New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2023.500

Generalized Ostrowski type results for twice differentiable functions on fractal sets

Samet Erden[1], *M. Adil Khan*[2], *Funda Türk*[1], *M. Zeki Sarıkaya*[3], *Rumeysa Erden*[1], *Neslihan Sümer*[1]

[1] Department of Mathematics, Faculty of Science, Bartın University, Bartın, Turkey
[2]Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan
[3]Department of Mathematics, Faculty of Science and Arts, Düzce University, Konuralp Campus, Düzce, Turkey

Received: 29 May 2023, Accepted: 19 July 2023
Published online: 28 July 2023.

Abstract: We first obtain a new auxiliary identity by utilizing twice differentiable functions on $\alpha-$type sets. Afterwards, two generalized Ostrowski type estimations for mappings whose second local fractional derivatives are bounded are derived, and special cases of these comprehensive inequalities are observed. Finally, some related applications such as inequalities including special means and generalized quadrature rules are presented.

Keywords: Ostrowski inequality, Local fractional integral, Fractal space.

1 Introduction

Ostrowski inequality [12], which is introduced by Ostrowski in 1938, have a great importance in many fields of mathematic due to the abundance of application areas. This inequality can be stated as follows.

Suppose that $\phi:[\sigma,\rho]\to\mathbb{R}$ is a differentiable function on (σ,ρ) such that its derivative $\phi^{'}:(\sigma,\rho)\to\mathbb{R}$ is bounded on (σ,ρ), i.e., $\|\phi'\|_\infty=\sup\limits_{\xi\in(\sigma,\rho)}|\phi'(\xi)|<\infty$. Then, one has

$$\left|\phi(\varkappa)-\frac{1}{\rho-\sigma}\int_\sigma^\rho\phi(\xi)d\xi\right|\leq\left[\frac{1}{4}+\frac{\left(\varkappa-\frac{\sigma+\rho}{2}\right)^2}{(\rho-\sigma)^2}\right](\rho-\sigma)\left\|\phi'\right\|_\infty \tag{1}$$

for all $\varkappa\in[\sigma,\rho]$. The constant $\frac{1}{4}$ is the best possible?

Inequality (1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Hence, inequality (1) has attracted considerable attention and interest from mathematicians and researchers. To exemplify, a generalized version of Ostrowski's inequality was established by Dragomir et al. in [5]. Dragomir expanded the inequality (1) by using an absolutely continuous function and a convex mapping in [3], and he gave applications for power and exponantial mappings. In addition, An integral inequality of Ostrowski type for functions the second derivatives of which is bounded are provided by Dragomir and Barnett in [4]. In [6]-[10], [13], and [25], the authors examined Ostrowski and Mid-point type inequalities for twice differentiable functions. In particular, Cerone et al. [1] proved the following result which is related to our paper.

* Corresponding author e-mail: erdensmt@gmail.com

Theorem 1. *Let $\phi : [\sigma,\rho] \to \mathbb{R}$ be a twice differentiable mapping such that $\phi'' : (\sigma,\rho) \to \mathbb{R}$ is bounded on (σ,ρ), i.e., $\|\phi''\|_\infty = \sup_{\xi\in(\sigma,\rho)} |\phi''(\xi)| < \infty$. Then we have the inequality*

$$\left|\phi(\varkappa) - \left(\varkappa - \frac{\sigma+\rho}{2}\right)\phi'(\varkappa) - \frac{1}{\rho-\sigma}\int_\sigma^\rho \phi(\xi)\mathrm{d}\xi\right| \qquad (2)$$

$$\leq \left[\frac{(\rho-\sigma)^2}{24} + \frac{1}{2}\left(\varkappa - \frac{\sigma+\rho}{2}\right)^2\right]\|\phi''\|_\infty \leq \frac{(\rho-\sigma)^2}{6}\|\phi''\|_\infty$$

for all $\varkappa \in [\sigma,\rho]$.

Whereas many mathematical problems can be solved by using classical methods or some approximate integral equations, domains of some mathematical issues are fractal curves, which are everywhere continuous but nowhere differentiable.The local fractional theory, which is introduced by Yang in [20] and [21], are one of beneficial tools to deal with the fractal and continuously nondifferentiable mappings. Therewith, a great number of researchers worked on local fractional theory in different areas such as mathematical physics, engineering problems and applied sciences. For more theoretical information and application areas on local fractional, you can see the references [20]-[24]. In addition, many researchers have worked on local fractional versions of integral inequalities which possess a very important role in applied and theoretical mathematics. For illustrate, authors provided local fractional versions of some significant inequalities including Hölder's inequality, Hermite-Hadamard inequality, Simpson and Newton in [2], [9], [11] and [16]. Furthermore, generalized Ostrowski type inequalities for mappings whose local fractional derivatives are bounded are obtained in [14] and [15]. The authors worked on fractional integral inequalities in [18] and [19], Tingsong et. al also gave certain integral inequalities for a different kind of convex functions in [17].

