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Note on symplectic SVD-like decomposition

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Abstract: The aim of this study was to introduce a constructive method to compute a symplectic singular value decomposition (SVD-like decomposition) of a $2n$-by-m rectangular real matrix A, based on symplectic reflectors. This approach used a canonical Schur form of skew-symmetric matrix and it allowed us to compute eigenvalues for the structured matrices as Hamiltonian matrix JAA^T.

Keywords: Singular value decomposition (SVD), Hamiltonian matrix, skew-symmetric matrix, skew-Hamiltonian matrix, symplectic matrix, Schur form, symplectic reflector.

1 Introduction

Singular Value Decomposition (SVD) has been used in many fields of scientific computing such as data compression, signal processing, automatic control working on applied linear algebra, signal and image processing [13, 14]. An example is about the eigenvalue problem of the matrix

$$F = \begin{bmatrix} -C & -G \\ I & 0 \end{bmatrix} = \begin{bmatrix} -C & -I0 \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & G \end{bmatrix}$$

which is related to the gyroscopic system [10, 11, 12]

$$q" + Cq' + Gq = 0\ ;\ q(0) = q_0\ ;\ q'(0) = q_1.$$

A matrix $G \in \mathbb{R}^{m\times m}$ is symmetric and positive semi-definite it has a full rank factorization $G = LL^T$. And $C \in \mathbb{R}^{m\times m}$ is skew-symmetric. By using the equality

$$\begin{bmatrix} -C & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}C & I \\ I & 0 \end{bmatrix} J \begin{bmatrix} \frac{1}{2}C & I \\ I & 0 \end{bmatrix}$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, I_n denotes the $n\times n$ identity matrix, F is similar to the Hamiltonian matrix

$$J \begin{bmatrix} \frac{1}{2}C & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & LL^T \end{bmatrix} = J \begin{bmatrix} -\frac{1}{2}C & I \\ L^T & 0 \end{bmatrix}^T \begin{bmatrix} -\frac{1}{2}C & I \\ L^T & 0 \end{bmatrix}.$$

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Therefore the eigenvalue problem of F can be solved by computing the *SVD*-like decomposition of $\begin{pmatrix} -\frac{1}{2}C & I \\ L^T & 0 \end{pmatrix}$.

This paper makes a constructive and significant contribution to this area of research for computing a symplectic SVD-like decomposition of $2n$-by-m real matrix based on a canonical Schur form of skew-symmetric matrix [16,17] and by the use of symplectic reflectors [1,2,3]. A method for computing an SVD-like decomposition was given by Hongguo Xu [16,17] of a n-by-$2m$ real matrix.

Symplectic SVD-like decomposition is effective for computing the structured canonical form of the Hamiltonian matrix $A^J A$. Most eigenvalue problems that arise in practice are known to be structured. Therefore, preserving the structure can help preserve physically relevant symmetries in the eigenvalues of the matrix and may improve the accuracy and efficiency of eigenvalue computation. Hamiltonian and skew-Hamiltonian eigenvalue problems arise from a number of applications, particularly in systems and control theory [7,12,15].

The paper is organized as follows: section 2 introduces some notation and some basic result; a symplectic SVD-like decomposition is proposed in section 3; and section 4 we give a numerical results to demonstrate the effectiveness of the proposed algorithm.

2 Terminology, notation and some basic facts

In this section, we recall some notations and necessary tools which will be used throughout this paper. The J-transpose of any $2n$-by-$2p$ matrix M is defined by $M^J = J_{2p}^T M^T J_{2n} \in \mathbb{R}^{2p\times 2n}$ where $J_{2n} = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$, with I_n and O_n are the $n\times n$ identity and zero matrix respectively. A Hamiltonian matrix $M \in \mathbb{R}^{2n\times 2n}$ has the explicit block structure $M = \begin{pmatrix} A & R \\ G & -A^T \end{pmatrix}$, where A, G, R are real $n\times n$ matrices and $G = G^T$, $R = R^T$. By straightforward algebraic manipulation, we can show that a Hamiltonian matrix M is equivalently defined by the property $M^J = -M$. Likewise, a matrix M is skew-Hamiltonian if and only if $M^J = -M$, it has the explicit block structure $W = \begin{pmatrix} A & R \\ G & A^T \end{pmatrix}$, where A, G, R are real $n\times n$ matrices and $G = -G^T$, $R = -R^T$. Any matrix $S \in \mathbb{R}^{2n\times 2p}$ that satisfies this property $S^T J_{2n} S = J_{2p}$ ($S^J S = I_{2p}$) is called symplectic matrix. This property is also called J-orthogonality. The symplectic similarity transformations preserve Hamiltonian and skew-Hamiltonian structures.

