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Simpson type integral inequalities for generalized strongly convex functions and applications

Samet Erden[1], *Artion Kashuri*[2], *Sevgi Kılıçer*[1]

[1]Department of Mathematics, Faculty of Science, Bartın University, Bartın, Turkey
[2]Department of Mathematics, Faculty of Technical Science, University Ismail Qemali, Vlora, Albania

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Abstract: We firstly establish an identity of Simpson type involving local fractional integral. By using the obtained result, some new Simpson type integral inequalities for mappings whose certain powers of the local fractional derivatives in modulus are generalized strongly convex are derived. Finally, some error estimations for local fractional integrals and inequalities involving generalized special means are given.

Keywords: Simpson's inequality, Generalized convex functions, Fractal spaces, Numerical integration, Generalized special means.

1 Introduction

The inequality theory is known to play an important role in almost all areas of pure and applied sciences. One of the results in this theory is the following inequality known as Simpson's inequality.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

We also note that the first fundamental result for convex mappings is Hermite-Hadamard inequality. Because this result has led to many new inequalities obtained by using convex functions, a number of researchers have devoted to finding new Hermite–Hadamard, Ostrowski and Simpson type inequalities for different classes of convex functions, please see [1]-[5],[7]-[14]. For example, Alomari et al. in [1], proved the following Lemma to establish novel Simpson's type inequalities based on s-convexity and concavity.

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on interior I° of an interval I and $a,b\in I$ with $a<b$. Then, the following equality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx=(b-a)\int_0^1 m(t)f'(bt+(1-t)a)dt,$$

* Corresponding author e-mail: erdensmt@gmail.com

where

$$m(t)=\begin{cases} t-\frac{1}{6}\ t\in[0,\frac{1}{2}); \\ \\ t-\frac{5}{6}\ t\in[\frac{1}{2},1]. \end{cases}$$

In the following section, we recall some basic definitions and properties of local fractional derivative and local fractional integral that we will use throughout this paper.

2 Preliminaries

First of all, we give the set R^α of real line numbers to describe the definitions of the local fractional derivative and integral. For $0<\alpha\le 1$, we have the following α-type set.

Z^α : The α-type set of integer is defined as the set $\{0^\alpha,\pm1^\alpha,\pm2^\alpha,\ldots,\pm n^\alpha,\ldots\}$.

Q^α : The α-type set of the rational numbers is defined as the set $\{m^\alpha=\left(\frac{p}{q}\right)^\alpha : p,q\in Z,\ q\neq 0\}$.

J^α : The α-type set of the irrational numbers is defined as the set $\{m^\alpha\neq\left(\frac{p}{q}\right)^\alpha : p,q\in Z,\ q\neq 0\}$.

R^α : The α-type set of the real line numbers is defined as the set $R^\alpha=Q^\alpha\cup J^\alpha$.

If a^α, b^α and c^α belongs the set R^α of real line numbers, then

(1) $a^\alpha+b^\alpha$ and $a^\alpha b^\alpha$ belongs the set R^α;

(2) $a^\alpha+b^\alpha=b^\alpha+a^\alpha=(a+b)^\alpha=(b+a)^\alpha$;

(3) $a^\alpha+(b^\alpha+c^\alpha)=(a+b)^\alpha+c^\alpha$;

(4) $a^\alpha b^\alpha=b^\alpha a^\alpha=(ab)^\alpha=(ba)^\alpha$;

(5) $a^\alpha(b^\alpha c^\alpha)=(a^\alpha b^\alpha)c^\alpha$;

(6) $a^\alpha(b^\alpha+c^\alpha)=a^\alpha b^\alpha+a^\alpha c^\alpha$;

(7) $a^\alpha+0^\alpha=0^\alpha+a^\alpha=a^\alpha$ and $a^\alpha 1^\alpha=1^\alpha a^\alpha=a^\alpha$.

The definition of the local fractional derivative is given as follows.

Definition 1.([16]) *A non-differentiable function $f:R\to R^\alpha$, $x\to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that*

$$|f(x)-f(x_0)|<\varepsilon^\alpha$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_\alpha(a,b)$.

Definition 2.([16]) *The local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by*

$$f^{(\alpha)}(x_0)=\left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^\alpha(f(x)-f(x_0))}{(x-x_0)^\alpha},$$

where $\Delta^\alpha(f(x)-f(x_0))\cong\Gamma(\alpha+1)(f(x)-f(x_0))$.

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^\alpha\ldots D_x^\alpha}^{k+1\ times}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,\ldots$.

Definition 3.([16]) *Let $f(x)\in C_\alpha[a,b]$. Then the local fractional integral is defined by*

$${}_aI_b^\alpha f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^\alpha=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,\ldots,\Delta t_{N-1}\}$, where $[t_j,t_{j+1}]$, $j=0,\ldots,N-1$ and $a=t_0<t_1<\ldots<t_{N-1}<t_N=b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^{\alpha}f(x)=0$ *if* $a=b$ *and* ${}_aI_b^{\alpha}f(x)=-{}_bI_a^{\alpha}f(x)$ *if* $a<b$. *If for any* $x\in[a,b]$, *there exists* ${}_aI_x^{\alpha}f(x)$, *then we denoted by* $f(x)\in I_x^{\alpha}[a,b]$.