The main purpose of this study is to establish two inequalities that are connected with the celebrated generalized Ostrowski type inequalities using functions whose second local fractional derivatives are bounded. Also, some applications for generalized special means and local fractional quadrature formula are given by using these two results improved in this paper.

2 Preliminaries

We give some important concepts, definitions and rules for local fractional theory. We first recall the set $\mathbb{R}^\alpha$ of real line numbers and the other $\alpha-$type sets which was introduced by Yang in [21]. These sets are of great significance to describe Yang's local fractional derivative and integral which will use throughout this article.

For $0 < \alpha \leq 1$, we have the following α-type sets:
$\mathbb{Z}^\alpha =: \{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, ..., \pm n^\alpha, ...\}$ is α-type integer set.
$\mathbb{Q}^\alpha =: \{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p,q \in Z,\ q \neq 0\}$ is α-type rational numbers set.
$J^\alpha =: \{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p,q \in Z,\ q \neq 0\}$ α-type irrational numbers set.
$\mathbb{R}^\alpha = \mathbb{Q}^\alpha \cup J^\alpha$ The α-type real line numbers set.
If $\sigma^\alpha, \rho^\alpha$ and ς^α belongs the set $\mathbb{R}^\alpha$ of real line numbers, then
(1) $\sigma^\alpha + \rho^\alpha$ and $\sigma^\alpha\rho^\alpha$ belongs the set $\mathbb{R}^\alpha$;
(2) $\sigma^\alpha + \rho^\alpha = \rho^\alpha + \sigma^\alpha = (\sigma+\rho)^\alpha = (\rho+\sigma)^\alpha$;
(3) $\sigma^\alpha + (\rho^\alpha + \varsigma^\alpha) = (\sigma+\rho)^\alpha + \varsigma^\alpha$;
(4) $\sigma^\alpha\rho^\alpha = \rho^\alpha\sigma^\alpha = (\sigma\rho)^\alpha = (\rho\sigma)^\alpha$;
(5) $\sigma^\alpha(\rho^\alpha\varsigma^\alpha) = (\sigma^\alpha\rho^\alpha)\varsigma^\alpha$;

(6) $\sigma^{\alpha}(\rho^{\alpha}+\varsigma^{\alpha})=\sigma^{\alpha}\rho^{\alpha}+\sigma^{\alpha}\varsigma^{\alpha}$;
(7) $\sigma^{\alpha}+0^{\alpha}=0^{\alpha}+\sigma^{\alpha}=\sigma^{\alpha}$ and $\sigma^{\alpha}1^{\alpha}=1^{\alpha}\sigma^{\alpha}=\sigma^{\alpha}$.
We can define Yang's local fractional derivative and integral as follows.

Definition 1.[21] *Assume that $\psi:\mathbb{R}\to\mathbb{R}^{\alpha}$ is a non-differentiable function, $\varkappa\to\psi(\varkappa)$ is named to be local fractional continuous at $\varkappa_0$, if there exists $\delta>0$ for any $\varepsilon>0$, such that*

$$|\psi(\varkappa)-\psi(\varkappa_0)|<\varepsilon^{\alpha}$$

holds for $|\varkappa-\varkappa_0|<\delta$, where $\varepsilon,\delta\in\mathbb{R}$. We denote $\psi(\varkappa)\in C_{\alpha}(\sigma,\rho)$, if $\psi(\varkappa)$ is local continuous on the interval (σ,ρ).

Definition 2.[21] *We define Yang's local fractional derivative of $\psi(\varkappa)$ of order α at $\varkappa=\varkappa_0$ by*

$$\psi^{(\alpha)}(\varkappa_0)=\left.\frac{\mathrm{d}^{\alpha}\psi(\varkappa)}{\mathrm{d}\varkappa^{\alpha}}\right|_{\varkappa=\varkappa_0}=\lim_{\varkappa\to\varkappa_0}\frac{\Delta^{\alpha}(\psi(\varkappa)-\psi(\varkappa_0))}{(\varkappa-\varkappa_0)^{\alpha}},$$

where $\Delta^{\alpha}(\psi(\varkappa)-\psi(\varkappa_0))\cong\Gamma(\alpha+1)(\psi(\varkappa)-\psi(\varkappa_0))$.

We can denote $\psi\in D_{(k+1)\alpha}(I)$ with $k=0,1,2,...,$ if there exists $\psi^{(k+1)\alpha}(\varkappa)=\overbrace{D_{\varkappa}^{\alpha}...D_{\varkappa}^{\alpha}}^{k+1\text{ times}}\psi(\varkappa)$ for any $\varkappa\in I\subseteq\mathbb{R}$