Proposition 1. *An augmented matrix*

$$S = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & P_{11} & 0 & P_{12} \\ 0 & 0 & I & 0 \\ 0 & P_{21} & 0 & P_{22} \end{pmatrix}$$

is symplectic if and only if $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ *is also symplectic.*

Setting $E_i = [e_i \; e_{n+i}] \in \mathbb{R}^{2n\times 2}$ for $i = 1, \cdots, n$, we obtain

$$E_i^J = E_i^T \text{ and } E_i^J E_j = \delta_{ij} I_2,$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Proposition 2. *Let $U = [u_1\ u_2]$ be a 2n-by-2 real matrix, where $u_1 = \sum_{i=1}^{2n} u_i^{(1)} e_i$ and $u_2 = \sum_{j=1}^{2n} u_j^{(2)} e_j$. Then, U is written uniquely as linear combination of $(E_i)_{1\le i\le n}$ on the ring $\mathbb{R}^{2\times 2}$.*

$$U = \sum_{i=1}^{n} E_i M_i \text{ where } M_i = \begin{pmatrix} u_i^{(1)} & u_i^{(2)} \\ u_{n+i}^{(1)} & u_{n+i}^{(2)} \end{pmatrix}.$$

Proposition 3. *Let M be a 2n-by-2n real matrix. Then, M is expressed uniquely as $M = \sum_{i=1}^{n}\sum_{j=1}^{n} E_i M_{ij} E_j^T$ where $M_{ij} \in \mathbb{R}^{2s\times 2s}$ is given by,*

$$\left(\begin{array}{c|c} m_{i,j} & m_{i,n+j} \\ \hline m_{n+i,j} & m_{n+i,n+j} \end{array}\right).$$

Proposition 4. *With the notations of the previous proposition, a matrix $M \in \mathbb{R}^{2n\times 2n}$ is Hamiltonian (skew-Hamiltonian) if $M_{ij}^J = -M_{ji}$ ($M_{ij}^J = M_{ji}$).*

Proof. The result is obvious, as $M^J = \sum_{i=1}^{n}\sum_{j=1}^{n} E_i M_{ji}^J E_j^T$ and $M^J = -M$.

Definition 1. *A matrix $M = \sum_{i=1}^{n}\sum_{j=1}^{n} E_i M_{ij} E_j^T \in \mathbb{R}^{2n\times 2n}$ is called in upper J-bidiagonal form if $M_{ij} = 0_2$ for $j \notin \{i, i+1\}$ and, in addition, M_{ii} and M_{ii+1} are diagonal.*

2.1 Symplectic reflectors

The symplectic reflector [2,3] in $\mathbb{R}^{2n\times 2}$ is defined in parallel with elementary reflectors.

Proposition 5. [3] *Let U and V be two 2n-by-2 real matrices that satisfy $U^J U = V^J V = I_2$. If the 2-by-2 matrix $C = I_2 + V^J U$ is nonsingular, the transformation $S = (U+V)C^{-1}(U+V)^J - I_{2n}$ is symplectic and takes U to V. This is called a symplectic reflector. Additionally, if $U^J = U^T$ and $V^J = V^T$, then S is orthogonal and symplectic.*

Remark. The proposition above remains true only if $U^J U = V^J V$. In this case, $C = U^J U + V^J U$.

