Coefficient bounds for certain subclasses of $m$-fold symmetric bi-univalent functions

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Received: 28 August 2018, Accepted: 6 February 2019
Published online: 17 March 2019.

Abstract: We consider two new subclasses $S_\Sigma^\sum (\tau, \lambda, \alpha)$ and $S_\Sigma^\sum (\tau, \lambda, \beta)$ of $\Sigma_m$ consisting of analytic and $m$-fold symmetric bi-univalent functions in the open unit disk $U$. Furthermore, we establish bounds for the coefficients of functions in these subclasses and several related classes are also considered. In addition to these, connections to earlier known results are presented.

Keywords: Analytic, bi-univalent, $m$-fold symmetric, coefficient bounds.

1 Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^\infty a_n z^n$$

(1)

which are analytic in the open unit disk $U = \{z : |z| < 1\}$, and Let $S$ be the subclass of $A$ consisting of from (1) which is also univalent in $U$ (for details, see [6]).

The Koebe one-quarter theorem [6] states that the image of $U$ under every function $f$ from $S$ contains a disk of Radius $1/4$. Thus, every such univalent function has inverse $f^{-1}$ which satisfies

$$f^{-1} (f(z)) = z (z \in U), f^{-1} (f(w)) = w, \left( |w| < r_0(f), r_0 \geq \frac{1}{4} \right) ,$$

(2)

where

$$f^{-1} (w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2a_3 + a_4) w^4 + ....$$

(3)

A Function $f \in A$ is said to be bi univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ denote the class of bi-univalent functions defined in unit disk $U$. For a brief history and interesting examples in class $\Sigma$, see [17]. Examples of functions in the class $\Sigma$ are

$$\frac{z}{1-z}, -\log (1-z), \frac{1}{2} \log \left( \frac{1+z}{1-z} \right),$$

(4)

and so on. However, the familiar Koebe functions is not a member of $\Sigma$. Other common examples of functions in such as

$$z - \frac{z^2}{2}, \frac{z}{1-z^2},$$

(5)

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are also not members of $\Sigma$(see [17]). For each function $f \in S$, function
\[
h(z) = \sqrt{f(z^m)} (z \in U, m \in \mathbb{N})
\]
is univalent and maps the unit disk $U$ into a region with $m$-fold symmetry. A function is said to be $m$-fold symmetric (see[11],[16]) if it has the following normalized form:
\[
f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} (z \in U, m \in \mathbb{N}).
\]
We denote by $S_m$ the class of $m$-fold symmetric univalent functions in $U$, which are normalized by the series expansion (7). In fact, the functions in the class $S$ are one-fold symmetric. Analogous to the concept of $m$-fold symmetric univalent functions, we here introduced the concept of $m$-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an $m$-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of $f$ is given as in (7) and the series expansion for $f^{-1}$, which has been recently proven by Srivastava et al. [18], is given as follows:
\[
g(w) = w - a_{m+1} w^{m+1} + \left[(m+1)a^2_{m+1} - a_{2m+1}^2\right] w^{2m+1} - \frac{1}{2} (m+1)(3m+2)a^3_{m+1} - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \right] w^{3m+1} + ...
\]
where $f^{-1} = g$. We denote by $\Sigma_m$ the class of $m$-fold symmetric bi-univalent functions in $U$. Some examples of $m$-fold symmetric bi-univalent functions are given as follows:
\[
\left(\frac{z^m}{1 - z^m}\right)^{1/m}, \left[-\log (1 - z^m)\right]^{1/m}, \left[\frac{1}{2} \log \left(\frac{1 + z^m}{1 - z^m}\right)\right]^{1/m}.
\]
Lewin [12] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [3] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [15] showed that $\max |a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha[4] introduced certain subclasses of bi-univalent function class $\Sigma$ similar to the familiar subclasses. $S^*(\beta)$ and $K^*(\beta)$ are of starlike and convex function order $\beta$ $(0 \leq \beta < 1)$, respectively (see[15]).