Lemma 2.([16])
(1) (Local fractional integration is anti-differentiation) Suppose that $f(x)=g^{(\alpha)}(x)\in C_{\alpha}[a,b]$, *then we have*

$${}_aI_b^{\alpha}f(x)=g(b)-g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x),g(x)\in D_{\alpha}[a,b]$ *and* $f^{(\alpha)}(x)$, $g^{(\alpha)}(x)\in C_{\alpha}[a,b]$, *then we have*

$${}_aI_b^{\alpha}f(x)g^{(\alpha)}(x)=f(x)g(x)|_a^b-{}_aI_b^{\alpha}f^{(\alpha)}(x)g(x).$$

Lemma 3.([16]) *We have*

i) $\dfrac{d^{\alpha}x^{k\alpha}}{dx^{\alpha}}=\dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}$;

ii) $\dfrac{1}{\Gamma(\alpha+1)}\int_a^b x^{k\alpha}(dx)^{\alpha}=\dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha}-a^{(k+1)\alpha}\right)$, $k\in R$.

As well as all these definitions, we should mention some properties of the local fractional derivative in order to apply the change of the variable in the integrals.

Lemma 4.([17]) *Suppose that* $f(x)\in C_{\alpha}[a,b]$ *and* $f(x)\in D_{\alpha}(a,b)$, *then, for* $0<\alpha\leq 1$, *we have the following* $\alpha-$*differential form*

$$d^{\alpha}f(x)=f^{(\alpha)}(x)dx^{\alpha}.$$

Lemma 5.([17]) *Let* I *be an interval,* $f,g:I\subset R\to R^{\alpha}$ *(*I° *is the interior of* I*) such that* $f,g\in D_{\alpha}(I^{\circ})$. *Then, we have*

$$\frac{d^{\alpha}y(x)}{dx^{\alpha}}=f^{(\alpha)}(g(x))\left(g^{(1)}(x)\right)^{\alpha}.$$

The generalized convex function which are examined in this paper are defined as follows:

Definition 4.([16]) *(Generalized convex function) Let* $f:I\subseteq R\to R^{\alpha}$. *For any* $x_1,x_2\in I$ *and* $t\in[0,1]$, *if the following inequality holds*

$$f(tx_1+(1-t)x_2)\leq t^{\alpha}f(x_1)+(1-t)^{\alpha}f(x_2),$$

then f *is called a generalized convex function on* I.

Recenlty, Anastassiou et al. in [2], gives the following new class of convex functions.

Definition 5. *A function* $f:I\subseteq R\to R^{\alpha}$ *is called generalized strongly m-convex with* $m\in(0,1]$ *and modulus* $c\in R^{+}$, *if*

$$f(tx_1+m(1-t)x_2)\leq t^{\alpha}f(x_1)+m^{\alpha}(1-t)^{\alpha}f(x_2)-(cm)^{\alpha}t^{\alpha}(1-t)^{\alpha}(x_1-x_2)^{2\alpha} \tag{1}$$

holds for any $x_1,x_2\in I$ *and* $t\in[0,1]$.

Taking $m=1$ in Definition 5, we have the following special case.

Definition 6. *A function* $f:I\subseteq R\to R^{\alpha}$ *is called generalized strongly convex with modulus* $c\in R^{+}$, *if*

$$f(tx_1+(1-t)x_2)\leq t^{\alpha}f(x_1)+(1-t)^{\alpha}f(x_2)-c^{\alpha}t^{\alpha}(1-t)^{\alpha}(x_1-x_2)^{2\alpha} \tag{2}$$

holds for any $x_1,x_2\in I$ *and* $t\in[0,1]$.

Here are two basic examples of generalized convex functions:
(1) $f(x) = x^{\alpha p}$, $x \geq 0$, $p > 1$;
(2) $f(x) = E_\alpha(x^\alpha)$, $x \in R$, where

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$$

is the well-known Mittag-Leffler function.

Motivated by the above literatures, the main objective of this paper is to discover in section 3, an interesting identity of Simpson type involving local fractional integral. By using the obtained result, some new Simpson type integral inequalities for mappings whose certain powers of the local fractional derivatives in modulus are generalized strongly convex are derived. In section 4, some inequalities involving generalized special means are given. In section 5, some error estimations applying Simpson's formula for local fractional integrals are obtained as well. The ideas and techniques of this paper may stimulate further research in the fascinating field of integral inequalities.

3 Inequalities for Generalized Strongly Convex Functions

We fistly establish an identity in the following Lemma in order to attain new inequalities involving local fractional integrals.