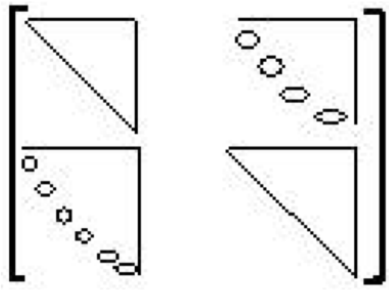
Lemma 1. *Let $U = [u_1\ \ u_2] \in \mathbb{R}^{2n\times 2}$ be a non-isotropic matrix ($U^J U \neq 0_2$) and $V = Uq(U)^{-1}$ its normalized matrix. Then, there is a symplectic reflector S takes V to E_1 and therefore U to $E_1 q(U)$, which in turn takes the following form:*

$$SU = \begin{pmatrix} * & \mathbf{0} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \mathbf{0} & * \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \swarrow n+1$$

where

$$q(U) = \begin{cases} \sqrt{\alpha} I_2, & \text{if } \alpha > 0 \\ \sqrt{-\alpha}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \text{if } \alpha < 0 \end{cases}$$
$$\alpha = u_1^H J u_2.$$

*Remark.*Using symplectic reflectors with a matrix $A \in \mathbb{R}^{2n\times 2n}$, we obtain the factorization $A = SR$, where $S \in \mathbb{R}^{2n\times 2n}$ is symplectic and $R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$. R is J-triangular and, in addition, R_{12} is a strictly n-by-n upper triangular matrix. R is as follows:



We discuss below some useful properties of symplectic reflectors.

Proposition 6. *Let S be a $2n$-by-$2n$ real symplectic matrix. There is then a sequence of symplectic reflectors $S_1, S_2, \cdots, S_n$, such that $S = S_1 S_2 \cdots S_n$.*

Proof. **Step 1:** Set $U_1 = [q_1, q_{n+1}] \in \mathbb{R}^{2n\times 2}$. As S is symplectic, then $U_1^J U_1 = I_2$. Then, the symplectic reflector $P_1 = (U_1 + E_1)(I_2 + E_1^J U_1)^{-1}(U_1 + E_1)^J - I_{2n}$ verifies $P_1 U_1 = E_1$. The $(n+1)^{th}$-component of both $(P_1 q_k)$ and $(P_1 q_{n+k})$ is equal to zero for $k = 2, 3, \ldots n$. On the one hand, $(P_1 q_1)^T J (P_1 q_k) = q_1^T J q_k = 0$, and on the other hand, $(P_1 q_1)^T J (P_1 q_k) = e_1^T J (P_1 q_k) = e_{n+1}^T (P_1 q_k)$ is simply the $(n+1)^{th}$-component of $(P_1 q_k)$. Likewise, the first component of both $(P_1 q_k)$ and $(P_1 q_{n+k})$ disppears. Finally, we obtain

$$P_1 S = \begin{array}{cc} & \begin{array}{cc} \hspace{4em} n \hspace{4em} & \hspace{4em} n \hspace{4em} \end{array} \\ \begin{array}{c} n \\[5em] n \end{array} & \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * & 0 & * & \cdots & * \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * & 0 & * & \cdots & * \end{array}\right) \end{array}$$

Thereafter, we continue to update the value of q_i: $q_i \longleftarrow P_1 q_i$ by varying i from 1 to $2n$. Note that now we have $q_1 = e_1$ and $q_{n+1} = e_{n+1}$.

Step 2: Set $U_1 = [q_1, q_{n+1}] \in \mathbb{R}^{2n\times 2}$. As S is symplectic, then $U_1^J U_1 = I_2$ and the symplectic reflector allows us to set $U_2 = [q_2, q_{n+2}] \in \mathbb{R}^{2n\times 2}$. As $P_1 S$ is still symplectic, U_2 verifies $U_2^J U_2 = I_2$, and the symplectic reflector

$P_2 = (U_2+E_2)(I_2+E_2^J U_2)^{-1}(U_2+E_2)^J - I_{2n}$ has the following form:

$$P_2 = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & * & * & * & \vdots & * & * & * \\ \vdots & * & * & * & \vdots & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ \hline 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & * & * & * & 0 & * & * & * \\ \vdots & * & * & * & \vdots & * & * & * \\ 0 & * & * & * & 0 & * & * & * \end{array}\right)$$

and verifies $P_2U_2 = E_2$. As in step 1, we obtain

$$P_2P_1S = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & * & \dots & * & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \cdots & * & 0 & 0 & * & \dots & * \\ \hline 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & * & \dots & * & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \dots & * & 0 & 0 & * & \dots & * \end{array}\right).$$

We thereby obtain $P_n \cdots P_2P_1S = I_{2n}$, and then $S = S_1S_2\cdots S_n$ where $S_k = P_k^J$, which achieves the desired result.

