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Generalization of the fejer-hadamard type inequalities for p-convex functions via k-fractional integrals

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Abstract: The aim of this paper is to obtain some more general fractional integral inequalities of Fejer-Hadamard type for p-convex functions via Riemann-Liouville k-fractional integrals. Also in particular fractional inequalities for p-convex functions via Riemann-Liouville fractional integrals have been deduced.

Keywords: p-convex functions, Hadamard inequality, Fejer- Hadamard Inequality, k-Fractional Integrals, k-Gamma Function.

1 Introduction

The advantages of fractional calculus have been described and pointed out in the last few decades by many authors. Fractional calculus is based on derivatives and integrals of fractional order, fractional differential equations and methods of their solution. It has been shown that the non integer order models of real systems are regularly more adequate than usually used integer order models [17].

Let $f \in L[a,b]$. The Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as follows:

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, x > a \tag{1}$$

and

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, x < b. \tag{2}$$

For further details one may see [7,15,20]. In [16], there is given definition of k-fractional Riemann-Liouville fractional integrals as follows.

Let $f \in L[a,b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)\, dt, x > a \tag{3}$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)\, dt, x < b. \tag{4}$$

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where $\Gamma_k(\alpha)$ is the k-Gamma function defined as

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{5}$$

One can note that

$$\Gamma_k(\alpha+k) = \alpha \Gamma_k(\alpha)$$

and

$$I_{a+}^{0,1} f(x) = I_{b-}^{0,1} f(x) = f(x).$$

For $k=1$, k-fractional integrals give Riemann-Liouville fractional integrals.

This always has been effort by researchers to see the things of one taste with respect to other. Fractional calculus needs the fractional inequalities to obtain solutions of fractional optimization problems (see [21,22,23,33] and references there in). In recent decades many fractional integral inequalities of Hadamard type have been established via fractional integral operators, for instance for Riemman-Liouville fractional integrals [13,26]. Hadamard inequality plays very important role in non linear analysis and optimization. In recent years this famous inequality has been generalized, refined and extended by many researchers using fractional calculus. They also prove some interesting related inequalities (see [3,4, 13,26,27,28,29,30,31,32] and references there in). In the following we give the well known Hadamard inequality for convex function.

Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$. Then the following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a)+f(b)}{2}. \tag{6}$$

It is well known in the literature as Hadamard inequality. In [6], Fejér established the following inequality which is the weighted generalization of the Hadamard inequality.

Theorem 1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a convex function. Then the inequalities*

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)\, dx \leq \int_a^b f(x) g(x)\, dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)\, dx \tag{7}$$

hold, where $g : [a,b] \longrightarrow \mathbb{R}$ is nonnegative, integrable and symmteric with respect to $\frac{a+b}{2}$.

In [9], Iscan gave the definition of harmonically convex functions.

Definition 1. *Let $I \subset \mathbb{R} \backslash \{0\}$ be a real interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq t f(y) + (1-t) f(x) \tag{8}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (8) is reversed, then f is said to be harmonically concave.

Example 1. Let $f : (0,\infty) \longrightarrow \mathbb{R}$, $f(x) = x$ and $g : (-\infty,0) \longrightarrow \mathbb{R}, g(x) = x$, then f is a harmonically convex function and g is a harmonically convcave function.

The following preposition is obvious from above example.

Proposition 1. *Let $I \subset \mathbb{R} \backslash \{0\}$ be a real interval and $f : I \longrightarrow \mathbb{R}$, is a function, then the following statements are true.*

(1) *if $I \subset (0,\infty)$ and f is a convex and nondecreasing function, then f is harmonically convex.*
(2) *if $I \subset (0,\infty)$ and f is a harmonically convex and nonincreasing function, then f is convex.*

(3) *if* $I \subset (-\infty, 0)$ *and* f *is a harmonically convex and nondecreasing function, then f is convex.*
(4) *if* $I \subset (-\infty, 0)$ *and* f *is a convex and nonincreasing function, then f is harmonically convex.*

In [9], İscan proved the following Hadamard type inequality for harmonically convex functions.

Theorem 2. *Let* $f : I \subset \mathbb{R} \backslash \{0\} \longrightarrow \mathbb{R}$ *be a harmonically convex function and* $a, b \in I$ *with* $a < b$*. If* $f \in L[a,b]$*, then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{9}$$

In [1], Chen and Wu represented the Fejér-Hadamard inequality for harmonically convex functions as follows.

Theorem 3. *Let* $f : I \subset \mathbb{R} \backslash \{0\} \longrightarrow \mathbb{R}$ *be a harmonically convex function and* $a, b \in I$*, with* $a < b$*. If* $f \in L[a,b]$ *and* $g : [a,b] \subset \mathbb{R} \backslash \{0\} \longrightarrow \mathbb{R}$ *is nonnegative, integrable and harmonically symmetric with respect to* $\frac{2ab}{a+b}$*, then*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx \leq \int_a^b \frac{f(x)g(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx. \tag{10}$$

In [25], Sarikaya et al. proved the Hadamard inequality for convex functions via fractional integrals as follows.

Theorem 4. *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be a positive function with* $0 \leq a < b$ *and* $f \in L[a,b]$*. If* f *is a convex function on* $[a,b]$*, then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{11}$$

with $\alpha > 0$.

In [11], İscan proved the Fejér-Hadmard inequality for convex functions via fractional integrals as follows.

Theorem 5. *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be a convex function with* $a < b$*. If g is nonnegative, integrable and symmetric with respect to* $\frac{a+b}{2}$*, then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] \leq \left[J_{a+}^{\alpha}(fg)(b) + J_{b-}^{\alpha}(fg)(a)\right] \leq \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] \tag{12}$$

with $\alpha > 0$.

In [12], İscan proved the Fejér-Hadamard inequality for harmonically convex functions via fractional integrals as follows.