Definition 3.[21] *If $\psi(\varkappa)$ is element of $C_{\alpha}[\sigma,\rho]$, then we define Yang's local fractional integral by*

$${}_{\sigma}I_{\rho}^{\alpha}\psi(\varkappa)=\frac{1}{\Gamma(\alpha+1)}\int_{\sigma}^{\rho}\psi(\xi)(\mathrm{d}\xi)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta\xi\to0}\sum_{j=0}^{N-1}\psi(\xi_j)(\Delta\xi_j)^{\alpha},$$

with $\Delta\xi_j=\xi_{j+1}-\xi_j$ and $\Delta\xi=\max\{\Delta\xi_1,\Delta\xi_2,...,\Delta\xi_{N-1}\}$, where $[\xi_j,\xi_{j+1}]$, $j=0,...,N-1$ and $\sigma=\xi_0<\xi_1<...<\xi_{N-1}<\xi_N=\rho$ is a division of interval $[\sigma,\rho]$. Here, it follows that ${}_{\sigma}I_{\rho}^{\alpha}\psi(\varkappa)=0$ if $\sigma=\rho$ and ${}_{\sigma}I_{\rho}^{\alpha}\psi(\varkappa)=-{}_{\rho}I_{\sigma}^{\alpha}\psi(\varkappa)$ if $\sigma<\rho$. If for any $\varkappa\in[\sigma,\rho]$, there exists ${}_{\sigma}I_{\varkappa}^{\alpha}\psi(\varkappa)$, then we denoted by $\psi(\varkappa)\in I_{\varkappa}^{\alpha}[\sigma,\rho]$.

Lemma 1.[21] *We should note that local fractional integration is anti-differentiation.*
(1) Suppose that $\psi(\varkappa)=\varphi^{(\alpha)}(\varkappa)\in C_{\alpha}[\sigma,\rho]$, then we have

$${}_{\sigma}I_{\rho}^{\alpha}\psi(\varkappa)=\varphi(\rho)-\varphi(\sigma).$$

(2) (Integration by parts for Yang's local fractional integrals) Suppose that $\psi(\varkappa),\varphi(\varkappa)\in D_{\alpha}[\sigma,\rho]$ and $\psi^{(\alpha)}(\varkappa)$, $\varphi^{(\alpha)}(\varkappa)\in C_{\alpha}[\sigma,\rho]$, then we have

$${}_{\sigma}I_{\rho}^{\alpha}\psi(\varkappa)\varphi^{(\alpha)}(\varkappa)=\left.\psi(\varkappa)\varphi(\varkappa)\right|_{\sigma}^{\rho}-{}_{\sigma}I_{\rho}^{\alpha}\psi^{(\alpha)}(\varkappa)\varphi(\varkappa).$$

Lemma 2.[21] *We have the following formulas:*
i) $\dfrac{\mathrm{d}^{\alpha}\varkappa^{k\alpha}}{\mathrm{d}x^{\alpha}}=\dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}\varkappa^{(k-1)\alpha}$;
ii) $\dfrac{1}{\Gamma(\alpha+1)}\int_{\sigma}^{\rho}\varkappa^{k\alpha}(\mathrm{d}\varkappa)^{\alpha}=\dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(\rho^{(k+1)\alpha}-\sigma^{(k+1)\alpha}\right)$, $k\in R$.

Theorem 2.[2] *Let $\psi,\varphi\in C_{\alpha}[\sigma,\rho]$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\frac{1}{\Gamma(\alpha+1)}\int_{\sigma}^{\rho}|\psi(\varkappa)\varphi(\varkappa)|(\mathrm{d}\varkappa)^{\alpha}\le\left(\frac{1}{\Gamma(\alpha+1)}\int_{\sigma}^{\rho}|\psi(\varkappa)|^{p}(\mathrm{d}\varkappa)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\alpha+1)}\int_{\sigma}^{\rho}|\varphi(\varkappa)|^{q}(\mathrm{d}\varkappa)^{\alpha}\right)^{\frac{1}{q}}.$$

Ostrowski inequality for Yang's local fractional integrals was introduced by Sarikaya and Budak as follows.

Theorem 3(Generalized Ostrowski inequality). *[14] Supposing that $I \subseteq \mathbb{R}$ is an interval, $\varphi : I^0 \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ (I^0 is the interior of I) such that $\varphi \in D_\alpha(I^0)$ and $\varphi^{(\alpha)} \in C_\alpha[\sigma,\rho]$ for $\sigma,\rho \in I^0$ with $\sigma < \rho$. Then, for all $\varkappa \in [\sigma,\rho]$, one has*

$$\left|\varphi(\varkappa) - \frac{\Gamma(1+\alpha)}{(b-\sigma)^\alpha}\, {}_\sigma I_\rho^\alpha \varphi(t)\right| \le 2^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\frac{1}{4^\alpha} + \left(\frac{\varkappa - \frac{\sigma+\rho}{2}}{\rho-\sigma}\right)^{2\alpha}\right] (\rho-\sigma)^\alpha \left\|\varphi^{(\alpha)}\right\|_\infty. \tag{3}$$

3 Main Results

We observe how the inequalities will arise when we use twice local fractional differentiable functions in this section. First of all, we establish an identity which is required to deduce our main results in the following Lemma.