The classes $S_\Sigma^*(\alpha)$ and $K_\Sigma^*(\alpha)$ of bi-starlike functions of order $\alpha$ and bi-convex functions of order $\alpha$, corresponding to function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of function classes $S^*_\Sigma(\alpha)$ and $K^*_\Sigma(\alpha)$, they found nonsharp estimates on the initial coefficients. In fact, the aforementioned work of Srivastava et al. [17] essentially revived the investigation of various subclasses of bi-univalent function class $\Sigma$ in recent years. Recently, many authors investigated bounds for various subclasses of bi-univalent functions (see,[1],[2],[7],[8],[13],[17],[19]). Not much is known about the bounds on general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works to determine general coefficient bounds $|a_n|$ for the analytic bi-univalent functions (see [5],[9],[10]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1,2\}, \mathbb{N} = \{1,2,3,...\}$) is still an open problem.

The aim of the this paper is to introduce two new subclasses of the function class $\Sigma_m$ and derive estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses. We have to remember the following lemma here so as to derive our basic results.

**Lemma 1.** [16]. If $p \in P$, then
\[
|p_n| \leq 2, \quad (n \in \mathbb{N} = \{1,2,...\}) \text{ and } |p_2 - \frac{p_1^2}{2}| \leq 2 - \frac{|p_1|^2}{2},
\]
where the Carathéodory class $P$ is the family of all functions $p$ analytic in $U$ for which

$$\text{Re}\{p(z)\} > 0, p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ..., (z \in U).$$

### 2 Coefficient bounds for function class $S_{\lambda m}(\lambda, \tau, \alpha)$

**Definition 1.** A function $f \in \sum_m$ is said to be in the class $S_{\lambda m}(\tau, \lambda, \alpha)$, $(\tau \in \mathbb{C}/\{0\}, \ 0 < \alpha \leq 1, \ 0 \leq \lambda < 1)$ if the following conditions are satisfied:

$$\left| \arg \left\{ 1 + \frac{1}{\tau} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{zf'(z) + (1 - \lambda)f(z)} - 1 \right) \right\} \right| < \frac{\alpha \pi}{2}, z \in U$$

(11)

$$\left| \arg \left\{ 1 + \frac{1}{\tau} \left( \frac{wg'(w) + \lambda w^2 g''(w)}{\lambda wg'(w) + (1 - \lambda)g(w)} - 1 \right) \right\} \right| < \frac{\alpha \pi}{2}, w \in U$$

(12)

where the function $g = f^{-1}$.

**Theorem 1.** Let $f$ given by (7) be in the class $S_{\lambda m}(\tau, \lambda, \alpha), \ 0 < \alpha \leq 1$. Then,

$$|a_{m+1}| \leq \frac{2\alpha |\tau|}{\sqrt{2m(m + 2\lambda^2)\alpha \tau - (\alpha - 1)m^2(1 + m\lambda)^2}}$$

(13)

$$|a_{2m+1}| \leq \frac{2(m + 1)\alpha^2 \tau^2}{m^2(1 + m\lambda)^2} + \frac{\alpha |\tau|}{m(1 + 2m\lambda)}$$

(14)

**Proof.** Let $f \in S_{\lambda m}(\tau, \lambda, \alpha)$. Then we can write

$$1 + \frac{1}{\tau} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{zf'(z) + (1 - \lambda)f(z)} - 1 \right) = [p(z)]^\alpha,$$

(15)

$$1 + \frac{1}{\tau} \left( \frac{wg'(w) + \lambda w^2 g''(w)}{\lambda wg'(w) + (1 - \lambda)g(w)} - 1 \right) = [q(w)]^\alpha,$$

(16)

where $g = f^{-1}$ and $p, q \in P$ have the following forms:

$$p(z) = 1 + p_1 z + p_2 z^2 + ..., q(w) = 1 + q_1 w + q_2 w^2 + ....$$

(17)

Now, equating the coefficients (15) and (16) we get

$$\frac{1}{\tau}m(1 + m\lambda)a_{m+1} = \alpha p_m,$$

(18)

$$\frac{1}{\tau} \left[ 2m(1 + 2m\lambda)a_{2m+1} - m(1 + m\lambda)^2 a_{m+1}^2 \right] = \alpha p_{2m} + \frac{a(a - 1)}{2} p_m^2,$$