Theorem 6. *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be a harmonically convex function with* $a < b$ *and* $f \in L[a,b]$*. If* $g : [a,b] \longrightarrow \mathbb{R}$ *is nonnegative, integrable and harmonically symmetric with respect to* $\frac{2ab}{a+b}$*, then the following inequalities for fractional integrals hold*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)\left[J_{\frac{1}{b}+}^{\alpha}(goh)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha}(goh)\left(\frac{1}{b}\right)\right] &\leq \left[J_{\frac{1}{b}+}^{\alpha}(fgoh)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha}(fgoh)\left(\frac{1}{b}\right)\right] \\ &\leq \frac{f(a)+f(b)}{2}\left[J_{\frac{1}{b}+}^{\alpha}(goh)\left(\frac{1}{a}\right) + J_{\frac{1}{a}-}^{\alpha}(goh)\left(\frac{1}{b}\right)\right] \end{aligned} \tag{13}$$

with $\alpha > 0$ *and* $h(x) = \frac{1}{x}$, $x \in [\frac{1}{b}, \frac{1}{a}]$*. In [10], Iscan gave the definition of p-convex functions on* $I \subset (0,\infty)$ *as follows.*

Definition 2. *Let* $I \subset (0,\infty)$ *be a real interval and* $p \in \mathbb{R} \backslash \{0\}$*. A function* $f : I \longrightarrow \mathbb{R}$ *is said to be p-convex, if*

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq t f(x) + (1-t) f(y) \tag{14}$$

holds for all $x, y \in I$ *and* $t \in [0,1]$.

It can be easily seen that for $p=1$ and $p=-1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I\subset(0,\infty)$, respectively.

Definition 3. *Let $p\in\mathbb{R}\backslash\{0\}$. A function $g:[a,b]\subset(0,\infty)\longrightarrow\mathbb{R}$ is said to be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$ if*

$$g(x)=g\left(\left[a^p+b^p-x^p\right]^{\frac{1}{p}}\right) \tag{15}$$

holds for all $x\in[a,b]$.

If we take $I\subset(0,\infty)$, $p\in\mathbb{R}\backslash\{0\}$ and $h(t)=t$, in Theorem 5 of [5], then we have the following theorem.

Theorem 7. *Let $f:I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a p-convex function, $p\in\mathbb{R}\backslash\{0\}$, and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequalities hold.*

$$f\left(\left[\frac{a^P+b^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\leq\frac{f(a)+f(b)}{2}. \tag{16}$$

In [13], İscan and Wu presented the Hadamard inequality for harmonically convex functions via fractional integrals as follows.

Theorem 8. *Let $f:I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a function such that $f\in L[a,b]$, where $a,b\in I$ with $a<b$. If f is harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}-}(fog)\left(\frac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}+}(fog)\left(\frac{1}{a}\right)\right]\leq\frac{f(a)+f(b)}{2} \tag{17}$$

with $\alpha>0$ and $g(x)=\frac{1}{x}$, $x\in[\frac{1}{b},\frac{1}{a}]$.

In this paper we generalize the Fejér-Hadamard type inequalities for p-convex functions via Riemann-Liouville fractional integrals. Using k-fractional integrals we obtain inequalities of the Fejér-Hadamard type. Results obtained in this paper have connection with results proved in [1,8,9,11,13,25].

2 Main results

First we prove the following lemma which we have frequently used to prove general results.

Lemma 1. *Let $p\in\mathbb{R}\backslash\{0\}$ and $w:[a,b]\subset(0,\infty)\longrightarrow\mathbb{R}$ is integrable, p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$, then the following equalities hold for k-fractional integrals.*

(i) *If $p>0$, then*

$$J^{\alpha,k}_{a^p+}(woh)(b^p)=J^{\alpha,k}_{b^p-}(woh)(a^p)=\frac{1}{2}\left[J^{\alpha,k}_{a^p+}(woh)(b^p)+J^{\alpha,k}_{b^p-}(woh)(a^p)\right]\quad\text{with}\quad h(x)=x^{\frac{1}{p}},\quad x\in[a^p,b^p].$$

(ii) *If $p<0$, then*

$$J^{\alpha,k}_{b^p+}(woh)(a^p)=J^{\alpha,k}_{a^p-}(woh)(b^p)=\frac{1}{2}\left[J^{\alpha,k}_{b^p+}(woh)(a^p)+J^{\alpha,k}_{a^p-}(woh)(b^p)\right]\quad\text{with}\quad h(x)=x^{\frac{1}{p}},\quad x\in[b^p,a^p].$$

Proof.

(i) Let $p>0$. By the defintion of k-fractional integrals we have

$$J^{\alpha,k}_{a^p+}(woh)(b^p)=\frac{1}{k\Gamma_k(\alpha)}\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}(woh)(x)\,dx=\frac{1}{k\Gamma_k(\alpha)}\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}w\left(x^{\frac{1}{p}}\right)dx.$$

Since w is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$ we have

$$J^{\alpha,k}_{a^p+}(woh)(b^p)=\frac{1}{k\Gamma_k(\alpha)}\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}w\left([a^p+b^p-x]^{\frac{1}{p}}\right)dx,$$

setting $t=a^p+b^p-x$, we have

$$J^{\alpha,k}_{a^p+}(woh)(b^p)=\frac{1}{k\Gamma_k(\alpha)}\int_{a^p}^{b^p}(x-a^p)^{\frac{\alpha}{k}-1}w\left(x^{\frac{1}{p}}\right)dx=J^{\alpha,k}_{b^p-}(woh)(a^p).$$

This completes the proof of (i). The proof of (ii) is same as (i).

Theorem 9. *Let $f:I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a p-convex function, $p\in\mathbb{R}\backslash\{0\}$. Let $a,b\in(0,\infty)$ with $a<b$, $f\in L[a,b]$ and $w:[a,b]\longrightarrow\mathbb{R}$ is nonnegative, integrable and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$, then the following inequalties for k-fractional integrals hold.*

(i) *If $p>0$, then*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\left[J^{\alpha,k}_{a^p+}(woh)(b^p)+J^{\alpha,k}_{b^p-}(woh)(a^p)\right]\le\left[J^{\alpha,k}_{a^p+}(fwoh)(b^p)+J^{\alpha,k}_{b^p-}(fwoh)(a^p)\right]$$
$$\le\frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{a^p+}(woh)(b^p)+J^{\alpha,k}_{b^p-}(woh)(a^p)\right]\quad(18)$$

with $h(x)=x^{\frac{1}{p}}$, $x\in[a^p,b^p]$.