Remark. In lemma 2.8, by using $U = [u \ \ -Ju]$, where $u \in \mathbb{R}^{2n}$ with $\|u\| \neq 0$, we obtain S orthogonal and symplectic.

Lemma 2. *Let $u \in \mathbb{R}^{2s}$ be a nonzero $2s$-component real vector. The orthogonal symplectic reflector $S = (U+\sqrt{\alpha}E_1)(\alpha I_2 + \sqrt{\alpha}E_1^J U)^{-1}(U+\sqrt{\alpha}E_1)^J - I_{2s}$, where $U = [u \ \ -Ju]$ verifies $Su = \sqrt{\alpha}e_1$ with $\alpha = u^T u = \|u\|_2^2$.*

Proof. As $U^JU = \alpha I_2$ with $\alpha = u^T u = \|u\|_2^2 > 0$, then a simple calculation gives the result.

3 Symplectic SVD-like decomposition

We describe here a new approach to compute a symplectic SVD-like decomposition for a $2n$-by-m rectangular real matrix A. It's based on a Schur form of skew-symmetric matrix A^TJA. We obtained the following result:

$$SAQ = \left(\begin{array}{c|c|c|c} \Sigma_p & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & \Sigma_p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

where Q is an orthogonal matrix, S is symplectic, and $\Sigma_p = \{\sigma_1, \cdots, \sigma_p\}$, $p = \dfrac{rank(A^TJA)}{2}$.

Let A be $2n$-by-m rectangular real matrix. We recall here an useful result, which is the Schur like form of the real skew-symmetric matrix A^TJA [16,17].

Theorem 1. *Given a* $2n$*-by-*m *real matrix* A*, there is a real orthogonal matrix* U *such that*

$$A^T JA = U \left(\begin{array}{c|c|c} 0_p & \Sigma_p^2 & 0 \\ -\Sigma_p^2 & 0_p & 0 \\ 0 & 0 & 0_{m-2p} \end{array} \right) U^T$$

where $\Sigma_p = diag(\sigma_1, \sigma_2, \dots, \sigma_p)$, $\sigma_i > 0$, $\forall i$ *and* $2p = rank(A^T JA)$.

Xu [16,17] showed that for any n-by-$2m$ real matrix A, there exists an orthogonal matrix Q and a symplectic matrix S, such that $A = QDS^{-1}$, where D is in the following form,

$$D = \begin{pmatrix} \Sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where Σ is positive diagonal. Symplectic SVD-like decomposition is effective for computing the structured canonical form of the Hamiltonian matrix $JA^T A$. Xu also proposed an algorithm for computing eigenvalues of $JA^T A$ using block A_{11} and A_{23} in step 1 of the algorithm (for more details, see section 2 in [17]). Although he obtained the eigenvalues, his algorithm does not compute the full decomposition of the poorly scaled matrices. We present a new constructive approach which is main result of this paper, in order to obtain the symplectic SVD-like decomposition and to compute the eigenvalues of Hamiltonian matrix JAA^T.

Theorem 2. ***(Symplectic SVD-like decomposition)*** *Let* A *be a* $2n$*-by-*m *rectangular real matrix. For a symplectic real matrix* $S \in \mathbb{R}^{2n\times 2n}$ *and an orthogonal real matrix* $Q \in \mathbb{R}^{m\times m}$*,*

$$SAQ = \left(\begin{array}{c|c|c|c} \Sigma_p & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & \Sigma_p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Proof. Applying the real Schur decomposition to the skew-symmetric matrix $A^T JA$

$$A^T JA = U \left(\begin{array}{c|c|c} 0_p & \Sigma_p^2 & 0 \\ \hline -\Sigma_p^2 & 0_p & 0 \\ \hline 0 & 0 & 0_{m-2p} \end{array} \right) U^T,$$

we construct $V_k = \dfrac{1}{\sigma_k} AU_k J_2^T$, where $U_k = U \left[e_k \;\; e_{p+k} \right]$ for $k = 1, 2, \cdots, p$. As we can easily verify $V_k^J V_k = I_2$, the $2n$-by-2 matrix V_k is symplectic. We have, $\mathscr{R}(A) = span\{AU\} = span\left\{AU^{[p]}\right\} \oplus span\left\{AU^{[s]}\right\}$, where $U^{[p]} = U(:, 1:2p)$ and $U^{[s]} = U(:, 2p+1:m)$. Therefore,