(19)

$$\frac{1}{\tau}m(1 + m\lambda)a_{m+1} = \alpha q_m,$$

(20)

$$\frac{1}{\tau} \left[ 2m(1 + 2m\lambda)[(m + 1)a_{m+1}^2 - a_{2m+1}] - m(1 + m\lambda)^2 a_{m+1}^2 \right] = \alpha q_{2m} + \frac{a(a - 1)}{2} q_m^2.$$  

(21)

Form (18) and (20), we obtain

$$p_m = -q_m,$$

(22)
Also from (19), (21) and (23) we have

\[
\frac{1}{\tau} \left[ a_{m+1}^2 + 2m \left[ (1 + 2m\lambda)(m+1) - (1 + m\lambda)^2 \right] \right] = \alpha (p_{2m} + q_{2m}) + \frac{\alpha (\alpha - 1)}{2} (p_m^2 + q_m^2),
\]

(24)

\[
a_{m+1} = \frac{\alpha^2 \tau^2 (p_{2m} + q_{2m})}{2m (m + 2m^2\lambda - m^2\lambda^2) \alpha \tau - (\alpha - 1)m^2 (1 + m\lambda)^2}.
\]

(25)

Applying Lemma 1 for coefficients \(p_{2m}\) and \(q_{2m}\), we obtain

\[
|a_{m+1}| \leq \frac{2\alpha |\tau|}{\sqrt{2m (m + 2m^2\lambda - m^2\lambda^2) \alpha \tau - (\alpha - 1)m^2 (1 + m\lambda)^2}}.
\]

(26)

Next, in order to find the bound on \(|a_{2m+1}|\), by subtracting (21) from (19), we obtain

\[
\frac{1}{\tau} \left[ 2m (1 + 2m\lambda) (2a_{2m+1} - (m+1)a_{m+1}) \right] = \alpha (p_{2m} + q_{2m}) + \frac{\alpha (\alpha - 1)}{2} (p_{m}^2 - q_{m}^2).
\]

(27)

Then, in view of (22) and (23) and applying Lemma 1 for coefficients \(p_m, p_{2m}\) and \(q_m, q_{2m}\) we have

\[
|a_{2m+1}| \leq \frac{2(m + 1)\alpha^2 \tau^2}{m^2 (1 + m\lambda)^2} + \frac{\alpha |\tau|}{m (1 + 2m\lambda)}.
\]

(28)

3 Coefficient bounds for function class \(S_{\Delta \lambda}(\lambda, \tau, \beta)\)

Definition 2. A function \(f \in S_{\Delta \lambda}\) given by (7) is said to be in class \(S_{\Delta \lambda}(\lambda, \tau, \beta), (\tau \in \mathbb{C} \setminus \{0\}, 0 < \beta \leq 1, 0 \leq \lambda < 1)\) if the following conditions are satisfied:

\[
\text{Re} \left[ 1 + \frac{1}{\tau} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1 - \lambda) f(z)} - 1 \right) \right] > \beta, z \in U,
\]

(29)

\[
\text{Re} \left[ 1 + \frac{1}{\tau} \left( \frac{wg'(w) + \lambda w^2 g''(w)}{\lambda wg'(w) + (1 - \lambda) g(w)} - 1 \right) \right] > \beta, w \in U,
\]

(30)

where the function \(g = f^{-1}\).

Theorem 2. Let given by (7) be in class \(S_{\Delta \lambda}(\lambda, \tau, \beta)\), \(0 \leq \beta < 1\). Then,

\[
|a_{m+1}| \leq \sqrt{\frac{2|\tau|(1 - \beta)}{m(m + 2m^2\lambda - m^2\lambda^2)}},
\]

(31)

\[
|a_{2m+1}| \leq \frac{2(m + 1)\tau^2 (1 - \beta)^2}{m^2 (1 + m\lambda)^2} + \frac{|\tau|(1 - \beta)}{m (1 + 2m\lambda)}.
\]

(32)

Proof. Let \(f \in S_{\Delta \lambda}(\lambda, \tau, \beta)\). Then we can write

\[
1 + \frac{1}{\tau} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1 - \lambda) f(z)} - 1 \right) = \beta + (1 - \beta) p(z),
\]

