty} \sigma_n(x_1, x_2, x_3) t^n \]

Equation (9) yields
\[ \sum_{n=0}^{\infty} \left( x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) \sigma_n(x_1, x_2, x_3) t^n - \sum_{n=1}^{\infty} n \sigma_n(x_1, x_2, x_3) t^n \]
\[ \quad = - \sum_{n=0}^{\infty} \sigma_n(x_1, x_2, x_3) t^{n+1} \]
\[ \quad = - \sum_{n=1}^{\infty} \sigma_{n-1}(x_1, x_2, x_3) t^n \]

from which the theorem follows.

**Theorem 1.** From
\[ e^t \phi_1(x_1t) \phi_2(x_2t) \phi_3(x_3t) = \sum_{n=0}^{\infty} \sigma_n(x_1, x_2, x_3) t^n \] (11)

it follows that
\[ \frac{\partial}{\partial x_1} \sigma_n(x_1, x_2, x_3) + \frac{\partial}{\partial x_2} \sigma_n(x_1, x_2, x_3) + \frac{\partial}{\partial x_3} \sigma_n(x_1, x_2, x_3) = 0 \] (12)

and for \( n \geq 1 \)
\[ (x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}) \sigma_n(x_1, x_2, x_3) - n \sigma_n(x_1, x_2, x_3) = -\sigma_{n-1}(x_1, x_2, x_3) \] (13)

Next, let us assume that the functions \( \phi_1, \phi_2 \) and \( \phi_3 \) in (11) have the formal power-series expansions.
\[ \phi_1(u_1) = \sum_{k_1=0}^{\infty} \gamma_{k_1} u_1^{k_1} ; \gamma_0 \neq 0 \] (14)
\[ \phi_2(u_2) = \sum_{k_2=0}^{\infty} \delta_{k_2} u_2^{k_2} ; \delta_0 \neq 0 \] (15)

and
\[ \phi_3(u_3) = \sum_{k_3=0}^{\infty} \xi_{k_3} u_3^{k_3} ; \xi_0 \neq 0 \] (16)

Then (11) yields
\[ \sum_{n=0}^{\infty} \sigma_n(x_1, x_2, x_3) t^n = \left( \sum_{n=0}^{\infty} t^n \right) \left( \sum_{k_1=0}^{\infty} \gamma_{k_1} u_1^{k_1} \right) \left( \sum_{k_2=0}^{\infty} \delta_{k_2} t^{k_2} \right) \left( \sum_{k_3=0}^{\infty} \xi_{k_3} u_3^{k_3} \right) \]
Theorem 2 yields for 

$$\sigma_n(x_1, x_2, x_3) = \sum_{n=0}^{\infty} \sum_{k_1=0}^{n-k_2} \sum_{k_3=0}^{n-k_1-k_2} \frac{\gamma_1 \delta_1 \xi_1 u_1^{k_1} x_1^{k_2} x_2^{k_3}}{(n-k_1-k_2-k_3)!}$$

(17) 

so that 

$$\sigma_n(x_1, x_2, x_3) = \sum_{n=0}^{\infty} \sum_{k_1=0}^{n-k_2} \sum_{k_3=0}^{n-k_1-k_2} \frac{\gamma_1 \delta_1 \xi_1 u_1^{k_1} x_1^{k_2} x_2^{k_3}}{(n-k_1-k_2-k_3)!}$$

(18) 

Now consider the sum 

$$\sum_{n=0}^{\infty} \sum_{k_1=0}^{n-k_2} \sum_{k_3=0}^{n-k_1-k_2} \frac{(c)_n \gamma_1 \delta_1 \xi_1 u_1^{k_1} x_1^{k_2} x_2^{k_3}}{n!} \left( \frac{x_1 t}{1-t} \right)^{k_1} \left( \frac{x_2 t}{1-t} \right)^{k_2} \left( \frac{x_3 t}{1-t} \right)^{k_3} \sum_{n=0}^{\infty} \frac{(c + k_1 + k_2 + k_3)_n t^n}{n!}$$