(ii) *If $p<0$, then*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\left[J^{\alpha,k}_{b^p+}(woh)(a^p)+J^{\alpha,k}_{a^p-}(woh)(b^p)\right]\le\left[J^{\alpha,k}_{b^p+}(fwoh)(a^p)+J^{\alpha,k}_{a^p-}(fwoh)(b^p)\right]$$
$$\le\frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{b^p+}(woh)(a^p)+J^{\alpha,k}_{a^p-}(woh)(b^p)\right]\quad(19)$$

with $h(x)=x^{\frac{1}{p}}$, $x\in[b^p,a^p]$.

Proof.

(i) Let $p>0$. Since f is a p-convex function, setting $t=1/2$ in (14) we have

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{\frac{1}{p}}\right)\le\frac{f(x)+f(y)}{2}.$$

Setting $x=[ta^p+(1-t)b^p]^{\frac{1}{p}}$ and $y=[tb^p+(1-t)a^p]^{\frac{1}{p}}$, $t\in[0,1]$ in above. Then integrating over $[0,1]$ after multiplying with $2t^{\frac{\alpha}{k}-1}w\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)$, we get

$$2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_0^1 t^{\frac{\alpha}{k}-1}w\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)dt \leq \int_0^1 t^{\frac{\alpha}{k}-1}f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)w\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)dt$$
$$+\int_0^1 t^{\frac{\alpha}{k}-1}f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)w\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)dt.$$

Setting $ta^p+(1-t)b^p = x$ in above inequality we get

$$\frac{2}{b^p-a^p}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_{a^p}^{b^p}\left(\frac{b^p-x}{b^p-a^p}\right)^{\frac{\alpha}{k}-1}w\left(x^{\frac{1}{p}}\right)dx \leq \frac{1}{b^p-a^p}\int_{a^p}^{b^p}\left(\frac{b^p-x}{b^p-a^p}\right)^{\frac{\alpha}{k}-1}f\left(x^{\frac{1}{p}}\right)w\left(x^{\frac{1}{p}}\right)dx$$
$$+\frac{1}{b^p-a^p}\int_{a^p}^{b^p}\left(\frac{b^p-x}{b^p-a^p}\right)^{\frac{\alpha}{k}-1}f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)w\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)dx.$$

$$\frac{2}{(b^p-a^p)^{\frac{\alpha}{k}}}f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}w\left(x^{\frac{1}{p}}\right)dx$$
$$\leq \frac{1}{(b^p-a^p)^{\frac{\alpha}{k}}}\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}f\left(x^{\frac{1}{p}}\right)w\left(x^{\frac{1}{p}}\right)dx+\frac{1}{(b^p-a^p)^{\frac{\alpha}{k}}}\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}f\left([a^p+b^p-x]^{\frac{1}{p}}\right)w\left(x^{\frac{1}{p}}\right)dx$$
$$=\frac{1}{(b^p-a^p)^{\frac{\alpha}{k}}}\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}(fwoh)(x)dx+\frac{1}{(b^p-a^p)^{\frac{\alpha}{k}}}\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}f\left([a^p+b^p-x]^{\frac{1}{p}}\right)w\left([a^p+b^p-x]^{\frac{1}{p}}\right)dx$$
$$=\frac{1}{(b^p-a^p)^{\frac{\alpha}{k}}}\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}(fwoh)(x)dx+\frac{1}{(b^p-a^p)^{\frac{\alpha}{k}}}\int_{a^p}^{b^p}(x-a^p)^{\frac{\alpha}{k}-1}f\left(x^{\frac{1}{p}}\right)w\left(x^{\frac{1}{p}}\right)dx$$
$$=\frac{1}{(b^p-a^p)^{\frac{\alpha}{k}}}\left[\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}(fwoh)(x)dx+\int_{a^p}^{b^p}(x-a^p)^{\frac{\alpha}{k}-1}(fwoh)(x)dx\right].$$

From which we have

$$2f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)k\Gamma_k(\alpha)J_{a^p+}^{\alpha,k}(woh)(b^p) \leq k\Gamma_k(\alpha)\left[J_{a^p+}^{\alpha,k}(fwoh)(b^p)+J_{b^p-}^{\alpha,k}(fwoh)(a^p)\right].$$

By Lemma 1, the above will be of the form

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)k\Gamma_k(\alpha)\left[J_{a^p+}^{\alpha,k}(woh)(b^p)+J_{b^p-}^{\alpha,k}(woh)(a^p)\right] \leq k\Gamma_k(\alpha)\left[J_{a^p+}^{\alpha,k}(fwoh)(b^p)+J_{b^p-}^{\alpha,k}(fwoh)(a^p)\right].$$

First inequality of (18) proved.

For the proof of the second inequality in (18), we first note that if f is a p-convex function, then for all $t \in [0,1]$, it gives

$$\frac{f\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)+f\left([tb^p+(1-t)a^p]^{\frac{1}{p}}\right)}{2} \leq \frac{f(a)+f(b)}{2}.$$

Integrating above inequality over $[0,1]$ after multiplying with $2t^{\frac{\alpha}{k}-1}g\left([ta^p+(1-t)b^p]^{\frac{1}{p}}\right)$ and using Lemma 2, we get

$$\left[J_{a^p+}^{\alpha,k}(fwoh)(b^p)+J_{b^p-}^{\alpha,k}(fwoh)(a^p)\right]\le\frac{f(a)+f(b)}{2}\left[J_{a^p+}^{\alpha,k}(woh)(b^p)+J_{b^p-}^{\alpha,k}(woh)(a^p)\right].$$

That is the second inequality of (18). The proof of (19) is same as of (18).