$$\begin{aligned} w \in span\left\{AU^{[p]}\right\} &\Longrightarrow w = AU^{[p]} z, \text{ where } z \in \mathbb{R}^{2p} \\ &\Longrightarrow A^T Jw = \left(A^T JA\right) U^{[p]} z \\ &\Longrightarrow A^T Jw = U^{[p]} \left(\begin{array}{c|c} 0_p & \Sigma_p^2 \\ \hline -\Sigma_p^2 & 0_p \end{array} \right) z \end{aligned}$$

and

$$\begin{aligned} w \in span\left\{AU^{[s]}\right\} &\Longrightarrow w = AU^{[s]}y, \text{ where } y \in \mathbb{R}^{m-2p} \\ &\Longrightarrow A^TJw = \left(A^TJA\right)U^{[s]}y = 0_{\mathbb{R}^m}. \end{aligned}$$

Thus,

$$\begin{aligned} w \in span\left\{AU^{[p]}\right\} \cap span\left\{AU^{[s]}\right\} &\Longrightarrow \begin{cases} w = AU^{[p]}z, \text{ where } z \in \mathbb{R}^{2p} \\ A^TJw = U^{[p]}\left(\begin{array}{c|c} 0_p & \Sigma_p^2 \\ \hline -\Sigma_p^2 & 0_p \end{array}\right) z = 0_{\mathbb{R}^m} \end{cases} \\ &\Longrightarrow z = 0_{\mathbb{R}^{2p}} \text{ and then } w = 0_{\mathbb{R}^{2n}}. \end{aligned}$$

This proves $span\left\{AU^{[p]}\right\} \cap span\left\{AU^{[s]}\right\} = \{0_{\mathbb{R}^{2n}}\}$. As a consequence, we have $\dim(span\left\{AU^{[p]}\right\}) = 2p$ and $\dim(span\left\{AU^{[s]}\right\}) = q = r-2p$, where r is the rank of A. The null space $\mathcal{N}(A)$ of the matrix A verifies $\mathcal{N}(A) \subset span\left\{U^{[s]}\right\}$.

Suppose that $\mathcal{N}(A) = span\left\{U^{[s]}(:, q+1 : m-2p)\right\} = span\{U(:, 2p+q+1 : m)\}$. We then set $v_j = Au_j$ for $j = 2p+1, \cdots, r = 2p+q$. As $v_j = Au_j$ where $u_j \in \mathcal{N}(A^TJA)$, then $(v_j)_{2p+1 \le j \le r}$ are J-orthogonal and $V_k^TJv_j = 0$ for all $1 \le k \le p$ and $2p+1 \le j \le r$. Let us construct the $2n$-by-r matrix V as follow: $V = \sum_{k=1}^{p} V_k \left[e_k \;\; e_{p+k}\right]^T + \sum_{j=2p+1}^{r} v_j e_j^T$ where e_i is the i^{th} canonical basis vector of $\mathbb{R}^r$. We have $V^TJ_{2n}V = \begin{pmatrix} 0_p & I_p & 0 \\ -I_p & 0_p & 0 \\ 0 & 0 & 0_q \end{pmatrix}$. And let's the m-by-m orthogonal matrix Q given by $Q = \sum_{k=1}^{p} U_k J_2^T \left[e_k \;\; e_{p+k}\right]^T + \sum_{j=2p+1}^{m} u_j e_j^T$, here e_i is the i^{th} canonical basis vector of $\mathbb{R}^m$. We have

$$AQ = V \left(\begin{array}{c|c|c} \Sigma_p & 0 & 0 \\ \hline 0 & \Sigma_p & 0 \\ \hline 0 & 0 & I_{q\times(m-2p)} \end{array}\right).$$