Lemma 3.*Let $I \subseteq R$ be an interval, $\phi : I^0 \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ (I^0 is the interior of I) such that $\phi \in D_{2\alpha}(I^0)$ and $\phi^{(2\alpha)} \in C_{2\alpha}[\sigma,\rho]$ for $\sigma,\rho \in I^0$ with $\sigma < \rho$. Then one has the identity*

$$\begin{aligned}
&\frac{1}{2^\alpha(\rho-\sigma)^\alpha}\frac{1}{\Gamma(1+\sigma)}\int_\sigma^\rho \Lambda_h(x,\xi;\alpha)\,\phi^{(2\alpha)}(\xi)\,(d\xi)^\alpha \qquad (4)\\
&= \frac{(h-2)^\alpha}{2^\alpha}\left(\varkappa - \frac{\sigma+\rho}{2}\right)^\alpha \phi^{(\alpha)}(\varkappa) + \frac{\Gamma(1+2\alpha)}{2^\alpha\Gamma(1+\alpha)}\phi(\varkappa)\\
&\quad - \frac{\Gamma(1+\alpha)}{(\rho-\sigma)^\alpha}(m_h(\varkappa))^\alpha\left[\frac{\phi(\rho)-\phi(\sigma)}{2^\alpha}\right]\\
&\quad - \frac{1}{2^\alpha(\rho-\sigma)^\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\int_\sigma^\rho \phi(\xi)\,(d\xi)^\alpha\\
&=: \Delta_h(\phi;\varkappa,\alpha)
\end{aligned}$$

for

$$\Lambda_h(\varkappa,\xi;\alpha) := \begin{cases} (\sigma-\xi)^\alpha(\xi-\sigma-m_h(\varkappa))^\alpha, & \sigma \le \xi < \varkappa \\ (\rho-\xi)^\alpha(\xi-\rho-m_h(\varkappa))^\alpha, & \varkappa \le \xi \le \rho \end{cases}$$

where $m_h(\varkappa) = h\left(\varkappa - \frac{\sigma+\rho}{2}\right)$, $h \in [0,2]$ and $\varkappa \in [\sigma,\rho]$.

Proof. By definition of function $\Lambda_h(\varkappa,t;\alpha)$, we find that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_\sigma^\rho \Lambda_h(\varkappa,\xi;\alpha)\,\phi^{(2\alpha)}(\xi)\,(d\xi)^\alpha\\
&= \frac{1}{\Gamma(1+\alpha)}\int_\sigma^\varkappa (\sigma-\xi)^\alpha(\xi-\sigma-m_h(\varkappa))^\alpha\,\phi^{(2\alpha)}(\xi)\,(d\xi)^\alpha\\
&\quad + \frac{1}{\Gamma(1+\alpha)}\int_\varkappa^\rho (\rho-\xi)^\alpha(\xi-\rho-m_h(\varkappa))^\alpha\,\phi^{(2\alpha)}(\xi)\,(d\xi)^\alpha.
\end{aligned}$$

Applying local fractional integration by parts and using the Lemma 2, we obtain

$$\frac{1}{\Gamma(1+\alpha)}\int_{\sigma}^{\rho}\Lambda_h(\varkappa,\xi;\alpha)\,\phi^{(2\alpha)}(\xi)\,(d\xi)^{\alpha}$$
$$=(h-2)^{\alpha}(\rho-\sigma)^{\alpha}\left(\varkappa-\frac{\sigma+\rho}{2}\right)^{\alpha}\phi^{(\alpha)}(\varkappa)$$
$$+\frac{1}{\Gamma(1+\alpha)}\int_{\sigma}^{\varkappa}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(\xi-\sigma)^{\alpha}-\Gamma(1+\alpha)(m_h(\varkappa))^{\alpha}\right]\phi^{(\alpha)}(\xi)\,(d\xi)^{\alpha}$$
$$+\frac{1}{\Gamma(1+\alpha)}\int_{\varkappa}^{\rho}\left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}(\xi-\rho)^{\alpha}-\Gamma(1+\alpha)(m_h(\varkappa))^{\alpha}\right]\phi^{(\alpha)}(\xi)\,(d\xi)^{\alpha}.$$

If we apply local fractional integration by parts again, then we obtain desired equality (4).

We deduce new inequalities involving local fractional integrals by considering generalized convex function in the following theorem.

Theorem 4. *The assumptions of Lemma 3 are satisfied. If $\phi^{(2\alpha)}$ is bounded on (σ,ρ), i.e., $\left\|\phi^{(2\alpha)}\right\|_{\infty}<\infty$, then we possess*

$$|\Delta_h(\phi;\varkappa,\alpha)| \tag{5}$$
$$\leq\left\{\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{(\rho-\varkappa)^{3\alpha}+(\varkappa-\sigma)^{3\alpha}}{2^{\alpha}(\rho-\sigma)^{\alpha}}\right)-h^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\varkappa-\frac{\sigma+\rho}{2}\right)^{2\alpha}\right.$$
$$\left.-\frac{1}{(\rho-\sigma)^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right][m_h(\varkappa)]^{3\alpha}\right\}\left\|\phi^{(2\alpha)}\right\|_{\infty}$$