Remark. We can observe the following in Theorem 9.

(i) If we put $k=1$, we get [8, Theorem 9] .
(ii) If we put $k=1$ and $p=1$, we get (12).
(iii) If we put $k=1$, $p=1$, and $w(x)=1$, we get (11).
(iv) If we put $k=1$, $p=1$, and $\alpha=1$, we get (7).
(v) If we put $k=1$, $p=1$, $\alpha=1$ and $w(x)=1$, we get (6).
(vi) If we put $k=1$ and $p=-1$, we get (13).
(vii) If we put $k=1$, $p=-1$, and $w(x)=1$, we get (17).
(viii) If we put $k=1$, $p=-1$, and $\alpha=1$, we get (10).
(ix) If we put $k=1$, $p=-1$, $\alpha=1$ and $w(x)=1$, we get (9).
(x) If we put $k=1$, $\alpha=1$ and $w(x)=1$ we get (16).

Lemma 2. *Let $f:I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I^o and $a,b\in I^o$ with $a<b, p\in\mathbb{R}\backslash\{0\}$. If $f'\in L[a,b]$ and $w:[a,b]\longrightarrow\mathbb{R}$ is integrable and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, then the following equalities for k-fractional integrals hold.*

(i) *If $p>0$, then*

$$\frac{f(a)+f(b)}{2}\left[J_{a^p+}^{\alpha,k}(wog)(b^p)+J_{b^p-}^{\alpha,k}(wog)(a^p)\right]-\left[J_{a^p+}^{\alpha,k}(fwog)(b^p)+J_{b^p-}^{\alpha,k}(fwog)(a^p)\right]$$
$$=\frac{1}{k\Gamma_k(\alpha)}\int_{a^p}^{b^p}\left[\int_{a^p}^{t}(b^p-s)^{\frac{\alpha}{k}-1}(wog)(s)ds-\int_{t}^{b^p}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\right](fog)'(t)dt \tag{20}$$

with $g(x)=x^{1/p}, x\in[a^p,b^p]$.

(ii) *If $p<0$, then*

$$\frac{f(a)+f(b)}{2}\left[J_{b^p+}^{\alpha,k}(wog)(a^p)+J_{a^p-}^{\alpha,k}(wog)(b^p)\right]-\left[J_{b^p+}^{\alpha,k}(fwog)(a^p)+J_{a^p-}^{\alpha,k}(fwog)(b^p)\right] \tag{21}$$
$$=\frac{1}{k\Gamma_k(\alpha)}\int_{b^p}^{a^p}\left[\int_{b^p}^{t}(a^p-s)^{\frac{\alpha}{k}-1}(wog)(s)ds-\int_{t}^{a^p}(s-b^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\right](fog)'(t)dt$$

with $g(x)=x^{1/p}, x\in[b^p,a^p]$.

Proof.

(i) Let $p>0$. Taking and solving the first integral on right hand side of (20) we have

$$\int_{a^p}^{b^p}\left[\int_{a^p}^{t}(b^p-s)^{\frac{\alpha}{k}-1}(wog)(s)ds\right](fog)'(t)dt$$
$$=\left(\int_{a^p}^{t}(b^p-s)^{\frac{\alpha}{k}-1}(wog)(s)ds\right)(fog)(t)\Big|_{a^b}^{b^p}-\int_{a^p}^{b^p}(b^p-t)^{\frac{\alpha}{k}-1}(fwog)(t)dt$$
$$=f(b)\int_{a^p}^{b^p}(b^p-s)^{\frac{\alpha}{k}-1}(wog)(s)ds-\int_{a^p}^{b^p}(b^p-t)^{\frac{\alpha}{k}-1}(fwog)(t)dt$$
$$=k\Gamma_k(\alpha)\left[f(b)J_{a^p+}^{\alpha,k}(wog)(b^p)-J_{a^p+}^{\alpha,k}(fwog)(b^p)\right],$$

using Lemma 2 in above we have

$$\int_{a^p}^{b^p}\left[\int_{a^p}^{t}(b^p-s)^{\frac{\alpha}{k}-1}(wog)(s)ds\right](fog)^{'}(t)dt$$
$$=k\Gamma_k(\alpha)\left[f(b)\left[\frac{J_{a^p+}^{\alpha,k}(wog)(b^p)+J_{b^p-}^{\alpha,k}(wog)(a^p)}{2}\right]-J_{a^p+}^{\alpha,k}(fwog)(b^p)\right], \tag{22}$$

now solving the second integral in (20) we have

$$\int_{a^p}^{b^p}\left[\int_{t}^{b^p}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\right](fog)^{'}(t)dt$$
$$=\left(\int_{t}^{b^p}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\right)(fog)(t)|_{a^b}^{b^p}+\int_{a^p}^{b^p}(t-a^p)^{\frac{\alpha}{k}-1}(fwog)(t)dt$$
$$=-f(a)\int_{a^p}^{b^p}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds+\int_{a^p}^{b^p}(t-a^p)^{\frac{\alpha}{k}-1}(fwog)(t)dt$$
$$=k\Gamma_k(\alpha)\left[-f(a)J_{b^p-}^{\alpha,k}(wog)(a^p)+J_{b^p-}^{\alpha,k}(fwog)(a^p)\right],$$

using Lemma 2 in above we have

$$\int_{a^p}^{b^p}\left[\int_{t}^{b^p}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\right](fog)^{'}(t)dt$$
$$=k\Gamma_k(\alpha)\left[-f(a)\left[\frac{J_{a^p+}^{\alpha,k}(wog)(b^p)+J_{b^p-}^{\alpha,k}(wog)(a^p)}{2}\right]+J_{b^p-}^{\alpha,k}(fwog)(a^p)\right]. \tag{23}$$

Subtracting (23) from (22) completes the proof of (20). The proof of (21) is similar as of (20).