Step 1: As $V_1 = [v_1, v_{p+1}] \in \mathbb{R}^{2n\times 2}$ verifies $V_1^JV_1 = I_2$, then the symplectic reflector $S_1 = (V_1+E_1)(I_2+E_1^JV_1)^{-1}(V_1+E_1)^J - I_{2n}$ verifies $S_1V_1 = E_1$. From the J-orthogonality given above, we have $(S_1V_1)^T J(S_1v_j) = E_1^TJ(S_1v_j) = 0$ for all $j \neq 1$ and $j \neq p+1$. Thus, the 1^{st} and the $(n+1)^{th}$ components of (S_1v_j) are equal to zero. We obtain

$$S_1V = \begin{array}{c} \\ n \\ \\ n \end{array} \left(\begin{array}{cccc|cccc|cccc} \multicolumn{4}{c|}{\overleftrightarrow{\quad p \quad}} & \multicolumn{4}{c|}{\overleftrightarrow{\quad p \quad}} & \multicolumn{4}{c}{\overleftrightarrow{\quad r-2p \quad}} \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * & 0 & * & \cdots & * & * & \cdots & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & * & \cdots & * & 0 & * & \cdots & * & * & \cdots & \cdots & * \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * & 0 & * & \cdots & * & * & \cdots & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & * \\ 0 & * & \cdots & * & 0 & * & \cdots & * & * & \cdots & \cdots & * \end{array}\right).$$

We can therefore update the value of V: $V \longleftarrow S_1V$.

Step 2: $V_2 = [v_2, v_{p+2}] \in \mathbb{R}^{2n\times 2}$ again verifies $V_2^JV_2 = I_2$. Then, the symplectic reflector $S_2 = (V_2+E_2)(I_2+E_2^JV_2)^{-1}(V_2+E_2)^J - I_{2n}$ is such that $S_2V_2 = E_2$. Again, from the J-orthogonality, we have $E_2^TJ(S_2v_j) = 0$ for all $j \neq 2$ and $j \neq p+2$. Thus, the 2^{nd} and the $(n+2)^{th}$ components of (S_2v_j) are equal to zero. We obtain

$$S_2V=\begin{array}{cc}
 & \begin{array}{ccc} \overbrace{\qquad\qquad\qquad}^{p} & \overbrace{\qquad\qquad\qquad}^{p} & \overbrace{\qquad\qquad}^{r-2p} \end{array} \\
\begin{array}{c} n \\[8em] n \end{array} &
\left(\begin{array}{ccccc|ccccc|cccc}
1 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & * & \cdots & * & 0 & 0 & * & \cdots & * & * & * & * & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & * & * & * & * \\
0 & 0 & * & \cdots & * & 0 & 0 & * & \cdots & * & * & * & * & * \\
\hline
0 & 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & * & \cdots & * & 0 & 0 & * & \cdots & * & * & * & * & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & * & * & * & * \\
0 & 0 & * & \cdots & * & 0 & 0 & * & \cdots & * & * & * & * & *
\end{array}\right)
\end{array}.$$

Continuing until the p^{th} step, we obtain

$$S_pV=\begin{array}{cc}
 & \begin{array}{ccc} \overbrace{\qquad\qquad}^{p} & \overbrace{\qquad\qquad}^{p} & \overbrace{\qquad\qquad}^{r-2p} \end{array} \\
\begin{array}{c} n \\[8em] n \end{array} &
\left(\begin{array}{ccc|ccc|cccc}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
\hline
0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & *
\end{array}\right)
\end{array}.$$

We can therefore update the value of V: $V \longleftarrow S_pV$. As $rank(V) = rank(A) = r = 2p+q$ and according to the lemma above, there exist q reflectors $S_{p+1}, S_{p+2}, \cdots, S_{p+q}$ such that $V \longleftarrow S_{p+q}\left(\cdots S_{p+2}\left(S_{p+1}V\right)\right)$ is reduce as follows:

$$V=\begin{pmatrix}
 & & 0_{p\times q} \\
I_{n\times p} & 0_{n\times p} & R \\
 & & 0_{(n-(p+q))\times q} \\
0_{n\times p} & I_{n\times p} & 0_{n\times q}
\end{pmatrix}$$

where $R \in \mathbb{R}^{(r-2p)\times(r-2p)}$ is nonsingular triangular matrix. By setting

$$S_{p+q+1} = diag(I_p, R^{-1}, I_{(n-(p+q))}, I_p, R, I_{(n-(p+q))})$$

which is a symplectic matrix, $V \longleftarrow S_{p+q+1}V$ is finally in the following form:

$$V=\begin{pmatrix}
 & & 0_{p\times q} \\
I_{n\times p} & 0_{n\times p} & I_q \\
 & & 0_{(n-(p+q))\times q} \\
0_{n\times p} & I_{n\times p} & 0_{n\times q}
\end{pmatrix}.$$