for all $\sigma\leq\varkappa\leq\frac{\sigma+\rho}{2}$ with $h\in[0,2]$, and

$$|\Delta_h(\phi;\varkappa,\alpha)| \tag{6}$$
$$\leq\left\{\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{(\rho-\varkappa)^{3\alpha}+(\varkappa-\sigma)^{3\alpha}}{2^{\alpha}(\rho-\sigma)^{\alpha}}\right)-h^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\varkappa-\frac{\sigma+\rho}{2}\right)^{2\alpha}\right.$$
$$\left.+\frac{1}{(\rho-\sigma)^{\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right][m_h(\varkappa)]^{3\alpha}\right\}\left\|\phi^{(2\alpha)}\right\|_{\infty}$$

for all $\frac{\sigma+\rho}{2}<\varkappa\leq\rho$ with $h\in[0,2]$. Here $m_h(\varkappa)$ is defined by $m_h(\varkappa)=h\left(\varkappa-\frac{\sigma+\rho}{2}\right)$.

Proof. If we take absolute value of both sides of (4), owing to the conditions of mapping $\phi^{(2\alpha)}$, we attain

$$|\Delta_h(\phi;\varkappa,\alpha)| \tag{7}$$
$$\leq\frac{1}{2^{\alpha}(\rho-\sigma)^{\alpha}}\frac{1}{\Gamma(1+\sigma)}\int_{\sigma}^{\rho}|P_h(\varkappa,\xi;\alpha)|\left|\phi^{(2\alpha)}(\xi)\right|(d\xi)^{\alpha}$$
$$\leq\frac{\left\|\phi^{(2\alpha)}\right\|_{\infty}}{2^{\alpha}(\rho-\sigma)^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_{\sigma}^{\varkappa}|\sigma-\xi|^{\alpha}|\xi-\sigma-m_h(\varkappa)|^{\alpha}(d\xi)^{\alpha}\right.$$
$$\left.+\frac{1}{\Gamma(1+\alpha)}\int_{\varkappa}^{\rho}|\rho-\xi|^{\alpha}|\xi-\rho-m_h(\varkappa)|^{\alpha}(d\xi)^{\alpha}\right].$$

We observe that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_{\gamma}^{r}|\xi-\gamma|^{\alpha}|\xi-\delta|^{\alpha}(d\xi)^{\alpha} \qquad (8)\\
&=\frac{1}{\Gamma(1+\alpha)}\int_{\gamma}^{\delta}(\xi-\gamma)^{\alpha}(\delta-\xi)^{\alpha}(d\xi)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_{\delta}^{r}(\xi-\gamma)^{\alpha}(\xi-\delta)^{\alpha}(d\xi)^{\alpha}\\
&=2^{\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right](\delta-\gamma)^{3\alpha}\\
&\quad+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(r-\gamma)^{3\alpha}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(\delta-\gamma)^{\alpha}(r-\gamma)^{2\alpha}
\end{aligned}$$

for all r,γ,δ such that $\gamma\leq\delta\leq r$.

It should be calculated local fractional integrals in (7) for the cases $\sigma\leq\varkappa\leq\frac{\sigma+\rho}{2}$ and $\frac{\sigma+\rho}{2}<\varkappa\leq\rho$;

For the case when $\sigma\leq\varkappa\leq\frac{\sigma+\rho}{2}$, we find that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_{\sigma}^{\varkappa}|\sigma-\xi|^{\alpha}|\xi-\sigma-m_h(\varkappa)|^{\alpha}(d\xi)^{\alpha} \qquad (9)\\
&=\frac{1}{\Gamma(1+\alpha)}\int_{\sigma}^{\varkappa}(\xi-\sigma)^{\alpha}(\xi-\sigma-m_h(\varkappa))^{\alpha}(d\xi)^{\alpha}\\
&=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\varkappa-\sigma}u^{\alpha}(u-m_h(\varkappa))^{\alpha}(du)^{\alpha}\\
&=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(\varkappa-\sigma)^{3\alpha}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}[m_h(\varkappa)]^{\alpha}(\varkappa-\sigma)^{2\alpha}.
\end{aligned}$$

If we use also the equality (8) for the second integral in (7), then we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_{\varkappa}^{\rho}|\rho-\xi|^{\alpha}|\xi-\rho+m_h(\varkappa)|^{\alpha}(d\xi)^{\alpha} \qquad (10)\\
&=-2^{\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right][m_h(\varkappa)]^{3\alpha}\\
&\quad+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(\rho-\varkappa)^{3\alpha}+\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}[m_h(\varkappa)]^{\alpha}(\rho-\varkappa)^{2\alpha}.
\end{aligned}$$

Substituting the equalities (9) and (10) in (7), the inequality (5) can be readily captured.
For the case when $\frac{\sigma+\rho}{2}<\varkappa\leq\rho$, using the equality (8), it follows that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_{\sigma}^{\varkappa}|\sigma-\xi|^{\alpha}|\xi-\sigma-m_h(\varkappa)|^{\alpha}(d\xi)^{\alpha}=2^{\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right][m_h(\varkappa)]^{3\alpha} \qquad (11)\\
&+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(\varkappa-\sigma)^{3\alpha}-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}[m_h(\varkappa)]^{\alpha}(\varkappa-\sigma)^{2\alpha}.
\end{aligned}$$

We observe that

$$\frac{1}{\Gamma(1+\alpha)}\int_{\varkappa}^{\rho}|\rho-\xi|^{\alpha}|\xi-\rho+m_h(\varkappa)|^{\alpha}(d\xi)^{\alpha}=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(\rho-\varkappa)^{3\alpha}+\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}[m_h(\varkappa)]^{\alpha}(\rho-\varkappa)^{2\alpha}. \qquad (12)$$

Substituting the equalities (11) and (12) in (7), the inequality (6) can be easily obtained. The proof is thus completed.