Remark. We can observe the following in Lemma 2.

(i) If we put $k=1$, we get [8, Lemma 2].
(ii) If we put $k,p=1$, we get [11, Lemma 2.4].
(iii) If we put $w(x)=1$ along with $k,p=1$, we get [25, Lemma 2].
(iv) If we put $k,p=1$ and $\alpha=1$, we get [26, Lemma 2.6].
(v) If we put $w(x)=1$ along with $k,p,\alpha=1$, we get [2, Lemma 2].
(vi) If we put $k=1$ and $p=-1$, we get [12, Lemma 3].
(vii) If we put $w(x)=1$, $k=1$ and $p=-1$, we get [13, Lemma 3].
(viii) If we put $w(x)=1$, $k,\alpha=1$ and $p=-1$, we get [9, Lemma 5].
(ix) If we put $w(x)=1$ along with $k,\alpha=1$, we get [18, Lemma 4].

We use Lemma 2 to prove the following result.

Theorem 10. *Let $f:I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I^o. If $f^{'}\in L[a,b]$, where $a,b\in(0,\infty)$ with $a<b$, $|f^{'}|$ is p-convex function on $[a,b]$ for $p\in\mathbb{R}\backslash\{0\}$ and $w:[a,b]\longrightarrow\mathbb{R}$ is continuous and p-symmetric with respect to $[\frac{a^p+b^p}{2}]^{1/p}$, then the following inequality for k-fractional integrals holds.*

(i) *If $p>0$, then*

$$\left|\frac{f(a)+f(b)}{2}\left[J_{a^p+}^{\alpha,k}(wog)(b^p)+J_{b^p-}^{\alpha,k}(wog)(a^p)\right]-\left[J_{a^p+}^{\alpha,k}(fwog)(b^p)+J_{b^p-}^{\alpha,k}(fwog)(a^p)\right]\right| \tag{24}$$
$$\leq\frac{\|w\|_\infty(b^p-a^p)^{\frac{\alpha}{k}+1}}{\alpha\Gamma_k(\alpha)}\left[|f^{'}(a)|C_1(\alpha/k,p)+|f^{'}(b)|C_2(\alpha/k,p)\right],$$

where

$$C_1(\alpha/k,p)=\int_0^1 \frac{|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}|}{p[ua^p+(1-u)b^p]^{1-\frac{1}{p}}}udu,$$
$$C_2(\alpha/k,p)=\int_0^1 \frac{|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}|}{p[ua^p+(1-u)b^p]^{1-\frac{1}{p}}}(1-u)du,$$

with $g(x)=x^{1/p}, x\in[a^p,b^p]$ *and* $\| w \|_\infty = sup_{t\in[a,b]} \mid w(t) \mid$.

(ii) *If* $p<0$, *then*

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{b^p+}(wog)(a^p)+J^{\alpha,k}_{a^p-}(wog)(b^p)\right]-\left[J^{\alpha,k}_{b^p+}(fwog)(a^p)+J^{\alpha,k}_{a^p-}(fwog)(b^p)\right]\right| \tag{25}$$
$$\leq \frac{\|w\|_\infty(a^p-b^p)^{\frac{\alpha}{k}+1}}{\alpha\Gamma_k(\alpha)}\left[|f'(a)|C_3(\alpha/k,p)+|f'(b)|C_4(\alpha/k,p)\right],$$

where

$$C_3(\alpha/k,p)=\int_0^1 \frac{-|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}|}{p[ua^p+(1-u)b^p]^{1-\frac{1}{p}}}udu,$$
$$C_4(\alpha/k,p)=\int_0^1 \frac{-|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}|}{p[ua^p+(1-u)b^p]^{1-\frac{1}{p}}}(1-u)du,$$

with $g(x)=x^{1/p}, x\in[b^p,a^p]$ *and* $\| w \|_\infty = sup_{t\in[a,b]} \mid w(t) \mid$.

Proof.

(i) Let $p>0$. Using Lemma 4 we have

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{a^p+}(wog)(b^p)+J^{\alpha,k}_{b^p-}(wog)(a^p)\right]-\left[J^{\alpha,k}_{a^p+}(fwog)(b^p)+J^{\alpha,k}_{b^p-}(fwog)(a^p)\right]\right| \tag{26}$$
$$\leq \frac{1}{k\Gamma_k(\alpha)}\int_{a^p}^{b^p}\left|\int_{a^p}^{t}(b^p-s)^{\frac{\alpha}{k}-1}(wog)(s)ds-\int_t^{b^p}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\right||(fog)'(t)|dt.$$

Taking and substituting $s=a^p+b^p-x$ in the first modulus on right hand side of above inequality we have

$$\left|\int_{a^p}^{t}(b^p-s)^{\frac{\alpha}{k}-1}(wog)(s)ds-\int_t^{b^p}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\right| = \left|-\int_{b^p}^{a^p+b^p-t}(x-a^p)^{\frac{\alpha}{k}-1}w([a^p+b^p-x]^{1/p})dx\right.$$
$$\left.-\int_t^{b^p}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\right|.$$

Using p-symmetry of w with respect to $[\frac{a^p+b^p}{2}]^{1/p}$ we have

$$
\begin{aligned}
&\Big|\int_{a^p}^{t}(b^p-s)^{\frac{\alpha}{k}-1}(wog)(s)ds-\int_{t}^{b^p}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\Big| \\
&=\Big|\int_{a^p+b^p-t}^{b^p}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds+\int_{b^p}^{t}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\Big| \\
&=\Big|\int_{a^p+b^p-t}^{t}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\Big| \le \begin{cases}\int_{t}^{a^p+b^p-t}\left|(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)\right|ds, & t\in\left[a^p,\frac{a^p+b^p}{2}\right]\\ \int_{a^p+b^p-t}^{t}\left|(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)\right|ds, & t\in\left[\frac{a^p+b^p}{2},b^p\right].\end{cases}
\end{aligned}
$$