(33)
\[ 1 + \frac{1}{\tau} \left( \frac{w g^\prime(w) + \lambda w^2 g''(w)}{w g^\prime(w) + (1 - \lambda) g(w)} - 1 \right) = \beta + (1 - \beta) q(w), \] (34)

where \( p, q \in P \) and \( g = f^{-1} \). It follows from (33), (34) that

\[ \frac{1}{\tau} m (1 + m \lambda) a_{m+1} = (1 - \beta) p_m, \] (35)

\[ \frac{1}{\tau} \left[ 2m (1 + 2m \lambda) a_{2m+1} - m (1 + m \lambda)^2 a_{m+1}^2 \right] = (1 - \beta) p_{2m}, \] (36)

\[ -\frac{1}{\tau} m (1 + m \lambda) a_{m+1} = (1 - \beta) q_m, \] (37)

\[ \frac{1}{\tau} \left[ 2m (1 + 2m \lambda) [(m + 1) a_{m+1}^2 - a_{2m+1} - m (1 + m \lambda)^2 a_{m+1}^2] \right] = (1 - \beta) q_{2m}. \] (38)

From (35) and (37), we obtain

\[ p_m = -q_m, \] (39)

\[ \frac{2}{\tau^2} m^2 (1 + m \lambda)^2 a_{m+1}^2 = (1 - \beta)^2 (p_m^2 + q_m^2). \] (40)

Adding (36) and (38), we have

\[ \frac{1}{\tau} \left[ 2m (1 + 2m \lambda) (m + 1) - 2m (1 + m \lambda)^2 \right] a_{m+1}^2 = (1 - \beta) (p_{2m} + q_{2m}). \] (41)

Therefore, we obtain

\[ a_{m+1}^2 = \frac{\tau (1 - \beta) (p_{2m} + q_{2m})}{2m (1 + 2m \lambda - m^2 \lambda^2)}. \] (42)

Applying Lemma 1 for the coefficients \( p_{2m} \) and \( q_{2m} \), we obtain

\[ |a_{m+1}| \leq \sqrt{\frac{2 |\tau| (1 - \beta)}{m (m + 2m^2 \lambda - m^2 \lambda^2)}}. \] (43)

Next, in order to find the bound on \( |a_{2m+1}| \), by subtracting (38) from (36), we obtain

\[ \frac{1}{\tau} \left[ 4m (1 + 2m \lambda) a_{2m+1} - 2m (1 + 2m \lambda) (m + 1) a_{m+1}^2 \right] = (1 - \beta) (p_{2m} - q_{2m}). \] (44)

Then, in view of (39) and (40), applying Lemma 1 for coefficients \( p_m, p_{2m} \) and \( q_m, q_{2m} \) we have

\[ |a_{2m+1}| \leq \frac{2 (m + 1) \tau^2 (1 - \beta)^2}{m^2 (1 + m \lambda)^2} + \frac{|\tau| (1 - \beta)}{m (1 + 2m \lambda)}. \] (45)

This completes the proof of Theorem 2.
4 Coefficient bounds for function class $S_{\Sigma_m}^{\lambda, \tau, \beta}$

**Definition 3.** [3] Let $p_n(\beta)$ with $n \geq 2$ and $0 \leq \beta < 1$ denote the class of univalent analytic function $p$, normalized with $p(0) = 1$ and satisfying
\[
\int_0^{2\pi} \left| \frac{\text{Re} p(z) - \beta}{1 - \beta} \right| d\theta \leq k \pi,
\]
where $z = re^{i\theta}$. For $\beta = 0$, we denote, $p_n = p_n(0)$ hence the class $p_n$ represents the class of functions $p(z)$, analytic in $U$, normalized with $p(0) = 1$ and having the representation
\[
p(z) = \int_0^{2\pi} \frac{1 - ze^{i\theta}}{1 + ze^{i\theta}} du(t),
\]
where $u$ is a real valued function with bounded variation which satisfies
\[
\int_0^{2\pi} du(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |du(t)| \leq n, \quad n \geq 2.
\]
Note that $p = p_2$ is the well known class of Carathéodory function (the normalized functions with positive real part in the open unit disk $U$).

