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Some applications on tangent bundle with Kaluza-Klein metric

Murat Altunbas and Merve Tastan

Erzincan University, Erzincan, Turkey

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Abstract: In this paper, differential equations of geodesics; parallelism, incompressibility and closeness conditions of the horizontal and complete lift of the vector fields are investigated with respect to Kaluza-Klein metric on tangent bundle.

Keywords: Kaluza-Klein metric, tangent bundle, geodesics, harmonic vector fields

1 Introduction

The history of tangent bundle TM of a Riemannian manifold M goes back with the work of Sasaki [1]. After the Dombrowski's comparision with the geometric objects of the base manifold and of the tangent bundle with respect to Sasaki metric ${}^{S}g$ in [2], many studies have been done on this topic in this sense and it has been seen that the metric ${}^{S}g$ arises a kind of "rigidity", i.e., most of geometric properties of the TM can not be ensured unless the base manifold M (and hence $(TM,{}^{S}g)$) is flat. To eliminate this deficiency, Musso and Tricceri defined another metric and called as Cheeger-Gromoll metric ${}^{CG}g$ [3]. It is shown that $(TM,{}^{CG}g)$ is not flat even if M is flat [4]. Later, more general metrics are demonstrated with deformations ${}^{CG}g$ including both Sasaki and Cheeger-Gromoll metrics. One of them is the metric $g_{a,b}$, which is obtained from deformation of the vertical part of the metric ${}^{CG}g$ with two positive definite scalar functions a and b [5].

In this paper, the Kaluza-Klein metric ${}^{KK}g$, which is defined by rescaling the horizontal part of the metric $g_{a,b}$, is considered and various characterizations on geodesics and some special vector fields are given with respect to this metric on tangent bundle. Throughout the paper; manifolds, functions and vector fields are differentiable of class C^{∞}.

2 Preliminaries

Let M be a manifold with finite dimension n. Then the set $TM = \bigcup_{P\in M} T_P M$ is the tangent bundle on M, where $\cup$ denotes the disjoint union of the tangent spaces $T_1^1(P)$ for all $P \in M$. TM is a $2n-$dimensional manifold. The natural projection $\pi : T_1^1(M) \to M$ is defined for any point $\tilde{P}$ of $T_1^1(M)$ such that $\tilde{P} \in T_1^1(P)$ with the surjective correspondence $\tilde{P} \to P$. If x^j are local coordinates in a neighbourhood U of $P \in M$, then a vector field u at P which is an element of TM is expressed in the form (x^j, u^j), where u^j are components of t with respect to the natural base. We consider $(x^j, u^j) = (x^j, x^{\bar{j}}) = (x^J)$, $j = 1,...,n$, $\bar{j} = n+1,...,2n$, $J = 1,...,2n$ as local coordinates in a neighborhood $\pi^{-1}(U)$.

* Corresponding author e-mail: maltunbas@erzincan.edu.tr

Now, let we assume that the base manifold M is a Riemannian manifold with the metric g and denote by ∇ its Levi-Civita connection. Then the vertical distribution is $VTM = \ker \pi_*$ and the horizontal distribution is defined by ∇. From here, the direct sum decomposition can be written as

$$TTM = VTM \oplus HTM.$$

In each local chart, with putting $\{E_{(j)}\} = \partial_j - u^s \Gamma_{sj}^h \partial_{\bar{h}}$, $\{E_{(\bar{j})}\} = \partial_{\bar{j}}$, we introduce a useful frame $\{E_\beta\} = \{E_{(j)}, E_{(\bar{j})}\}$ adapted to the distributions. If $X = X^i \frac{\partial}{\partial x^i}$ is the local expressions in U of a vector field X on M, then the vertical lift ${}^V A$, the horizontal lift ${}^H X$ and the complete lift ${}^C X$ of X are given with respect to the adapted frame $\{E_\beta\}$ by

$$ {}^H X = \begin{pmatrix} {}^H X^j \\ {}^H X^{\bar{j}} \end{pmatrix} = \begin{pmatrix} X^j \\ 0 \end{pmatrix}, \tag{1}$$

$$ {}^C X = \begin{pmatrix} {}^C X^j \\ {}^C X^{\bar{j}} \end{pmatrix} = \begin{pmatrix} X^j \\ u^s \nabla_s X^j \end{pmatrix}. \tag{2}$$

Lemma 1. *The Lie brackets of the adapted frame of TM satisfy the following identities:*

$$\begin{cases} [E_j, E_i] = u^s R_{ijs}{}^m E_{\bar{h}}, \\ [E_j, E_{\bar{i}}] = \Gamma_{ji}^h E_{\bar{h}}, \\ [E_{\bar{j}}, E_{\bar{i}}] = 0. \end{cases}$$

where $R_{ijs}{}^m$ denotes the components of the curvature tensor of M.