(19) 

$$= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} (c)_{k_1+k_2+k_3} \gamma_1 \delta_1 \xi_1 u_1^{k_1} x_1^{k_2} x_2^{k_3} \left( \frac{x_1 t}{1-t} \right)^{k_1} \left( \frac{x_2 t}{1-t} \right)^{k_2} \left( \frac{x_3 t}{1-t} \right)^{k_3}$$

(20) 

$$\sum_{n=0}^{\infty} \sum_{k_1=0}^{n-k_2} \sum_{k_3=0}^{n-k_1-k_2} \frac{(c)_n \gamma_1 \delta_1 \xi_1 u_1^{k_1} x_1^{k_2} x_2^{k_3}}{(n-k_1-k_2-k_3)!} \left( \frac{x_1 t}{1-t} \right)^{k_1} \left( \frac{x_2 t}{1-t} \right)^{k_2} \left( \frac{x_3 t}{1-t} \right)^{k_3}$$

(21) 

We thus arrive at the following theorem:

**Theorem 2.** From 

$$e^c \phi_1(x_1 t) \phi_2(x_2 t) \phi_3(x_3 t) = \sum_{n=0}^{\infty} \sigma_n(x_1, x_2, x_3) t^n$$

$$\phi_1(u_1) = \sum_{k_1=0}^{\infty} \gamma_1 u_1^{k_1}, \quad \phi_2(u_2) = \sum_{k_2=0}^{\infty} \delta_2 u_2^{k_2}, \quad \phi_3(u_3) = \sum_{k_3=0}^{\infty} \xi_2 u_3^{k_3}$$

it follows that for arbitrary c.

$$\left( \frac{x_1 t}{1-t} \right)^{k_1} \left( \frac{x_2 t}{1-t} \right)^{k_2} \left( \frac{x_3 t}{1-t} \right)^{k_3}$$

(22) 

in which 

$$G(u_1, u_2, u_3) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} (c)_{k_1+k_2+k_3} \gamma_1 \delta_2 \xi_2 u_1^{k_1} u_2^{k_2} u_3^{k_3}$$

(23) 

### 3 Applications of Theorems 1 & 2

The role of theorem 2 is as follows: If a set $\sigma_n(x_1, x_2, x_3)$ has a generating function of the form $e^c \phi_1(x_1 t) \phi_2(x_2 t) \phi_3(x_3 t)$, Theorem 2 yields for $\sigma_n(x_1, x_2, x_3)$ another generating function of the form exhibited in (22). For instance, if $\phi_1(u_1), \phi_2(u_2)$ and $\phi_3(u_3)$ are specified, the theorem gives for $\sigma_n(x_1, x_2, x_3)$ a class (c arbitrary) of generating functions involving three variables hypergeometric functions.
The generating function is given by
\[
(\exp L_n \ln 1; a_1; 2 \kappa_1 + k_2; x_2; x_3) \times \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \sum_{k_3=0}^{n-k_1-k_2} \frac{(-n)_{k_1+k_2+k_3} k_1 k_2 k_3}{(\alpha_1 + 1)_{k_1} (\alpha_2 + 1)_{k_2} (\alpha_3 + 1)_{k_3}}
\]
(24)

The generating function is given by
\[
\psi_2^{(3)}(c; 1 + \alpha_1, 1 + \alpha_2, 1 + \alpha_3; \frac{-x_1 t}{1 - t}, \frac{-x_2 t}{1 - t}, \frac{-x_3 t}{1 - t})
\]
(31)

Therefore Theorem 2 yields
\[
\psi_2^{(3)} = \sum_{n=0}^{\infty} \frac{(n!)^2 (c)_n L_{-n}^{(a_1, a_2, a_3)}(x_1, x_2, x_3)}{(1 + \alpha_1)(1 + \alpha_2)(1 + \alpha_3)} n^n
\]

where \(\psi_2^{(3)}\) is given in the form [6, p. 62 (11)]

\[
\psi_2^{(3)} [a; b, c, d; x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}}{(b)m(c)n(d)_p} x^m y^n z^p m!n!p!
\]

References