Putting above in (26) we have

$$
\begin{aligned}
&\Big|\frac{f(a)+f(b)}{2}\left[J_{a^p+}^{\alpha,k}(wog)(b^p)+J_{b^p-}^{\alpha,k}(wog)(a^p)\right]-\left[J_{a^p+}^{\alpha,k}(fwog)(b^p)+J_{b^p-}^{\alpha,k}(fwog)(a^p)\right]\Big| \\
&\le\frac{1}{k\Gamma_k(\alpha)}\int_{a^p}^{b^p}\Big|\int_{a^p+b^p-t}^{t}(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)ds\Big||(fog)'(t)|dt. \\
&\le\frac{1}{k\Gamma_k(\alpha)}\left[\int_{a^p}^{\frac{a^p+b^p}{2}}\left(\int_{t}^{a^p+b^p-t}\left|(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)\right|ds\right)|(fog)'(t)|dt\right. \\
&\left.+\int_{\frac{a^p+b^p}{2}}^{b^p}\left(\int_{a^p+b^p-t}^{t}\left|(s-a^p)^{\frac{\alpha}{k}-1}(wog)(s)\right|ds\right)|(fog)'(t)|dt\right] \\
&\le\frac{\|w\|_\infty}{k\Gamma_k(\alpha)}\left[\int_{a^p}^{\frac{a^p+b^p}{2}}\left(\int_{t}^{a^p+b^p-t}(s-a^p)^{\frac{\alpha}{k}-1}ds\right)|(fog)'(t)|dt+\int_{\frac{a^p+b^p}{2}}^{b^p}\left(\int_{a^p+b^p-t}^{t}(s-a^p)^{\frac{\alpha}{k}-1}ds\right)|(fog)'(t)|dt\right] \\
&=\frac{\|w\|_\infty}{\alpha\Gamma_k(\alpha)}\left[\int_{a^p}^{\frac{a^p+b^p}{2}}\left((s-a^p)^{\frac{\alpha}{k}}|_{t}^{a^p+b^p-t}\right).\frac{t^{1/p-1}}{p}|f'(t^{1/p})|dt+\int_{\frac{a^p+b^p}{2}}^{b^p}\left((s-a^p)^{\frac{\alpha}{k}}|_{a^p+b^p-t}^{t}\right).\frac{t^{1/p-1}}{p}|f'(t^{1/p})|\right] \\
&=\frac{\|w\|_\infty}{\alpha\Gamma_k(\alpha)}\left[\int_{a^p}^{\frac{a^p+b^p}{2}}\left((b^p-t)^{\frac{\alpha}{k}}-(t-a^p)^{\frac{\alpha}{k}}\right).\frac{t^{1/p-1}}{p}|f'(t^{1/p})|dt+\int_{\frac{a^p+b^p}{2}}^{b^p}\left((t-a^p)^{\frac{\alpha}{k}}-(b^p-t)^{\frac{\alpha}{k}}\right).\frac{t^{1/p-1}}{p}|f'(t^{1/p})|\right].
\end{aligned}
$$

Substituting $t=ua^p+(1-u)b^p$ in above inequality we have

$$
\begin{aligned}
&\Big|\frac{f(a)+f(b)}{2}\left[J_{a^p+}^{\alpha,k}(wog)(b^p)+J_{b^p-}^{\alpha,k}(wog)(a^p)\right]-\left[J_{a^p+}^{\alpha,k}(fwog)(b^p)+J_{b^p-}^{\alpha,k}(fwog)(a^p)\right]\Big| \\
&\le\frac{\|w\|_\infty(b^p-a^p)^{\frac{\alpha}{k}+1}}{\alpha\Gamma_k(\alpha)}\times\left[\int_0^{\frac{1}{2}}\left((1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}\right).\frac{[ua^p+(1-u)b^p]^{1/p-1}}{p}|f'([ua^p+(1-u)b^p]^{1/p})|du\right. \\
&\left.+\int_{\frac{1}{2}}^{1}\left(u^{\frac{\alpha}{k}}-(1-u)^{\frac{\alpha}{k}}\right).\frac{[ua^p+(1-u)b^p]^{1/p-1}}{p}|f'([ua^p+(1-u)b^p]^{1/p})|du\right] \\
&=\frac{\|w\|_\infty(b^p-a^p)^{\frac{\alpha}{k}+1}}{\alpha\Gamma_k(\alpha)}\times\int_0^1\Big|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}\Big|\frac{[ua^p+(1-u)b^p]^{1/p-1}}{p}|f'([ua^p+(1-u)b^p]^{1/p})|du. \qquad (27)
\end{aligned}
$$

Since $|f^{'}|$ is p-convex function, we have

$$\Big|\frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{a^p+}(wog)(b^p)+J^{\alpha,k}_{b^p-}(wog)(a^p)\right]-\left[J^{\alpha,k}_{a^p+}(fwog)(b^p)+J^{\alpha,k}_{b^p-}(fwog)(a^p)\right]\Big|$$
$$\leq\frac{\|w\|_\infty(b^p-a^p)^{\frac{\alpha}{k}+1}}{\alpha\Gamma_k(\alpha)}\left[|f^{'}(a)|\int_0^1\Big|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}\Big|\frac{[ua^p+(1-u)b^p]^{1/p-1}}{p}udu\right.$$
$$\left.+|f^{'}(b)|\int_0^1\Big|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}\Big|\frac{[ua^p+(1-u)b^p]^{1/p-1}}{p}(1-u)du\right].$$

Putting $C_1(\alpha/k,p)$ and $C_2(\alpha/k,p)$ in above completes the proof of (24). Proof of (25) is similar as of (24).

Remark. We can observe the following in Theorem 10.