3 Kaluza-Klein metric on tangent bundle

Definition 1. *Let (M,g) be a Riemannian manifold, a metric ${}^{KK}g$ on TM will be called Kaluza-Klein if it takes the form*

$${}^{KK}g\left({}^H X, {}^H Y\right) = c(t)g(X,Y)),$$
$${}^{KK}g\left({}^H X, {}^V Y\right) = 0,$$
$${}^{KK}g\left({}^V X, {}^V Y\right) = a(t)g(X,Y) + b(t)g(X,Y)$$

where t is energy density in direction of u is defined by $t = g(u,u)/2$, c is strictly positive and a and b such that ${}^{KK}g$ is positive definite.

The metric ${}^{KK}g$ and its inverse have respectively the components with respect to adapted frame E_β as follows

$$ {}^{KK}g = \begin{pmatrix} cg_{ji} & 0 \\ 0 & ag_{ji} + bg_{js}g_{it}u^s u^t \end{pmatrix},$$

$$ ({}^{KK}g)^{-1} = \begin{pmatrix} \frac{1}{c}g^{ij} & 0 \\ 0 & \frac{1}{a}g^{ij} - \frac{b}{a(a+2bt)}u^i u^j \end{pmatrix}.$$

Standard computations with the Koszul's formula and having mind the Lemma 1 give the following proposition.

Proposition 1. *Let ^{KK}g be a Kaluza-Klein metric on TM, then the corresponding Levi-Civita connection coefficients are given by*

$$\begin{cases} ^{KK}\Gamma_{ji}^{h}=\Gamma_{ji}^{h}, \quad ^{KK}\Gamma_{ji}^{\bar{h}}=-\frac{1}{2}u^{k}R_{jik}{}^{h}-\frac{c'}{2(a+2tb)}g_{ji}u^{h}, \\ ^{KK}\Gamma_{j\bar{i}}^{h}=\frac{a}{2c}u^{k}R_{kij}{}^{h}+\frac{c'}{2c}u_{i}\delta_{j}^{h}, \quad ^{KK}\Gamma_{j\bar{i}}^{\bar{h}}=\Gamma_{ji}^{h}, \\ ^{KK}\Gamma_{\bar{j}i}^{h}=\frac{a}{2}u^{k}R_{kji}{}^{h}+\frac{c'}{2c}u_{j}\delta_{i}^{h}, \quad ^{KK}\Gamma_{\bar{j}i}^{\bar{h}}=0, \\ ^{KK}\Gamma_{\bar{j}\bar{i}}^{h}=0, \\ ^{KK}\Gamma_{\bar{j}\bar{i}}^{\bar{h}}=L(u_{j}\delta_{i}^{h}+u_{i}\delta_{j}^{h})+Mg_{ji}u^{h}+Nu_{j}u_{i}u^{h} \end{cases} \tag{3}$$

with respect to adapted frame E_β. Here, Γ_{ji}^{h} and $R_{kij}{}^{h}$ denotes the Levi-Civita connection and Riemannian curvature components of g respectively and $L=\frac{a'}{2a}$, $M=\frac{2b-a'}{2(a+2tb)}$, $N=\frac{ab'-2a'b}{2a(a+2tb)}$.

For the invariant expression of the Levi-Civita connection, see [6].

4 Some special vector fields on $(TM,{}^{KK}g)$

First, let us compute the covariant derivatives of horizontal lift and complete lift of the vector field X. Then we have

$$^{KK}\nabla^{H}X=\begin{pmatrix} \nabla_{i}X^{h} & -\frac{a}{2}R_{iks}{}^{h}u^{k}X^{s}+\frac{c'}{2c}u_{i}X^{h} \\ -\frac{1}{2}R_{isk}{}^{h}u^{k}X^{s}-\frac{c'}{2(a+2tb)}u^{h}X_{i} & 0 \end{pmatrix}, \tag{4}$$

$$^{KK}\nabla^{C}X=\begin{pmatrix} \nabla_{i}X^{h}-\frac{a}{2c}R_{mki}{}^{h}u^{k}u^{s}\nabla_{s}X^{m}+\frac{c'}{2c}u_{m}u^{s}\nabla_{s}X^{m}\delta_{i}^{h} & -\frac{a}{2}R_{ikm}{}^{h}u^{k}X^{m}+\frac{c'}{2c}u_{i}X^{h} \\ u^{s}(\nabla_{i}\nabla_{s}X^{h}-\frac{1}{2}R_{ims}{}^{h}X^{h})-\frac{c'}{2(a+2tb)}u^{h}X_{i} & \nabla_{i}X^{h}+(L(u_{i}\delta_{m}^{h}+u_{m}\delta_{i}^{h})+Mg_{mi}u^{h}+Nu_{m}u_{i}u^{h})u^{s}\nabla_{s}X^{m} \end{pmatrix}. \tag{5}$$

Since $R_{mki}{}^{h}X^{i}=0$ is a consequence of the fact that $\nabla_{i}X^{h}=0$, we obtain the following proposition from (4) and (5).