Remark. If we choose $\alpha = 1$ in the inequalities of theorem 4, then we capture the results obtained by Erden et al. in [8].

Corollary 1. *Under the same conditions of theorem 4 with $h = 0$, one has the inequality*

$$\left| \frac{\Gamma(1+2\alpha)}{2^{\alpha}\Gamma(1+\alpha)}\phi(\varkappa) + \left(\frac{\sigma+\rho}{2} - x\right)^{\alpha} \phi^{(\alpha)}(\varkappa) - \frac{\Gamma(1+2\alpha)}{2^{\alpha}(\rho-\sigma)^{\alpha}}\, {}_{\sigma}I_{\rho}^{\alpha}\phi(\xi) \right| \tag{13}$$
$$\leq \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(\frac{(\rho-\varkappa)^{3\alpha} + (\varkappa-\sigma)^{3\alpha}}{2^{\alpha}(\rho-\sigma)^{\alpha}}\right) \left\|\phi^{(2\alpha)}\right\|_{\infty}.$$

Corollary 2. *If we choose $x = \frac{\sigma+\rho}{2}$ in (13), then we possess the inequality*

$$\left| \frac{\Gamma(1+2\alpha)}{2^{\alpha}\Gamma(1+\alpha)}\phi\left(\frac{\sigma+\rho}{2}\right) - \frac{\Gamma(1+2\alpha)}{2^{\alpha}(\rho-\sigma)^{\alpha}}\, {}_{\sigma}I_{\rho}^{\alpha}\phi(\xi) \right| \leq 2\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \frac{(\rho-\sigma)^{2\alpha}}{2^{4\alpha}} \left\|\phi^{(2\alpha)}\right\|_{\infty}.$$

which is generalized Mid-point type inequalities for functions whose second local fractional derivatives are bounded.

Remark. If we take $\alpha = 1$ in (13), then the inequality (13) collapses to the previous well-known result (2).

Remark. Similar results can be achieved by choosing $h = 2$ in the inequalities (5) and (6).

4 Applications to Numerical Integration

In this part, we deal with the inequalities (5) and (6) in order to develope new composite quadrature rules which generalize the estimates given in the earlier works.

Supposing that $D_z : \sigma = \varkappa_0 < \varkappa_1 < ... < \varkappa_{z-1} < \varkappa_z = \rho$ is a division of the interval $[\sigma, \rho]$, $\zeta_k \in [\varkappa_k, \varkappa_{k+1}]$ for $k = 0, ..., z-1$. We also define the quadrature

$$\mathscr{S}\left(\phi, \phi^{(\alpha)}, \zeta, D_z\right) = \frac{(h-2)^{\alpha}}{\Gamma(1+2\alpha)} \sum_{k=0}^{z-1} \left(\zeta_k - \frac{\varkappa_k + \varkappa_{k+1}}{2}\right)^{\alpha} \phi^{(\alpha)}(\zeta_k)\, b_k^{\alpha} + \frac{1}{\Gamma(1+\alpha)} \sum_{k=0}^{z-1} \phi(\zeta_k)\, b_k^{\alpha} \tag{14}$$
$$-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} h^{\alpha} \sum_{k=0}^{z-1} \left(\zeta_k - \frac{\varkappa_k + \varkappa_{k+1}}{2}\right)^{\alpha} \left[\phi(\varkappa_{k+1}) - \phi(\varkappa_k)\right]$$

where $b_k = \varkappa_{k+1} - \varkappa_k$, $k = 0, ..., z-1$.

Theorem 5. *Let $I \subseteq \mathbb{R}$ be an interval, $\phi : I^0 \subseteq \mathbb{R} \to \mathbb{R}^{\alpha}$ (I^0 is the interior of I) such that $\phi \in D_{2\alpha}(I^0)$ and $\phi^{(2\alpha)} \in C_{2\alpha}[\sigma, \rho]$ for $\sigma, \rho \in I^0$ with $\sigma < \rho$. If $\phi^{(2\alpha)}$ is bounded on (σ, ρ), i.e., $\|f''\|_{\infty} < \infty$, then one has the representation*

$${}_{\sigma}I_{\rho}^{\alpha}\phi(\xi) = \frac{1}{\Gamma(1+\alpha)} \int_{\sigma}^{\rho} \phi(\xi)(d\xi)^{\alpha} = \mathscr{S}\left(\phi, \phi^{(\alpha)}, \zeta, D_z\right) + \mathscr{R}\left(\phi, \phi^{(\alpha)}, \zeta, D_z\right)$$