(i) If we put $k=1$, we get [8, Theorem 10].
(ii) If we put $k,p=1$, we get [11, Theorem 2.6].
(iii) If we put $w(x)=1$ along with $k,p=1$, we get [25, Theorem 3].
(iv) If we put $w(x)=1$ along with $k,p,\alpha=1$, we get [2, Theorem 2.2].
(v) If we put $w(x)=1$ along with $k,\alpha=1$, we get [18, Theorem 3.1].
(vi) If we put $\alpha,k=1$ and $p=1$, we get [8, Corollary 1 (1)].
(vii) If we put $k=1$ and $p=-1$, we get [8, Corollary 1 (2)].
(viii) If we put $w(x)=1$ along with $\alpha,k=1$ and $p=-1$, we get [8, Corollary 1 (3)].
(ix) If we put $\alpha,k=1$ and $p=-1$, we get [8, Corollary 1 (4)].
(x) If we put $w(x)=1$ along with $k=1$ and $p=-1$, we get [8, Corollary 1 (5)].

We use the power mean inequality and p-convexity of $|f^{'}|^q$ to prove the following result.

Theorem 11. *Let $f:I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I^o. If $f^{'}\in L[a,b]$, where $a,b\in I$ with $a<b$, $|f^{'}|^q$ is p-convex function on $[a,b]$ for $p\in\mathbb{R}\backslash\{0\}$ and $w:[a,b]\longrightarrow\mathbb{R}$ is continuous and p-symmetric with respect to $[\frac{a^p+b^p}{2}]^{\frac{1}{p}}$, then the following inequality for k-fractional integrals holds.*

(i) *If $p>0$, then*

$$\Big|\frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{a^p+}(wog)(b^p)+J^{\alpha,k}_{b^p-}(wog)(a^p)\right]-\left[J^{\alpha,k}_{a^p+}(fwog)(b^p)+J^{\alpha,k}_{b^p-}(fwog)(a^p)\right]\Big| \tag{28}$$
$$\leq\frac{\|w\|_\infty(b^p-a^p)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}C_5^{1-\frac{1}{q}}(\alpha/k,p)\left[|f^{'}(a)|^qC_1(\alpha/k,p)+|f^{'}(b)|^qC_2(\alpha/k,p)\right]^{\frac{1}{q}},$$

where

$$C_5(\alpha/k,p)=\int_0^1\frac{|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}|}{p[ua^p+(1-u)b^p]^{1-\frac{1}{p}}}du$$

with $g(x)=x^{1/p}, x\in[a^p,b^p]$ and $\|w\|_\infty=sup_{t\in[a,b]}|w(t)|$.

(ii) *If $p<0$, then*

$$\Big|\frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{b^p+}(wog)(a^p)+J^{\alpha,k}_{a^p-}(wog)(b^p)\right]-\left[J^{\alpha,k}_{b^p+}(fwog)(a^p)+J^{\alpha,k}_{a^p-}(fwog)(b^p)\right]\Big| \tag{29}$$
$$\leq\frac{\|w\|_\infty(a^p-b^p)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}C_6^{1-\frac{1}{q}}(\alpha/k,p)\left[|f^{'}(a)|^qC_3(\alpha/k,p)+|f^{'}(b)|^qC_4(\alpha/k,p)\right]^{\frac{1}{q}},$$

where

$$C_6(\alpha/k,p)=\int_0^1 \frac{-|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}|}{p[ua^p+(1-u)b^p]^{1-\frac{1}{p}}}du$$

with $g(x)=x^{1/p}, x\in[b^p,a^p]$ *and* $\| w \|_\infty = sup_{t\in[a,b]} | w(t) |$.

Proof. (i) Let $p>0$. Using power mean inequality in (27) we have

$$\begin{aligned}
&\Big| \frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{a^p+}(wog)(b^p)+J^{\alpha,k}_{b^p-}(wog)(a^p)\right]-\left[J^{\alpha,k}_{a^p+}(fwog)(b^p)+J^{\alpha,k}_{b^p-}(fwog)(a^p)\right]\Big| \\
&\le \frac{\|w\|_\infty (b^p-a^p)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}\left(\int_0^1 \left|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}\right| \frac{[ua^p+(1-u)b^p]^{1/p-1}}{p}du\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \left|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}\right| \frac{[ua^p+(1-u)b^p]^{1/p-1}}{p}|f^{'}([ua^p+(1-u)b^p]^{1/p})|^q du\right)^{1/q}.
\end{aligned}$$

By the p-convexity of $|f^{'}|^q$, the right hand side of above will be

$$\begin{aligned}
&\Big| \frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{a^p+}(wog)(b^p)+J^{\alpha,k}_{b^p-}(wog)(a^p)\right]-\left[J^{\alpha,k}_{a^p+}(fwog)(b^p)+J^{\alpha,k}_{b^p-}(fwog)(a^p)\right]\Big| \\
&\le \frac{\|w\|_\infty (b^p-a^p)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}C_5^{1-\frac{1}{q}}(\alpha/k,p)\times\Bigg[|f^{'}(a)|^q\int_0^1 \left|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}\right| \frac{[ua^p+(1-u)b^p]^{1/p-1}}{p}udu \\
&+|f^{'}(b)|^q\int_0^1 \left|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}\right| \frac{[ua^p+(1-u)b^p]^{1/p-1}}{p}|(1-u)du\Bigg]^{1/q} \\
&=\frac{\|w\|_\infty (b^p-a^p)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}C_5^{1-\frac{1}{q}}(\alpha/k,p)\left[|f^{'}(a)|^q C_1(\alpha/k,p)+|f^{'}(b)|^q C_2(\alpha/k,p)\right]^{1/q}.
\end{aligned}$$

The proof of (28) is completed and the proof of (29) is same as of (28).

Remark. We can observe the following in Theorem 11.