Proposition 2. *The complete (resp. horizontal) lift of a vector field on M to TM is parallel in $(TM,{}^{KK}g)$ if the given vector field is parallel and c is a constant function.*

Secondly, we consider the divergence and rotation of the lifts of vector fields. A vector field X on (M,g) is said to be *incompressible* if and only if its divergence $divX=\nabla_{j}X^{j}=0$. On the other hand, a vector field X on (M,g) is said to be *closed* if the rotation of associated covector field (whose components are $X_{j}=g_{ji}X^{i}$) is zero, i.e.,

$$(rotX)_{ji}=\partial_{j}X_{i}-\partial_{i}X_{j}=\nabla_{j}X_{i}-\nabla_{i}X_{j}=0. \tag{6}$$

X is said to be *harmonic* on (M,g) if it is both incompressible and closed. From (4) and (5), the divergence of the lifts are

$$div^{H}X=divX,\ div^{C}X=2divX+u_{m}[L(n+1)+M+N\|u\|^{2}]u^{s}\nabla_{s}X^{m}+\frac{nc'}{2c}u_{m}u^{s}\nabla_{s}X^{m}. \tag{7}$$

Thus, it is clear that the horizontal lift of X is incompressible if and only if it is incompressible in M. But, the complete lift of X is not incompressible even if it is incompressible in M in general. For the special case $a=cons.$, $b=0$ and $c=cons.$, we can say the complete lift of X is incompressible if and only if it is incompressible in M.

The rotation of associated covector field of a vector field $\tilde{X}$ on $(TM,{}^{KK}\nabla)$ is given by

$$^{KK}\nabla_{J}X_{I}-{}^{KK}\nabla_{I}X_{J}={}^{KK}g_{IM}{}^{KK}\nabla_{J}X^{M}-{}^{KK}g_{JM}{}^{KK}\nabla_{I}X^{M}. \tag{8}$$

Using (4) and (5), we obtain the components of $rot^H X$ and $rot^C X$ as

$$
\begin{aligned}
rot(^H X)_{ji} &= c(rotX)_{ji}, \qquad (9)\\
rot(^H X)_{\bar{j}i} &= (\frac{a(1-c)}{2}R_{jkbi} + \frac{b}{2}u_j u_s R_{ibk}{}^s)u^k X^b + \frac{c'}{2(a+2tb)}(a+b\|u\|^2)u_j X_i,\\
rot(^H X)_{j\bar{i}} &= (\frac{a(1-c)}{2}R_{bjki} + \frac{b}{2}u_i u_s R_{bjk}{}^s)u^k X^b - \frac{c'}{2(a+2tb)}(a+b\|u\|^2)u_j X_i,\\
rot(^H X)_{\bar{j}\bar{i}} &= 0;\\
rot(^C X)_{ji} &= c(rotX)_{ji} + aR_{jikm}u^k u^b(\nabla_b X^m), \qquad (10)\\
rot(^C X)_{\bar{j}i} &= (\frac{a(1-c)}{2}R_{ikbj} + \frac{b}{2}u_j u_s R_{ikb}{}^s)u^k X^b - u^s(a\nabla_i\nabla_s X_j + b\nabla_i\nabla_s X^a u_j u_a) + \frac{c'}{2(a+2tb)}(a+b\|u\|^2)u_j X_i,\\
rot(^C X)_{j\bar{i}} &= (\frac{a(1-c)}{2}R_{bjki} + \frac{b}{2}u_i u_s R_{bjk}{}^s)u^k X^b + u^s(a\nabla_j\nabla_s X_i + b\nabla_j\nabla_s X^a u_i u_a) - \frac{c'}{2(a+2tb)}(a+b\|u\|^2)u_i X_j,\\
rot(^C X)_{\bar{j}\bar{i}} &= a(rotX)_{ji} + bu^s(\nabla_j X_s u_i - \nabla_i X_s u_j).
\end{aligned}
$$

Thus, we get the following proposition.