**Definition 4.** For $0 \leq \lambda \leq 1$ and $0 \leq \beta \leq 1$, a function $f \in \Sigma_m$ given by (1) is said to be in the class $S_{\Sigma_m}^{\lambda, \tau, \beta}$, if the following two conditions are satisfied:
\[
1 + \frac{1}{\tau} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{\lambda zf'(z) + (1 - \lambda)f(z)} - 1 \right) \in p_n(\beta),
\]
(46)
\[
1 + \frac{1}{\tau} \left( \frac{wg'(w) + \lambda w^2 g''(w)}{\lambda wg'(w) + (1 - \lambda)g(w)} - 1 \right) \in p_n(\beta),
\]
(47)
where, $\tau \in C \setminus \{0\}$ the function $g = f^{-1}$ is given by (3), and $z, w \in U$. In order to derive Theorem 3, we shall need the following lemma:

**Lemma 2.** [20]. Let the function $\phi(z) = 1 + h_1z + h_2z^2 + \ldots \in U$ such that $\phi \in p_n(\beta)$ then, $|h_k| \leq n(1 - \beta)$; $k \geq 1$.

**Theorem 3.** If $f \in S_{\Sigma_m}^{\lambda, \tau, \beta}$, then
\[
|a_{m+1}| \leq \min \left\{ \sqrt{\frac{n|\tau|(1 - \beta)}{m(n + 2m\lambda - m^2\lambda^2)}}, \frac{n|\tau|(1 - \beta)}{m(1 + m\lambda)} \right\},
\]
(48)
\[
|a_{2m+1}| \leq \frac{(m + 1)n|\tau|(1 - \beta)}{2m(n + 2m\lambda - m^2\lambda^2)}.
\]
(49)

**Proof.** Since $f \in S_{\Sigma_m}^{\lambda, \tau, \beta}$, from the definition relations (46) and (47) it follows that
\[
\frac{1}{\tau}m(1 + m\lambda)a_{m+1} = p_m,
\]
(50)
\[
\frac{1}{\tau} \left[ 2m(1 + 2m\lambda)a_{2m+1} - m(1 + m\lambda)^2 a_{m+1}^2 \right] = p_{2m},
\]
(51)
\[
- \frac{1}{\tau}m(1 + m\lambda)a_{m+1} = q_m,
\]
(52)
\[
\frac{1}{\tau} \left[ 2m(1+2m\lambda) \left[ (m+1) a_{m+1}^a - a_{2m+1}^a \right] - m(1+m\lambda)^2 a_{m+1}^a \right] = q_{2m}.
\] (53)

From (50) and (52), it follows that
\[
a_{m+1} = \frac{\tau p_m}{m(1+m\lambda)} = -\frac{\tau q_m}{m(1+m\lambda)},
\] (54)
and (51), (53) yields
\[
a_{m+1}^2 = \frac{\tau (p_{2m} + q_{2m})}{2m(m + 2m^2\lambda - m^2\lambda^2)}.
\] (55)

Applying Lemma 2 for coefficients \(p_{2m}\) and \(q_{2m}\), we obtain
\[
|a_{m+1}| \leq \min \left\{ \frac{n|\tau| (1-\beta)}{m(m + 2m^2\lambda - m^2\lambda^2)}, \frac{n|\tau| (1-\beta)}{m(1+m\lambda)} \right\}.
\] (56)

Next, in order to find the bound on \(|a_{2m+1}|\), by subtracting (51) from (53), we obtain
\[
\frac{1}{\tau} \left[ 2m(1+2m\lambda)(2a_{2m+1} - (m+1)a_{m+1}^a) \right] = p_{2m} - q_{2m}.
\] (57)

Then, in view of (54) and (55) and applying Lemma 2 for coefficients \(p_m\), \(p_{2m}\) and \(q_m\), \(q_{2m}\) we have
\[
|a_{2m+1}| \leq \frac{(m+1)n|\tau| (1-\beta)}{2m(m + 2m^2\lambda - m^2\lambda^2)}.
\] (58)

This completes the proof of Theorem 3.

5 Conclusions

If we set \(\lambda = 0\) and \(\tau = 1\) in Theorems 1 and 2, then the classes \(S^\alpha_{\Sigma_m}(\tau, \lambda, \alpha)\) and \(S^\beta_{\Sigma_m}(\tau, \lambda, \beta)\) reduce to the classes \(S^\alpha_{\Sigma_m}\) and \(S^\beta_{\Sigma_m}\) respectively. Thus we obtain the following corollaries.