Let $S = S_{p+q+1}S_{p+q}\cdots S_2S_1$. We then have

$$SAQ = \begin{pmatrix} I_{n\times p} & 0_{n\times p} & \begin{matrix} 0_{p\times q} \\ I_q \\ 0_{(n-(p+q))\times q} \end{matrix} \\ 0_{n\times p} & I_{n\times p} & 0_{n\times q} \end{pmatrix} \left(\begin{array}{c|c|c} \Sigma_p & 0 & 0 \\ \hline 0 & \Sigma_p & 0 \\ \hline 0 & 0 & I_{q\times(m-2p)} \end{array}\right) = \begin{pmatrix} \begin{array}{c|} \Sigma_p \\ \hline 0_{(n-p)\times p} \end{array} & 0_{n\times p} & \begin{matrix} 0_{p\times(m-2p)} \\ I_q \; 0_{q\times(m-2p)} \\ 0_{(n-(p+q))\times(m-2p)} \end{matrix} \\ 0_{n\times p} & \begin{array}{|c|} \Sigma_p \\ \hline 0_{(n-p)\times p} \end{array} & 0_{n\times(m-2p)} \end{pmatrix}.$$

which corresponds to the hypothesized form.

4 Algorithm (Symplectic SVD-like algorithm)

Input : Matrix $A \in \mathbb{R}^{2n\times m}$.

Output : A symplectic matrix $S \in \mathbb{R}^{2n\times 2n}$, an orthogonal matrix $Q \in \mathbb{R}^{m\times m}$ and the desired SVD-like decomposition.

1. Compute a canonical Schur form of skew-symmetric matrix $M = A^TJA \in \mathbb{R}^{m\times m}$ such that

$$UMU^T = \left(\begin{array}{c|c|c} 0_p & \Sigma_p^2 & 0 \\ \hline -\Sigma_p^2 & 0_p & 0 \\ \hline 0 & 0 & 0_{m-2p} \end{array}\right).$$

2. For $k = 1,\cdots,p$, where $2p = rank(M)$
Compute $V_k = \dfrac{1}{\sigma_k}AU_kJ_2^T$ where $U_k = U[e_k\ e_{p+k}]$.
End For
3. Set $v_j = Au_j$ for $j = 2p+1,\cdots,r$ with $r = rank(A) = 2p+q$.
4. Set $V = \sum\limits_{k=1}^{p} V_k[e_k\ e_{p+k}]^T + \sum\limits_{k=2p+1}^{r} v_je_j^T$
and $Q = \sum\limits_{k=1}^{p} U_kJ_2^T[e_k\ e_{p+k}]^T + \sum\limits_{k=2p+1}^{m} u_je_j^T$.
5. Set $S = I_{2n}$ and
For $k = 1,\cdots,p$
Compute a symplectic reflector S_k associated to V_k
Update $V \leftarrow S_kV$ and $S \leftarrow SS_k$.
End For
6. **For** $k = 1,\cdots,q = r-2p$
Using lemma 2.2, we can compute a symplectic reflector S_k associated with v_k.
Update $V \leftarrow S_kV$ with

$$S_kV = \begin{pmatrix} I_{n\times p} & 0_{n\times p} & \begin{matrix} 0_{p\times q} \\ R \\ 0_{(n-(p+q))\times q} \end{matrix} \\ 0_{n\times p} & I_{n\times p} & 0_{n\times q} \end{pmatrix}$$

and $S \leftarrow SS_k$.
End For
7. Updating $V \leftarrow S_{p+q+1}V$ and $S \leftarrow SS_{p+q+1}$
where $S_{p+q+1} = diag(I_p, R^{-1}, I_{(n-(p+q))}, I_p, R, I_{(n-(p+q))})$.
with $R = V(p+1:p+q, 2p+1:r)$

8. $SAQ = \Sigma = \left(\begin{array}{c|c|c|c} \Sigma_p & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & \Sigma_p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