Lemma 6. *Let $P \subset \mathbb{R}$ be an interval (P° is the interior of P), and let $\varphi : P^\circ \subset \mathbb{R} \to \mathbb{R}^\alpha$ be a mapping such that $\varphi \in D_\alpha(P^\circ)$ and $\varphi^{(\alpha)} \in C_\alpha[\rho,\sigma]$ for $\rho,\sigma \in P^\circ$ with $\rho < \sigma$. Then, for all $x \in [\rho,\sigma]$, the following identity holds:*

$$\begin{aligned}
&\frac{1}{6^\alpha}\left[\varphi(\rho) + 4^\alpha \varphi\left(\frac{\rho+\sigma}{2}\right) + \varphi(\sigma)\right] - \frac{\Gamma(1+\alpha)}{(\sigma-\rho)^\alpha}\, {}_\rho I_\sigma{}^\alpha \varphi(x) \\
&= \frac{(\sigma-\rho)^\alpha}{2^\alpha \Gamma(1+\alpha)} \left\{ \int_0^1 \left(\frac{s}{2} - \frac{1}{3}\right)^\alpha \varphi^{(\alpha)}\left(\frac{1+s}{2}\sigma + \frac{1-s}{2}\rho\right)(ds)^\alpha \right. \\
&\left. + \int_0^1 \left(\frac{1}{3} - \frac{s}{2}\right)^\alpha \varphi^{(\alpha)}\left(\frac{1+s}{2}\rho + \frac{1-s}{2}\sigma\right)(ds)^\alpha \right\}.
\end{aligned} \tag{3}$$

Proof. Using integration by parts presented in the Lemma 2 for the first integral in right-hand side of the identity (3), and applying the change of variable $x = \frac{1+s}{2}\sigma + \frac{1-s}{2}\rho$ to the resulting integral, by means of Lemmas 4 and 5, and the second property of the fractal space $\rho^\alpha + \sigma^\alpha = (\rho+\sigma)^\alpha$, it follows that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \int_0^1 \left(\frac{s}{2} - \frac{1}{3}\right)^\alpha \varphi^{(\alpha)}\left(\frac{1+s}{2}\sigma + \frac{1-s}{2}\rho\right)(ds)^\alpha \\
&= \frac{2^\alpha}{(\sigma-\rho)^\alpha}\left(\frac{s}{2} - \frac{1}{3}\right)^\alpha \varphi\left(\frac{1+s}{2}\sigma + \frac{1-s}{2}\rho\right)\Bigg|_0^1 \\
&- \frac{2^\alpha}{(\sigma-\rho)^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)} \int_0^1 \frac{1}{2^\alpha}\varphi\left(\frac{1+s}{2}\sigma + \frac{1-s}{2}\rho\right)(ds)^\alpha \\
&= \frac{2^\alpha}{(\sigma-\rho)^\alpha}\left[\frac{1}{6^\alpha}\varphi(\sigma) + \frac{1}{3^\alpha}\varphi\left(\frac{\rho+\sigma}{2}\right)\right] - \frac{2^\alpha}{(\sigma-\rho)^{2\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)} \int_{\frac{\rho+\sigma}{2}}^{\sigma} \varphi(x)(dx)^\alpha.
\end{aligned} \tag{4}$$

Similarly, calculating the other integral in (3), we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{1}{3}-\frac{s}{2}\right)^{\alpha}\varphi^{(\alpha)}\left(\frac{1+s}{2}\rho+\frac{1-s}{2}\sigma\right)(ds)^{\alpha} \tag{5}$$

$$=\frac{2^{\alpha}}{(\sigma-\rho)^{\alpha}}\left[\frac{1}{6^{\alpha}}\varphi(\rho)+\frac{1}{3^{\alpha}}\varphi\left(\frac{\rho+\sigma}{2}\right)\right]-\frac{2^{\alpha}}{(\sigma-\rho)^{2\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)}\int_{\rho}^{\frac{\rho+\sigma}{2}}\varphi(x)(dx)^{\alpha}.$$

Multiplying the resulting identity by the factor $\frac{(\sigma-\rho)^{\alpha}}{2^{\alpha}}$ after adding the equalities (4) and (5), the required identity (3) can be obtained.

In the following theorem, we give a result for functions whose local fractional derivatives in modulus are generalized strongly convex by using the Lemma 6.

Theorem 2.*Suppose that the assumptions of Lemma 6 are satisfied. If $|\varphi^{(\alpha)}|$ is a generalized strongly convex on $[\rho,\sigma]$, then for $c\in\mathbb{R}^{+}$, the following inequality holds:*

$$\left|\frac{1}{6^{\alpha}}\left[\varphi(\rho)+4^{\alpha}\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{\Gamma(1+\alpha)}{(b-\rho)^{\alpha}}{}_{\rho}I_{\sigma}{}^{\alpha}\varphi(x)\right| \tag{6}$$

$$\leq\left(\frac{5}{36}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(\sigma-\rho)^{\alpha}\left[|\varphi^{(\alpha)}(\rho)|+|\varphi^{(\alpha)}(\sigma)|\right]-\frac{c^{\alpha}(\sigma-\rho)^{3\alpha}}{4^{\alpha}}K(\alpha).$$

Here, $K(\alpha)$ is defined by

$$K(\alpha)=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{5}{18}\right)^{\alpha}+\left(\frac{11}{81}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\left(\frac{49}{162}\right)^{\alpha}\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}. \tag{7}$$

*Proof.*Taking modulus in both sides of Lemma 6, we have

$$\left|\frac{1}{6^{\alpha}}\left[\varphi(\rho)+4^{\alpha}\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{\Gamma(1+\alpha)}{(\sigma-\rho)^{\alpha}}{}_{\rho}I_{\sigma}{}^{\alpha}\varphi(x)\right| \tag{8}$$

$$\leq\frac{(\sigma-\rho)^{\alpha}}{2^{\alpha}\Gamma(1+\alpha)}\left\{\int_0^1\left|\frac{s}{2}-\frac{1}{3}\right|^{\alpha}\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\sigma+\frac{1-s}{2}\rho\right)\right|(ds)^{\alpha}\right.$$

$$\left.+\int_0^1\left|\frac{1}{3}-\frac{s}{2}\right|^{\alpha}\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\rho+\frac{1-s}{2}\sigma\right)\right|(ds)^{\alpha}\right\}.$$