where $\mathscr{S}\left(\phi, \phi^{(\alpha)}, \zeta, D_z\right)$ is defined as in (14), and the remainder $\mathscr{R}\left(\phi, \phi^{(\alpha)}, \zeta, D_z\right)$ gives the estimations:

$$\left|\mathscr{R}\left(\phi, \phi^{(\alpha)}, \zeta, D_z\right)\right| \leq \left\|\phi^{(2\alpha)}\right\|_{\infty} \left\{ \frac{1}{\Gamma(1+3\alpha)} \sum_{k=0}^{z-1} \left((\varkappa_{k+1} - \zeta_k)^{3\alpha} + (\zeta_k - \varkappa_k)^{3\alpha} \right) \right. \tag{15}$$
$$-2^{\alpha} h^{\alpha} \frac{\Gamma(1+\alpha)}{\Gamma^2(1+2\alpha)} \sum_{k=0}^{z-1} \left(\zeta_k - \frac{\varkappa_k + \varkappa_{k+1}}{2}\right)^{2\alpha} b_k^{\alpha}$$
$$\left. -\frac{2^{\alpha} h^{3\alpha}}{\Gamma(1+2\alpha)} \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \sum_{k=0}^{z-1} \left(\zeta_k - \frac{\varkappa_k + \varkappa_{k+1}}{2}\right)^{3\alpha} \right\}$$

for $\varkappa_k \leq \zeta_k \leq \frac{\varkappa_k+\varkappa_{k+1}}{2}$ *with* $h \in [0,2]$, *and*

$$\left|\mathscr{R}\left(\phi,\phi^{(\alpha)},\zeta,D_z\right)\right| \leq \left\|\phi^{(2\alpha)}\right\|_\infty \left\{\frac{1}{\Gamma(1+3\alpha)}\sum_{k=0}^{z-1}\left((\varkappa_{k+1}-\zeta_k)^{3\alpha}+(\zeta_k-\varkappa_k)^{3\alpha}\right)\right. \tag{16}$$
$$-2^\alpha h^\alpha \frac{\Gamma(1+\alpha)}{\Gamma^2(1+2\alpha)}\sum_{k=0}^{z-1}\left(\zeta_k-\frac{\varkappa_k+\varkappa_{k+1}}{2}\right)^{2\alpha} b_k^\alpha$$
$$\left.+\frac{2^\alpha h^{3\alpha}}{\Gamma(1+2\alpha)}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\sum_{k=0}^{z-1}\left(\zeta_k-\frac{\varkappa_k+\varkappa_{k+1}}{2}\right)^{3\alpha}\right\}$$

for $\frac{\varkappa_k+\varkappa_{k+1}}{2} \leq \zeta_k \leq \varkappa_{k+1}$ *with* $h \in [0,2]$, $k = 0,...,z-1$.

*Proof.*If we apply the inequality (5) on $[\varkappa_k, \varkappa_{k+1}]$ for $k = 0, ..., z-1$, then we attain

$$\left|\frac{(h-2)^\alpha}{2^\alpha}\left(\zeta_k-\frac{\varkappa_k+\varkappa_{k+1}}{2}\right)^\alpha \phi^{(\alpha)}(\zeta_k)+\frac{\Gamma(1+2\alpha)}{2^\alpha\Gamma(1+\alpha)}\phi(\zeta_k)\right.$$
$$-\frac{\Gamma(1+\alpha)}{2^\alpha(\varkappa_{k+1}-\varkappa_k)^\alpha}h^\alpha\left(\zeta_k-\frac{\varkappa_k+\varkappa_{k+1}}{2}\right)^\alpha[\phi(\varkappa_{k+1})-\phi(\varkappa_k)]$$
$$\left.-\frac{\Gamma(1+2\alpha)}{2^\alpha(\varkappa_{k+1}-\varkappa_k)^\alpha\Gamma(1+\alpha)}\int_{\varkappa_k}^{\varkappa_{k+1}}\phi(\xi)(d\xi)^\alpha\right|$$
$$\leq \left\|\phi^{(2\alpha)}\right\|_\infty\left\{\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{(\varkappa_{k+1}-\zeta_k)^{3\alpha}+(\zeta_k-\varkappa_k)^{3\alpha}}{2^\alpha(\varkappa_{k+1}-\varkappa_k)^\alpha}\right)\right.$$
$$-h^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\zeta_k-\frac{\varkappa_k+\varkappa_{k+1}}{2}\right)^{2\alpha}$$
$$\left.-\frac{h^{3\alpha}}{(\varkappa_{k+1}-\varkappa_k)^\alpha}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\left(\zeta_k-\frac{\varkappa_k+\varkappa_{k+1}}{2}\right)^{3\alpha}\right\}$$

for $\varkappa_k \leq \zeta_k \leq \frac{\varkappa_k+\varkappa_{k+1}}{2}$ with $h \in [0,2]$. Later, summing the above inequality over k from 0 to $z-1$ and using the triangle inequality, the estimations (15) can be readily obtained. If we follow similar operations for the case when $\frac{\varkappa_k+\varkappa_{k+1}}{2} \leq \zeta_k \leq \varkappa_{k+1}$, then we obtain the estimation (16).