Proposition 3. *Let (M,g) be a flat Riemannian manifold and c is a constant function. Then the followings hold.*

(i) *The horizontal lift of a vector field X on M to $(TM,^{KK}g)$ is harmonic if and only if X is harmonic on M.*

(ii) *The complete lift of a vector field X is harmonic if and and only if it is parallel in (M,g).*

5 Geodesics on $(TM,^{KK}g)$

It is known that a curve $\tilde{\gamma}$ is a geodesic on TM with respect to $^{KK}\nabla$ if and only if it satisfies the following differential equations

$$\frac{d}{dt}(\frac{\theta^\alpha}{dt}) +^{KK} \Gamma^\alpha_{\gamma\beta}\frac{\theta^\gamma}{dt}\frac{\theta^\beta}{dt} = 0$$

with respect to adapted frame, where

$$\frac{\theta^h}{dt} = \frac{dx^h}{dt}, \quad \frac{\theta^{\bar{h}}}{dt} = \frac{\delta u^h}{dt} = \frac{du^h}{dt} + \Gamma^h_{ji}\frac{dx^j}{dt}u^i.$$

By direct computations with using (4), then we have the following proposition.

Proposition 4. *Let $\tilde{\gamma}$ be a curve on TM with locally expressions $x^h = x^h(t), x^{\bar{h}} = u^h(t)$ with respect to induced coordinates $(x^h, x^{\bar{h}})$ in $\pi^{-1}(U) \subset TM$. Then the curve $\tilde{\gamma}$ is a geodesic on $(TM,^{KK}g)$ if and only if it satisfies the following equations*

$$\begin{cases} \frac{\delta^2 x^h}{dt^2} + \frac{a(c+1)}{2c}(u^k R_{kji}{}^h + \frac{c'}{c}u_j\delta^h_i)\frac{\delta u^j}{dt}\frac{dx^i}{dt} = 0, \\ \frac{\delta^2 u^h}{dt^2} + [L(u_j\delta^h_i + u_i\delta^h_j) + Mg_{ji}u^h + Nu_j u_i u^h]\frac{\delta u^j}{dt}\frac{\delta u^i}{dt} - \frac{c'}{2(a+2tb)}g_{ji}u^h\frac{dx^j}{dt}\frac{dx^i}{dt} = 0. \end{cases} \qquad (11)$$

If a curve $\tilde{\gamma}$ satisfying the equations (11) lies on a fibre given by $x^h = cons.$, then by virtue of $\frac{dx^h}{dt} = 0$ and $\frac{\delta u^h}{dt} = \frac{du^h}{dt}$, the equations (11) reduces to

$$\frac{\delta^2 u^h}{dt^2} + [L(u_j\delta^h_i + u_i\delta^h_j) + Mg_{ji}u^h + Nu_j u_i u^h]\frac{du^j}{dt}\frac{du^i}{dt} = 0. \qquad (12)$$

Hence, we get the proposition below.

Proposition 5. *If a geodesic lies on a fibre of $(TM,^{KK}g)$,it's local expression as in (12).*

Now, let $\gamma = \pi \circ^H \gamma$ be a geodesic on (M,∇), i.e. $\frac{\delta^2 x^h}{dt^2} = 0$. Using this condition and $\frac{\delta u^j}{dt} = \frac{\delta X^h}{dt} = 0$, the following proposition is obtained.

Proposition 6. *The horizontal lift of a geodesic on* (M,∇) *is not a geodesic* $(TM,^{KK}g)$.

Finally, let $\gamma = \pi \circ \tilde{\gamma}$ be a geodesic on (M,∇), i.e. $\frac{\delta^2 x^h}{dt^2} = \frac{\delta}{dt}(\frac{dx^h}{dt}) = 0$. On the other hand, from the definition of the natural lift of the curve $(x^h = x^h(t),\, u^h = \frac{dx^h}{dt})$, we obtain

$$\frac{\delta^2 u^h}{dt^2} = \frac{c'}{2(a+2tb)} g_{ji} u^h \frac{dx^j}{dt}\frac{dx^i}{dt} \tag{13}$$

By the virtue of (13) and (11), it is easily seen that the natural lift of a curve on M is a not a geodesic on $(TM,^{KK}g)$. This corollary is given as a proposition below.

Proposition 7. *The natural lift* $\tilde{\gamma}$ *any geodesic on M is not a geodesic on* $(TM,^{KK}g)$.

6 Conclusions

In this study, some characterizations on specific vector fields and geodesics are given on tangent bundle with respect to a Cheeger-Gromoll type metric, say Kaluza-Klein metric. The results in this paper have two significant difference from previous papers. Firstly, the proposition 3 was given as follows with respect to Sasaki metric in [7] : ”The complete (resp. horizontal) lift of a vector field X on M to TM is harmonic with in $(TM,^{S}g)$ if and only if X is harmonic and has vanishing second covariant derivative (resp. harmonic) in (M,g).” Here, this proposition is written under the additional conditions such that flatness of the base manifold and the constancy of the function c. Secondly, the last two propositions on geodesics were written affirmatively with respect to the metrics Sasaki [8], Cheeger-Gromoll [9] and the metric $g_{a,b}$ [10].

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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