Because $|\varphi^{(\alpha)}|$ is a generalized strongly convex on $[\rho,\sigma]$, it follows that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{s}{2}-\frac{1}{3}\right|^{\alpha}\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\sigma+\frac{1-s}{2}\rho\right)\right|(ds)^{\alpha} \tag{9}$$

$$+\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{1}{3}-\frac{s}{2}\right|^{\alpha}\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\rho+\frac{1-s}{2}\sigma\right)\right|(ds)^{\alpha}$$

$$\leq\left[|\varphi^{(\alpha)}(\rho)|+|\varphi^{(\alpha)}(\sigma)|\right]\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{s}{2}-\frac{1}{3}\right|^{\alpha}(ds)^{\alpha}-\frac{c^{\alpha}(\sigma-\rho)^{2\alpha}}{2^{\alpha}\Gamma(1+\alpha)}\int_0^1\left|\frac{s}{2}-\frac{1}{3}\right|^{\alpha}\left(1-s^2\right)^{\alpha}(ds)^{\alpha}.$$

Applying the change of variables $\left(\frac{1}{3}-\frac{s}{2}\right)^{\alpha}=u^{\alpha}$ and $\left(\frac{s}{2}-\frac{1}{3}\right)^{\alpha}=v^{\alpha}$, by means of Lemmas 4 and 5, it is found that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{s}{2}-\frac{1}{3}\right|^{\alpha}(ds)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{2}{3}}\left(\frac{1}{3}-\frac{s}{2}\right)^{\alpha}(ds)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{2}{3}}^1\left(\frac{s}{2}-\frac{1}{3}\right)^{\alpha}(ds)^{\alpha} \tag{10}$$

$$=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{5}{18}\right)^{\alpha}.$$

Also, it is easy to see that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{s}{2}-\frac{1}{3}\right|^{\alpha}(1-s^2)^{\alpha}(ds)^{\alpha}=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{5}{18}\right)^{\alpha}+\left(\frac{11}{81}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\left(\frac{49}{162}\right)^{\alpha}\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}. \tag{11}$$

If we substitute the equalities (10) and (11) in (9), and multiply the resulting inequality by the factor $(\sigma-\rho)^{\alpha}/2^{\alpha}$, then we capture the desired inequality (6), which completes the proof.

Corollary 1.*Suppose that all the assumptions of Theorem 2 hold. If we choose $\alpha=1$, then for $c\in\mathbb{R}^{+}$, we have the inequality*

$$\left|\frac{1}{6}\left[\varphi(\rho)+4\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{1}{\sigma-\rho}\int_{\rho}^{\sigma}\varphi(x)dx\right|\leq\frac{5}{72}(\sigma-\rho)[|\varphi'(\rho)|+|\varphi'(\sigma)|]-\frac{169}{1296}c(\sigma-\rho)^3.$$

Corollary 2.*Suppose that all the assumptions of Theorem 2 hold. If $|\varphi^{(\alpha)}|$ is a generalized convex on $[\rho,\sigma]$, then we get the inequality*

$$\left|\frac{1}{6^{\alpha}}\left[\varphi(\rho)+4^{\alpha}\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{\Gamma(1+\alpha)}{(\sigma-\rho)^{\alpha}}{}_{\rho}I_{\sigma}{}^{\alpha}\varphi(x)\right| \tag{12}$$

$$\leq\left(\frac{5}{36}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(\sigma-\rho)^{\alpha}\left[|\varphi^{(\alpha)}(\rho)|+|\varphi^{(\alpha)}(\sigma)|\right].$$

*Remark.*If we choose $\alpha=1$ in (12), then we obtain the result

$$\left|\frac{1}{6}\left[\varphi(\rho)+4\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{1}{\sigma-\rho}\int_{\rho}^{\sigma}\varphi(x)dx\right|\leq\frac{5}{72}(\sigma-\rho)\left[|\varphi'(\rho)|+|\varphi'(\sigma)|\right],$$

which is presented by Sarikaya et al. in [12].

Now, we give some inequalities for local fractional differentiable mappings whose certain powers in the absolute value are generalized strongly convex.

Theorem 3. *Supposing that all the assumptions of the Lemma 6 hold. If $|\varphi^{(\alpha)}|^q$ is a generalized strongly convex function on $[\rho,\sigma]$, for $p,q>1$ where $p^{-1}+q^{-1}=1$ and $c\in\mathbb{R}^+$, the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6^\alpha}\left[\varphi(\rho)+4^\alpha\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{\Gamma(1+\alpha)}{(\sigma-\rho)^\alpha}{}_\rho I_\sigma{}^\alpha\varphi(x)\right|\\&\leq(\sigma-\rho)^\alpha\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\frac{2^{\alpha(p+1)}+1}{6^{\alpha(p+1)}}\right)^{\frac{1}{p}}\times\left\{\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{|\varphi^{(\alpha)}(\rho)|^q+3^\alpha|\varphi^{(\alpha)}(\sigma)|^q}{2^\alpha}-M_\alpha(\rho,\sigma,c)\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{3^\alpha|\varphi^{(\alpha)}(\rho)|^q+|\varphi^{(\alpha)}(\sigma)|^q}{2^\alpha}-M_\alpha(\rho,\sigma,c)\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Here, $M_\alpha(\rho,\sigma,c)$ is defined by

$$M_\alpha(\rho,\sigma,c)=\frac{c^\alpha(\sigma-\rho)^{2\alpha}}{4^\alpha}\left[\frac{1}{\Gamma(1+\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right].$$