Computation of the symplectic SVD-like decomposition of a $2n$-by-m matrix A can be viewed as solving an eigenvalue problem of Hamiltonain JAA^T without having to compute the product of the full matrix, as we shown in the first example below. Let A be a $2n$-by-m rectangular matrix. Using algorithm 3.1, we obtain the following result:

$$SAQ = \Sigma = \left(\begin{array}{c|c|c|c} \Sigma_p & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & \Sigma_p & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore,

$$JAA^T = \Sigma = S^J \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & \Sigma_p^2 & 0 & 0 \\ qw < 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -\Sigma_p^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) S$$

which is the structured canonical form of Hamiltonian matrix JAA^T.

5 Numerical examples

We show here the results of numerical tests to compare the method shown in algorithm 3.1 with that of Xu [17]. We computed the eigenvalues of Hamiltonian matrix JAA^T and calculated the error of symplectic SVD-like decomposition. these experiments were carried out with Matlab 7.8.0 (R2009a) and run on a Core Duo Pentium processor.

Let A be a rectangular matrix, defined as follows:

$$A = Q \left(\begin{array}{ccc|ccc} \Sigma & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \Sigma & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) U^T$$

where Q is a random orthogonal matrix and U is a 14×14 random orthogonal symplectic matrix. We calculated the error that occurred when computing the symplectic SVD-like decomposition and compared the relative errors in the computed eigenvalues for Hamiltonian matrix JAA^T obtained by the proposed method and that of Xu [17].

• $\Sigma = diag(4,3,2,1)$, the error in computing symplectic SVD-like decomposition was $1.1743e-014$ with our method and $1.1897e-014$ with that of Xu. The relative errors for nonzero eigenvalues with the two methods are shown in the table below:

eigenvalue	Algorithm3.1	Xu [17]
$\pm 16i$	$2.2204e-015$	$2.8e-014$
$\pm 9i$	$1.1842e-015$	$5.3e-015$
$\pm 4i$	$2.2204e-015$	$1.2e-013$
$\pm i$	$1.3323e-015$	$6.9e-015$

• $\Sigma = diag(10^{-2}, 10^{-1}, 1, 10^{2})$, the error in computing symplectic SVD-like decomposition was $4.3215e-009$ with our method and 148.4 with that of Xu. As in the previous example, although the eigenvalues are computed correctly by Xu's method, computation of the symplectic SVD-like decomposition was incomplete. The relative errors for nonzero eigenvalues with the two methods are shown in the table bellow:

eigenvalue	Algorithm3.1	Xu [17]
$\pm 10^{4}i$	$1.5421e-015$	$6.4681e-015$
$\pm i$	$3.2846e-014$	$6.4567e-014$
$\pm 10^{-2}i$	$5.4357e-012$	$8.5701e-012$
$\pm 10^{-4}i$	$3.5356e-009$	$5.1430e-009$

• $\Sigma = diag(10^{-4}, 10^{-2}, 1, 10^{2})$, the error in computing symplectic SVD-like decomposition was $1.9411e-008$ with our method and 172.4 with that of Xu. As in the previous example, although the eigenvalues are computed correctly by Xu's method, computation of the symplectic SVD-like decomposition was incomplete. The relative errors for nonzero eigenvalues with the two methods are shown in the table below:

eigenvalue	Alg3.1	Xu method[17]
$\pm 10^{4}i$	$1.0914e-015$	$9.0541e-015$
$\pm i$	$1.2679e-013$	$1.3e-015$
$\pm 10^{-4}i$	$1.5679e-009$	$5.7e-012$
$\pm 10^{-8}i$	$1.6254e-006$	$4.3456e-010$

In this example, the relative error of the computed eigenvalue corresponding to $\lambda = 10^{-8}$ given by matlab was 12.6377 and the absolute error was $1.2638e-007$.

6 Conclusion

We have presented a numerical method for computing symplectic SVD-like decomposition. It is based on the canonical Schur-form of skew-symmetric matrix, as used by Xu [16,17]. The canonical form of the skew-symmetric matrix AJA^{T}, the Hamiltonian matrix $JA^{T}A$ and the skew-Hamiltonian matrix $A^{J}A$ can be derived from such a decomposition. The numerical examples presented show the effectiveness of proposed algorithm.

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