**Corollary 1.** [2]. Let \(f\) given by (7) be in the class \(S^\alpha_{\Sigma_m}\) \((0 < \alpha \leq 1)\). Then,
\[
|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{\alpha + 1}} \quad \text{and} \quad |a_{2m+1}| \leq \frac{\alpha}{m} + \frac{2(m+1)a^2}{m^2}.
\] (59)

**Corollary 2.** [2]. Let \(f\) given by (7) be in the class \(S^\beta_{\Sigma_m}\) \((0 \leq \beta < 1)\). Then,
\[
|a_{m+1}| \leq \sqrt{\frac{2(1-\beta)}{m}} \quad \text{and} \quad |a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m}.
\] (60)

Classes \(S^\alpha_{\Sigma_m}\) and \(S^\beta_{\Sigma_m}\) are, respectively, defined as follows.

**Definition 5.** [2]. A function \(f \in S_{\Sigma_m}\) given by (7) is said to be in class \(S^\alpha_{\Sigma_m}\) if the following conditions are satisfied:
\[
\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2}, f \in \Sigma_m, (0 < \alpha \leq 1, z \in U),
\] (61)

\[
\left| \arg \left( \frac{wg'(w)}{g(w)} \right) \right| < \frac{\alpha \pi}{2}, g \in \Sigma_m, (0 < \alpha \leq 1, w \in U),
\] (62)
where the function \( g = f^{-1} \).

**Definition 6.** [2]. A function \( f \in S_{\Sigma} \) given by (7) is said to be in class \( S_{\Sigma}^\beta \) if the following conditions are satisfied:

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \beta, f \in \Sigma, \quad (0 \leq \beta < 1, \ z \in U),
\]

\[
\Re \left( \frac{wg'(w)}{g(w)} \right) > \beta, g \in \Sigma, \quad (0 \leq \beta < 1, \ w \in U),
\]

where the function \( g = f^{-1} \).

For one-fold symmetric bi-univalent functions and \( \lambda = 0 \), Theorems 2 and 3 reduce to Corollaries 6 and 7, respectively, which were proven earlier by Murugusundaramoorty et. al. [14].

**Corollary 3.** [14]. Let \( f \) given by (7) be in class \( S_{\Sigma}^\alpha \) \((0 < \alpha \leq 1)\). Then,

\[
|a_2| \leq \frac{2\alpha}{\sqrt{\alpha + 1}} \quad \text{and} \quad |a_3| \leq 4\alpha^2 + \alpha.
\]

**Corollary 4.** [14]. Let \( f \) given by (7) be in the class \( S_{\Sigma}^\beta \) \((0 \leq \alpha < 1)\). Then,

\[
|a_2| \leq \sqrt{2(1 - \beta)} \quad \text{and} \quad |a_3| \leq 4(1 - \beta)^2 + (1 - \beta).
\]

If we set \( \lambda = 0 \), \( \lambda = 1 \) and \( \tau = 1 \) in Theorem 1, then the classes \( S_{\Sigma_{\tau}}^\alpha(\lambda, \beta) \) reduce to the class \( S_{\Sigma}^\beta \). Thus, we obtain the following corollaries.

**Corollary 5.** [20]. If \( 1 + \frac{1}{\tau} \left[ \frac{zf''(z)}{f'(z)} - 1 \right] \in p_n(\beta) \) and \( 1 + \frac{1}{\tau} \left[ \frac{wg''(w)}{g'(w)} - 1 \right] \in p_n(\beta) \) then,

\[
|a_2| \leq \min \left\{ \tau n|\lambda|(1 - \beta), \quad n|\tau|(1 - \beta) \right\} \quad \text{and} \quad |a_3| \leq n|\tau|(1 - \beta).
\]

**Corollary 6.** [20]. If \( 1 + \frac{1}{\tau} \left[ \frac{zf''(z)}{f'(z)} \right] \in p_n(\beta) \) and \( 1 + \frac{1}{\tau} \left[ \frac{wg''(w)}{g'(w)} \right] \in p_n(\beta) \) then,

\[
|a_2| \leq \min \left\{ \frac{n|\tau|(1 - \beta)}{2}, \quad \frac{n|\tau|(1 - \beta)}{2} \right\} \quad \text{and} \quad |a_3| \leq \frac{n|\tau|(1 - \beta)}{2}.
\]

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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