Proof. Applying the Hölder's inequality to the result (8), we find that

$$\begin{aligned}&\left|\frac{1}{6^\alpha}\left[\varphi(\rho)+4^\alpha\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{\Gamma(1+\alpha)}{(\sigma-\rho)^\alpha}{}_\rho I_\sigma{}^\alpha\varphi(x)\right| \qquad (13)\\&\leq\frac{(\sigma-\rho)^\alpha}{2^\alpha}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{s}{2}-\frac{1}{3}\right|^{\alpha p}(ds)^\alpha\right)^{\frac{1}{p}}\times\left\{\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\sigma+\frac{1-s}{2}\rho\right)\right|^q(ds)^\alpha\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\rho+\frac{1-s}{2}\sigma\right)\right|^q(ds)^\alpha\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Since $|\varphi^{(\alpha)}|^q$ is a generalized strongly convex function, with the help of Lemmas 4 and 5, we have

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\sigma+\frac{1-s}{2}\rho\right)\right|^q(ds)^\alpha\leq\frac{1}{\Gamma(1+\alpha)}\int_0^1\left[\left(\frac{1+s}{2}\right)^\alpha|\varphi^{(\alpha)}(\sigma)|^q+\left(\frac{1-s}{2}\right)^\alpha|\varphi^{(\alpha)}(\rho)|^q\right.\\&\left.-c^\alpha\left(\frac{1+s}{2}\right)^\alpha\left(\frac{1-s}{2}\right)^\alpha(\sigma-\rho)^{2\alpha}\right](ds)^\alpha\\&=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{|\varphi^{(\alpha)}(\rho)|^q+3^\alpha|\varphi^{(\alpha)}(\sigma)|^q}{2^\alpha}-\frac{c^\alpha(\sigma-\rho)^{2\alpha}}{4^\alpha}\left[\frac{1}{\Gamma(1+\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\rho+\frac{1-s}{2}\sigma\right)\right|^q(ds)^\alpha\\&\leq\frac{1}{\Gamma(1+\alpha)}\int_0^1\left[\left(\frac{1+s}{2}\right)^\alpha|\varphi^{(\alpha)}(\rho)|^q+\left(\frac{1-s}{2}\right)^\alpha|\varphi^{(\alpha)}(\sigma)|^q-c^\alpha\left(\frac{1+s}{2}\right)^\alpha\left(\frac{1-s}{2}\right)^\alpha(\sigma-\rho)^{2\alpha}\right](ds)^\alpha\\&=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{|\varphi^{(\alpha)}(\sigma)|^q+3^\alpha|\varphi^{(\alpha)}(\rho)|^q}{2^\alpha}-\frac{c^\alpha(\sigma-\rho)^{2\alpha}}{4^\alpha}\left[\frac{1}{\Gamma(1+\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right].\end{aligned}$$

Applying the change of variables to the rest integral in right side of (13), from Lemmas 3, 4 and 5, one can easily see that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{s}{2}-\frac{1}{3}\right|^{\alpha p}(ds)^\alpha=2^\alpha\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\frac{2^{\alpha(p+1)}+1}{6^{\alpha(p+1)}}. \tag{14}$$

Hence, the proof is completed.

*Remark.*Suppose that all the assumptions of Theorem 3 hold. If we take $\alpha=1$, then for $c\in\mathbb{R}^+$, we have the inequality

$$\left|\frac{1}{6}\left[\varphi(\rho)+4\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{1}{\sigma-\rho}\int_\rho^\sigma\varphi(x)dx\right|\leq(\sigma-\rho)\left(\frac{2^{p+1}+1}{6^{(p+1)}(p+1)}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\frac{|\varphi'(\rho)|^q+3|\varphi'(\sigma)|^q}{4}-\frac{c(\sigma-\rho)^2}{6}\right)^{\frac{1}{q}}+\left(\frac{3|\varphi'(\rho)|^q+|\varphi'(\sigma)|^q}{4}-\frac{c(\sigma-\rho)^2}{6}\right)^{\frac{1}{q}}\right\}.$$

Corollary 3.*Suppose that all the assumptions of Theorem 3 hold. If $|\varphi^{(\alpha)}|^q$ is a generalized convex function on $[\rho,\sigma]$, then we get the inequality*

$$\left|\frac{1}{6^\alpha}\left[\varphi(\rho)+4^\alpha\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{\Gamma(1+\alpha)}{(\sigma-\rho)^\alpha}{}_\rho I_\sigma{}^\alpha\varphi(x)\right| \tag{15}$$
$$\leq(\sigma-\rho)^\alpha\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\frac{2^{\alpha(p+1)}+1}{6^{\alpha(p+1)}}\right)^{\frac{1}{p}}\times\left\{\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{|\varphi^{(\alpha)}(\rho)|^q+3^\alpha|\varphi^{(\alpha)}(\sigma)|^q}{2^\alpha}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{3^\alpha|\varphi^{(\alpha)}(\rho)|^q+|\varphi^{(\alpha)}(\sigma)|^q}{2^\alpha}\right)^{\frac{1}{q}}\right\}.$$