It is clear that the special cases of the estimations (15) and (16) give previously well-known quadrature formulas such as midpoint quadrature rule.

*Remark.*If we choose $\alpha = 1$ in the results of theorem 5, then we reach the estimates given by Erden et al. in [8].

Corollary 3.*Under the same conditions of theorem 5 with* $\zeta_k = \frac{\varkappa_k+\varkappa_{k+1}}{2}$ *and* $h = 0$, *then one obtain the Mid-point quadrature formula for local fractional integrals*

$${}_\sigma I_\rho^\alpha \phi(\xi) = \frac{1}{\Gamma(1+\alpha)}\int_\sigma^\rho \phi(\xi)(d\xi)^\alpha = \mathscr{S}_M(\phi,D_z)+\mathscr{R}_M(\phi,D_z)$$

where the remainder $\mathscr{R}_M(\phi,D_z)$ *satisfies the estimation*

$$|\mathscr{R}_M(\phi,D_z)| \leq \frac{2}{\Gamma(1+3\alpha)}\frac{\left\|\phi^{(2\alpha)}\right\|_\infty}{2^{3\alpha}}\sum_{k=0}^{z-1} b_k^{3\alpha}. \tag{17}$$

*Remark.*If we take $\alpha = 1$ in (17), we recapture the Mid-point quadrature rule which was presented by Cerone et al. in [1].

5 Applications to Some Special Means

In this section, we obtain some inequalities including generalized means.For this, we first recall certain generalized special means.

Generalized Arithmatic mean is defined by

$$A(\sigma,\rho)=\frac{\sigma^{\alpha}+\rho^{\alpha}}{2^{\alpha}},$$

and generalized Logartihmic mean is also defined by

$$L_s(\sigma,\rho)=\left[\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left[\frac{\rho^{(s+1)\alpha}-\sigma^{(s+1)\alpha}}{(\rho-\sigma)^{\alpha}}\right]\right]^{\frac{1}{s}},$$

for $s\in Z\backslash\{-1,0\}$, $\sigma,\rho\in\mathbb{R}$ with $\sigma\neq\rho$.

We consider the function $\phi:(0,\infty)\to\mathbb{R}^{\alpha}$, $\phi(\xi)=\xi^{s\alpha}$, $s\in Z\backslash\{-1,0\}$. If we use the Lemma 2 and the above definitions, then, we possess

$$\phi^{(\alpha)}(\varkappa)=\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-1)\alpha)}\varkappa^{(s-1)\alpha},\ \ \phi\left(\frac{\sigma+\rho}{2}\right)=[A(\sigma,\rho)]^{s}$$

and

$$\frac{1}{(\rho-\sigma)^{\alpha}}\ {}_{\sigma}I_{\rho}^{\alpha}\phi(\xi)=[L_s(\sigma,\rho)]^{s}$$

for $0<\sigma<\rho$. Also, one has

$$\begin{aligned}\left\|\phi^{(2\alpha)}\right\|_{\infty}&=\begin{cases}\left|\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-2)\alpha)}\right|\rho^{(s-2)\alpha}, & s>1\\ \left|\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-2)\alpha)}\right|\sigma^{(s-2)\alpha}, & s\in(-\infty,1]\backslash\{-1,0\}\end{cases}\\ &=:\Upsilon_s(\sigma,\rho).\end{aligned}$$

If we reconsider the inequality (13) by considering generalized special means, then we have the inequality

$$\begin{aligned}&\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}\Gamma(1+\alpha)}\varkappa^{s\alpha}+\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-1)\alpha)}\left[A(\sigma,\rho)\varkappa^{(s-1)\alpha}-\varkappa^{s\alpha}\right]-\frac{\Gamma(1+2\alpha)}{2^{\alpha}}[L_s(\sigma,\rho)]^{s}\right|\\ &\leq\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{(\rho-\varkappa)^{3\alpha}+(\varkappa-\sigma)^{3\alpha}}{2^{\alpha}(\rho-\sigma)^{\alpha}}\right)\Upsilon_s(\sigma,\rho),\end{aligned}\tag{18}$$

for $0<\sigma<\rho$. If we also choose $\varkappa=\frac{\sigma+\rho}{2}$ in (18), then we possess

$$\left|\frac{\Gamma(1+2\alpha)}{2^{\alpha}\Gamma(1+\alpha)}[A(\sigma,\rho)]^{s}-\frac{\Gamma(1+2\alpha)}{2^{\alpha}}[L_s(\sigma,\rho)]^{s}\right|\leq2\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{(\rho-\sigma)^{2\alpha}}{2^{4\alpha}}\Upsilon_s(\sigma,\rho),$$

which is the Mid-point type inequality involving generalized special means.

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