(i) If we put $k=1$, we get [8, Theorem 11].
(ii) If we put $k,p=1$, we get [11, Theorem 2.8].
(iii) If we put $w(x)=1$ along with $k,p=1$ and $\alpha=1$, we get [19, Theorem 1].
(iv) If we put $w(x)=1$ along with $p=-1$ and $k=-1$, we get [2, Theorem 5].
(v) If we put $w(x)=1$ along with $k,\alpha=1$ and $p=-1$, we get [18, Theorem 2.6].
(vi) If we put $w(x)=1$ along with $k,\alpha=1$, we get [18, Theorem 3.2].
(vii) If we put $w(x)=1$ along with $p,k=1$, we get [8, Corollary 2 (1)].
(viii) If we put $\alpha,k=1$ and $p=1$, we get [8, Corollary 2 (2)].
(ix) If we put $k=1$ and $p=-1$, we get [8, Corollary 2 (3)].
(x) If we put $\alpha,k=1$ and $p=-1$, we get [8, Corollary 2 (4)].

We use the Hölder's inequality and the p-convexity of $|f^{'}|^q$ to prove the following result.

Theorem 12. *Let $f:I\subset(0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on I^o. If $f^{'}\in L[a,b]$, where $a,b\in I$ with $a<b$, $|f^{'}|^q, q>1$ is p-convex function on $[a,b]$ for $p\in\mathbb{R}\backslash\{0\}$, $\frac{1}{q}+\frac{1}{r}=1$ and $w:[a,b]\longrightarrow\mathbb{R}$ is continuous and p-symmetric with respect to $[\frac{a^p+b^p}{2}]^{\frac{1}{p}}$, then the following inequality for k-fractional integrals holds.*

(i) *If* $p>0$, *then*

$$\Big|\frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{a^p+}(wog)(b^p)+J^{\alpha,k}_{b^p-}(wog)(a^p)\right]-\left[J^{\alpha,k}_{a^p+}(fwog)(b^p)+J^{\alpha,k}_{b^p-}(fwog)(a^p)\right]\Big| \le \frac{\|w\|_\infty(b^p-a^p)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}C_7^{\frac{1}{r}}(\alpha/k,p,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}, \tag{30}$$

where

$$C_7(\alpha/k,p,r)=\int_0^1\left(\frac{|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}|}{p[ua^p+(1-u)b^p]^{1-\frac{1}{p}}}\right)^r du$$

with $g(x)=x^{1/p}, x\in[a^p,b^p]$ *and* $\| w \|_\infty = sup_{t\in[a,b]} | w(t) |$.

(ii) *If* $p<0$, *then*

$$\Big|\frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{b^p+}(wog)(a^p)+J^{\alpha,k}_{a^p-}(wog)(b^p)\right]-\left[J^{\alpha,k}_{b^p+}(fwog)(a^p)+J^{\alpha,k}_{a^p-}(fwog)(b^p)\right]\Big| \le \frac{\|w\|_\infty(a^p-b^p)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}C_8^{\frac{1}{r}}(\alpha/k,p,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}, \tag{31}$$

where

$$C_8(\alpha/k,p,r)=\int_0^1\left(-\frac{|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}|}{p[ua^p+(1-u)b^p]^{1-\frac{1}{p}}}\right)^r du$$

with $g(x)=x^{1/p}, x\in[b^p,a^p]$ *and* $\| w \|_\infty = sup_{t\in[a,b]} | w(t) |$.

Proof. (i) Let $p>0$. Using the Hölder's inequality in (27) and the p-convexity of $|f'|^q$ we have

$$\frac{\Gamma_k(\alpha+k)}{\|w\|_\infty(b^p-a^p)^{\frac{\alpha}{k}+1}}\times\Big|\frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{a^p+}(wog)(b^p)+J^{\alpha,k}_{b^p-}(wog)(a^p)\right]-\left[J^{\alpha,k}_{a^p+}(fwog)(b^p)+J^{\alpha,k}_{b^p-}(fwog)(a^p)\right]\Big|$$
$$\le\left(\int_0^1\left(\Big|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}\Big|\frac{[ua^p+(1-u)b^p]^{1/p-1}}{p}\right)^r du\right)^{1/r}\times\left(\int_0^1|f'([ua^p+(1-u)b^p]^{1/p})|^q du\right)^{1/q}$$
$$\le\left(\int_0^1\left(\Big|(1-u)^{\frac{\alpha}{k}}-u^{\frac{\alpha}{k}}\Big|\frac{[ua^p+(1-u)b^p]^{1/p-1}}{p}\right)^r du\right)^{1/r}\times\left(|f'(a)|^q\int_0^1 udu+|f'(b)|^q\int_0^1(1-u)du\right)^{1/q}.$$

Simple computation gives

$$\Big|\frac{f(a)+f(b)}{2}\left[J^{\alpha,k}_{a^p+}(wog)(b^p)+J^{\alpha,k}_{b^p-}(wog)(a^p)\right]-\left[J^{\alpha,k}_{a^p+}(fwog)(b^p)+J^{\alpha,k}_{b^p-}(fwog)(a^p)\right]\Big| \le \frac{\|w\|_\infty(b^p-a^p)^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+k)}C_7^{\frac{1}{r}}(\alpha/k,p,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

This completes the proof of (30) and the proof of (31) is same as of (30).

Remark. We can observe the following in Theorem 12.

(i) If we put $k=1$, we get [8, Theorem 12].
(ii) If we put $k,p=1$, we get [11, Theorem 2.9 (i)].

(iii) If we put $w(x) = 1$ along with $k, p = 1$ and $\alpha = 1$, we get [2, Theorem 2.3].
(iv) If we put $k, p = 1$ and $\alpha = 1$, we get [26, Theorem 2.8].
(v) If we put $w(x) = 1$ along with $p, k = 1$, we get [8, Corollary 3 (1)].
(vi) If we put $w(x) = 1$ along with $k = 1$ and $p = -1$, we get [8, Corollary 3 (2)].
(vii) If we put $w(x) = 1$ along with $\alpha, k = 1$ and $p = -1$, we get [8, Corollary 3 (3)].
(viii) If we put $k = 1$ and $p = -1$, we get [8, Corollary 3 (4)].
(ix) If we put $\alpha, k = 1$ and $p = -1$, we get [8, Corollary 3 (5)].
(x) If we put $w(x) = 1$ along with $\alpha, k = 1$, we get [8, Corollary 3 (6)].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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