Corollary 4.*If we choose $\alpha=1$ in (15), then we obtain the result*

$$\left|\frac{1}{6}\left[\varphi(\rho)+4\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{1}{\sigma-\rho}\int_\rho^\sigma\varphi(x)dx\right|$$
$$\leq(\sigma-\rho)\left(\frac{2^{(p+1)}+1}{6^{(p+1)}(p+1)}\right)^{\frac{1}{p}}\times\left\{\left(\frac{|\varphi'(\rho)|^q+3|\varphi'(\sigma)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\varphi'(\rho)|^q+|\varphi'(\sigma)|^q}{4}\right)^{\frac{1}{q}}\right\},$$

which was given by Sarikaya et al. in [12].

Theorem 4.*Suppose that all the assumptions of the Lemma 6 hold. If $|\varphi^{(\alpha)}|^q$ is a generalized strongly convex function on $[\rho,\sigma]$, for $p,q>1$ where $p^{-1}+q^{-1}=1$ and $c\in\mathbb{R}^+$, the following inequality holds:*

$$\left|\frac{1}{6^\alpha}\left[\varphi(\rho)+4^\alpha\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{\Gamma(1+\alpha)}{(\sigma-\rho)^\alpha}{}_\rho I_\sigma{}^\alpha\varphi(x)\right|\leq(\sigma-\rho)^\alpha\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\frac{2^{\alpha(p+1)}+1}{6^{\alpha(p+1)}}\right)^{\frac{1}{p}} \tag{16}$$
$$\times\left\{\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\left|\varphi^{(\alpha)}\left(\frac{\rho+\sigma}{2}\right)\right|^q+|\varphi^{(\alpha)}(\sigma)|^q\right]-N_\alpha(\rho,\sigma,c)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[|\varphi^{(\alpha)}(\rho)|^q+\left|\varphi^{(\alpha)}\left(\frac{\rho+\sigma}{2}\right)\right|^q\right]-N_\alpha(\rho,\sigma,c)\right)^{\frac{1}{q}}\right\}.$$

Here, $N_\alpha(\rho,\sigma,c)$ is defined by

$$N_\alpha(\rho,\sigma,c)=\frac{c^\alpha(\sigma-\rho)^{2\alpha}}{4^\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right].$$

*Proof.*Considering the inequality (13), since $|\varphi^{(\alpha)}|^q$ is a generalized strongly convex function and from the Lemmas 3, 4 and 5, it follows that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\sigma+\frac{1-s}{2}\rho\right)\right|^q(ds)^\alpha=\frac{2^\alpha}{(\sigma-\rho)^\alpha}\frac{1}{\Gamma(1+\alpha)}\int_{\frac{\rho+\sigma}{2}}^{\sigma}\left|\varphi^{(\alpha)}(x)\right|^q(dx)^\alpha \tag{17}$$

$$=\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\varphi^{(\alpha)}\left((1-t)\frac{(\rho+\sigma)}{2}+t\sigma\right)\right|^q(dt)^\alpha\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left|\varphi^{(\alpha)}\left(\frac{\rho+\sigma}{2}\right)\right|^q+\left|\varphi^{(\alpha)}(\sigma)\right|^q\right)$$
$$-c^\alpha\frac{(\sigma-\rho)^{2\alpha}}{2^{2\alpha}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)$$

and

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\rho+\frac{1-s}{2}\sigma\right)\right|^q(ds)^\alpha=\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\varphi^{(\alpha)}\left((1-t)\rho+t\frac{(\rho+\sigma)}{2}\right)\right|^q(dt)^\alpha \tag{18}$$
$$\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left|\varphi^{(\alpha)}\left(\frac{\rho+\sigma}{2}\right)\right|^q+\left|\varphi^{(\alpha)}(\rho)\right|^q\right)-c^\alpha\frac{(\sigma-\rho)^{2\alpha}}{2^{2\alpha}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right).$$

If we substitute the results (14), (17) and (18) in (13), then we can easily capture the required inequality (16). Hence, the proof is completed.

Corollary 5.*Suppose that all the assumptions of Theorem 4 hold. $If|\varphi^{(\alpha)}|^q$ is a generalized convex function on $[\rho,\sigma]$, then we have the inequality*

$$\left|\frac{1}{6^\alpha}\left[\varphi(\rho)+4^\alpha\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{\Gamma(1+\alpha)}{(\sigma-\rho)^\alpha}{}_\rho I_\sigma{}^\alpha\varphi(x)\right| \tag{19}$$
$$\leq(\sigma-\rho)^\alpha\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\frac{2^{\alpha(p+1)}+1}{6^{\alpha(p+1)}}\right)^{\frac{1}{p}}\times\left\{\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\left|\varphi^{(\alpha)}\left(\frac{\rho+\sigma}{2}\right)\right|^q+|\varphi^{(\alpha)}(\sigma)|^q\right]\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[|\varphi^{(\alpha)}(\rho)|^q+\left|\varphi^{(\alpha)}\left(\frac{\rho+\sigma}{2}\right)\right|^q\right]\right)^{\frac{1}{q}}\right\}.$$

*Remark.*Special cases of the inequality (16) and (19) can also be obtained by choosing $\alpha=1$.

Theorem 5. *Assume that all the assumptions of the Lemma 6 hold. If $|\varphi^{(\alpha)}|^q$ is a generalized strongly convex function on $[\rho,\sigma]$, then for $q\geq 1$ and $c\in\mathbb{R}^+$, the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6^\alpha}\left[\varphi(\rho)+4^\alpha\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{\Gamma(1+\alpha)}{(\sigma-\rho)^\alpha}{}_\rho I_\sigma{}^\alpha\varphi(x)\right|\\&\leq\frac{(\sigma-\rho)^\alpha}{2^\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{5}{18}\right)^\alpha\right)^{1-\frac{1}{q}}\\&\times\left\{\left(\left|\varphi^{(\alpha)}(\sigma)\right|^qP(\alpha)+\left|\varphi^{(\alpha)}(\rho)\right|^qQ(\alpha)-c^\alpha\frac{(\sigma-\rho)^{2\alpha}}{4^\alpha}K(\alpha)\right)^{\frac{1}{q}}\right.\\&\left.+\left(\left|\varphi^{(\alpha)}(\sigma)\right|^qP(\alpha)+\left|\varphi^{(\alpha)}(\rho)\right|^qQ(\alpha)-c^\alpha\frac{(\sigma-\rho)^{2\alpha}}{4^\alpha}K(\alpha)\right)^{\frac{1}{q}}\right\},\end{aligned}\tag{20}$$

where

$$P(\alpha)=\frac{1}{2^\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{5}{18}\right)^\alpha+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{11}{54}\right)^\alpha-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{1}{27}\right)^\alpha\right],\tag{21}$$

$$Q(\alpha)=\frac{1}{2^\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{5}{18}\right)^\alpha+\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{1}{27}\right)^\alpha-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{11}{54}\right)^\alpha\right]\tag{22}$$

and $K(\alpha)$ is defined as in (7).

Proof. Applying generalized power mean inequality to the inequality (8), we find that

$$\begin{aligned}&\left|\frac{1}{6^\alpha}\left[\varphi(\rho)+4^\alpha\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{\Gamma(1+\alpha)}{(\sigma-\rho)^\alpha}{}_\rho I_\sigma{}^\alpha\varphi(x)\right|\\&\leq\frac{(\sigma-\rho)^\alpha}{2^\alpha}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{s}{2}-\frac{1}{3}\right|^\alpha(ds)^\alpha\right)^{1-\frac{1}{q}}\times\left\{\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{s}{2}-\frac{1}{3}\right|^\alpha\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\sigma+\frac{1-s}{2}\rho\right)\right|^q(ds)^\alpha\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{1}{3}-\frac{s}{2}\right|^\alpha\left|\varphi^{(\alpha)}\left(\frac{1+s}{2}\rho+\frac{1-s}{2}\sigma\right)\right|^q(ds)^\alpha\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

For the rest of this proof, using the same strategy which is used in the proof of Theorem 2 by taking into account generalized strongly convexity of $|\varphi^{(\alpha)}|^q$, the desired inequality can be attained.

Corollary 6. *Suppose that all the assumptions of Theorem 5 hold. If $|\varphi^{(\alpha)}|^q$ is a generalized convex function on $[\rho,\sigma]$, then we have the inequality*

$$\begin{aligned}&\left|\frac{1}{6^\alpha}\left[\varphi(\rho)+4^\alpha\varphi\left(\frac{\rho+\sigma}{2}\right)+\varphi(\sigma)\right]-\frac{\Gamma(1+\alpha)}{(\sigma-\rho)^\alpha}{}_\rho I_\sigma{}^\alpha\varphi(x)\right|\\&\leq\frac{(\sigma-\rho)^\alpha}{2^\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{5}{18}\right)^\alpha\right)^{1-\frac{1}{q}}\times\left\{\left(\left|\varphi^{(\alpha)}(\sigma)\right|^qP(\alpha)+\left|\varphi^{(\alpha)}(\rho)\right|^qQ(\alpha)\right)^{\frac{1}{q}}\right.\\&\left.+\left(\left|\varphi^{(\alpha)}(\sigma)\right|^qP(\alpha)+\left|\varphi^{(\alpha)}(\rho)\right|^qQ(\alpha)\right)^{\frac{1}{q}}\right\},\end{aligned}\tag{23}$$

where $P(\alpha)$ and $Q(\alpha)$ are defined as in (21) and (22), respectively.

*Remark.*Special case of the inequality (20) can also be captured by choosing $\alpha = 1$.

*Remark.*If we take $\alpha = 1$ in (23), then we attain the result given in [12].

4 Applications for Special Means

Let us recall some generalized special means:

1. Generalized arithmetic mean: $A(\mu,\nu) = \frac{\mu^{\alpha}+\nu^{\alpha}}{2^{\alpha}}$;
2. Generalized logarithmic mean: $L_n(\mu,\nu) = \left(\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}\left[\frac{\nu^{(n+1)\alpha}-\mu^{(n+1)\alpha}}{(\nu-\mu)^{\alpha}}\right]\right)^{\frac{1}{n}}$,
   where $n \in \mathbb{Z}\setminus\{-1,0\}, \mu,\nu \in \mathbb{R}, \mu \neq \nu$.

We handle the mapping $\varphi : (0,+\infty) \to R^{\alpha}$, $\varphi(x) = x^{n\alpha}$, $n > 1$. Then, for $0 < \mu < \nu$, we have

$$\varphi\left(\frac{\mu+\nu}{2}\right) = [A(\mu,\nu)]^n, \quad \frac{\varphi(\mu)+\varphi(\nu)}{6^{\alpha}} = \frac{A(\mu^n,\nu^n)}{3^{\alpha}}$$

and

$$\frac{1}{(\nu-\mu)^{\alpha}}\ {}_{\mu}I_{\nu}^{\alpha}\varphi(x) = [L_n(\mu,\nu)]^n.$$

Also, using the Lemma 3, one has

$$|\varphi^{(\alpha)}(\mu)| = \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}\mu^{(n-1)\alpha}$$

and

$$|\varphi^{(\alpha)}(\nu)| = \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}\nu^{(n-1)\alpha}.$$

Now, we reconsider the inequality (12) by taking into account $\varphi(x) = x^{n\alpha}$, then we get

$$\left|\frac{A(\mu^n,\nu^n)}{3^{\alpha}} + \left(\frac{2}{3}\right)^{\alpha}[A(\mu,\nu)]^n - \Gamma(1+\alpha)[L_n(\mu,\nu)]^n\right|$$
$$\leq \left(\frac{5}{36}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(\sigma-\rho)^{\alpha}\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}\left[\mu^{(n-1)\alpha}+\nu^{(n-1)\alpha}\right],$$

which is an inequality involving above generalized special means.

*Remark.*Using Theorems 3, 4 and 5, we can get some new inequalities involving generalized special means. We omit their proofs and the details are left to the interested readers.

5 Applications to Simpson's Formula

In this last section, we give some estimates of Simpson's quadrature formula by using inequalities developed in the section 3. Assume that d is a division of the interval $[\rho,\sigma]$, i.e.,

$$d := \rho = x_0 < x_1 < x_2, \ldots, < x_{m-1} < x_m = \sigma.$$

Simpson's quadrature formula is defined by

$$S(\varphi,d)=\frac{1}{\Gamma(1+\alpha)6^{\alpha}}\sum_{i=0}^{m-1}\left[\varphi(x_i)+4^{\alpha}\varphi\left(\frac{x_i+x_{i+1}}{2}\right)+\varphi(x_{i+1})\right]h_i(\alpha), \tag{24}$$

where $h_i(\alpha)=(x_{i+1}-x_i)^{\alpha}$ and $i=0,1,\ldots,m-1$.

Proposition 1. *The assumptions of Lemma 6 are satisfied. If $|\varphi^{(\alpha)}|$ is a generalized strongly convex on $[\rho,\sigma]$, then, for every division d of $[\rho,\sigma]$ and $c\in\mathbb{R}^{+}$, we have the following representation*

$${}_{\rho}I_{\sigma}{}^{\alpha}\varphi(x)=\frac{1}{\Gamma(1+\alpha)}\int_{\rho}^{\sigma}\varphi(x)(\mathrm{d}x)^{\alpha}=S(\varphi,d)+R(\varphi,d),$$

where $S(\varphi,d)$ is defined as in (24) and the remainder satisfies the estimation:

$$|R(\varphi,d)|\leq\frac{5^{\alpha}}{\Gamma(1+2\alpha)36^{\alpha}}\sum_{i=0}^{m-1}\left[|\varphi^{(\alpha)}(x_i)|+|\varphi^{(\alpha)}(x_{i+1})|\right][h_i(\alpha)]^2-\frac{c^{\alpha}K(\alpha)}{\Gamma(1+\alpha)4^{\alpha}}\sum_{i=0}^{m-1}[h_i(\alpha)]^4. \tag{25}$$

Here, $K(\alpha)$ is defined as in (7).

Proof. Applying Theorem 2 on the subinterval $[x_i,x_{i+1}]$, $i=0,1,\ldots,m-1$ of the division d, we find that

$$\begin{aligned}&\left|\frac{1}{6^{\alpha}}\left[\varphi(x_i)+4^{\alpha}\varphi\left(\frac{x_i+x_{i+1}}{2}\right)+\varphi(x_{i+1})\right]-\frac{\Gamma(1+\alpha)}{h_i(\alpha)}{}_{\rho}I_{\sigma}{}^{\alpha}\varphi(x)\right|\\ \leq&\left(\frac{5}{36}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}h_i(\alpha)\left[|\varphi^{(\alpha)}(x_i)|+|\varphi^{(\alpha)}(x_{i+1})|\right]-\frac{c^{\alpha}[h_i(\alpha)]^3}{4^{\alpha}}K(\alpha),\end{aligned}$$

for all $i=0,1,\ldots,m-1$. Summing over i from 0 to $m-1$ and multiplying by the factor $h_i(\alpha)/\Gamma(1+\alpha)$, we obtain the estimation (25).

Remark. Using Theorems 3, 4 and 5, we can obtain some new estimations like as in the Proposition 1. We omit their proofs and the details are left to the interested readers.

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