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On ψ-type fractional differential equations with measure of noncompactness in Banach space

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Abstract: This paper is concerned with the fractional differential equations with initial conditions. Results are established by using the concept of fractional theory, semigroup, the Monch fixed point theorem and measure of noncompactness

Keywords: ψ-fractional derivative; Initial value problem; Noncompactness; Monch's fixed point theorem

1 Introduction

In this paper, we are concerned with the initial value problem

$$\begin{cases} {}^c\mathscr{D}^{\alpha;\psi} y(t) = f(t,y), \quad \text{for} \quad t \in J := [0,T], \ 1 < r < 2, \\ y(0) = y_0, \quad y^{'}(0) = y_1, \end{cases} \tag{1}$$

where ${}^c\mathscr{D}^{\alpha}$ is the ψ-type Caputo fractional derivative, $f : J \times E \to E$ is a given function satisfying some assumptions that will be specified later, and E is a Banach space with norm $\|\cdot\|$. The technique used here is the measure of noncompactness associated with Monch's fixed point theorem.

The fractional derivative is understood in the Caputo sense. The origin of fractional calculus goes back to Newton and Leibnitz in the seventeenth century. One observes that fractional order can be very complex in the viewpoint of mathematics and they have recently proved to be valuable in various fields of science and engineering. In fact, one can find numerous applications in electrochemistry, electromagnetism, viscoelasticity, biology and hydrogeology. For example space-fractional diffusion equations have been used in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium, for details, see [5,6,8] and the references therein. Differential equations of fractional order have appeared in many branches of physics and technical sciences [4, 7]. It has seen considerable development in the last decade, see [10,11] and the references therein.

To the best of our knowledge, the existence of solutions for the ψ-type fractional differential equation (1) with initial conditions using the theory of measure of noncompactness is a subject that has not been treated in the literature. Our purpose in this paper is to establish some results concerning the existence of solutions for equations that can be modeled in the form (1) by virtue of the theory of measure of noncompactness associated with Monch's fixed point theorem.

Throughout this paper, we consider the problem (1) under the following hypotheses:

(H1) $f : J \times E \to E$ satisfies the Caratheodory conditions.

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(H2)There exists $p \in L^1(J,\mathbb{R}_+)\bigcap C(J,\mathbb{R}_+)$, such that,

$$\|f(t,y)\| \leq p(t)\,\|y\|, \quad \text{for} \quad t\in J \quad \text{and each} \quad y\in E.$$

(H3)For each $t \in J$ and each bounded set $B \subset E$, we have

$$\lim_{h\to 0^+} r\left(f(J_{t,h}\times B)\right) \leq p(t)r(B); \quad \text{here} \quad J_{t,h} = [t-h,t]\bigcap J.$$

Upon making some appropriate assumptions, some sufficient conditions for the existence of solutions for the ψ-type fractional differential Eq. (1) are given.

2 Notations, definitions and auxiliary facts

Denote by $C(J,E)$ the Banach space of continuous functions $y: J\to E$, with the usual supremum norm

$$\|y\|_\infty = \sup\{\|y\|, \quad t\in J\}.$$

Let $L^1(J,E)$ be the Banach space of measurable functions $y: J\to E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_J y(t)dt.$$

$AC^1(J,E)$ denotes the space of functions $y: J\to E$, whose first derivative is absolutely continuous.

Moreover, for a given set V of functions $v: J\to E$, let us denote by

$$V(t) = \{v(t): \quad v\in V\}, \quad t\in J,$$

and

$$V(J) = \{v(t): \quad v\in V\}, \quad t\in J.$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness, for more information in [3].

Definition 21 *[1, 3] Let E be a Banach space and Ω_E the bounded subset of E. The Kuratowski measure of noncompactness is the map $r:\Omega_E\to[0,\infty]$ defined by*

$$r(B) = \inf\left\{\varepsilon>0: \quad B\subseteq\bigcup_{i=1}^{n} B_i \quad for \quad diam(B_i)\leq\varepsilon\right\};$$

here $B\in\Omega_E$. This measure of noncompactness satisfies some important properties;

(a)$r(B)=0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).
(b)$r(B) = r(\overline{B})$.
(c)$A\subset B \Rightarrow r(A)\leq r(B)$.
(d)$r(A+B)\leq r(A)+r(B)$.
(e)$r(cB) = |c|\,r(B)$; $c\in\mathbb{R}$.
(f)$r(convB) = r(B)$.

To discuss the problem in this paper, we need the following results.

Definition 22 *[2] The ψ-type fractional order integral of the function $h\in L^1([a,b])$ of order $\alpha\in\mathbb{R}_+$ is defined by*

$$\mathscr{I}_a^{\alpha;\psi}h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}h(s)ds,$$

where Γ is the gamma function. When $a=0$, we write $\mathscr{I}^{\alpha;\psi}h(t) = h(t)*_\psi\varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{(\psi(t))^{\alpha-1}}{\Gamma(\alpha)}$ for $t>0$, and $\varphi_\alpha(t)=0$ for $t\leq 0$, and $\varphi_\alpha\to\delta(t)$ as $\alpha\to 0$, where δ is the delta function.

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Definition 23 *[2] For a function h defined on the interval $[a,b]$, the ψ-type Caputo fractional-order derivative of h, is defined by*

$$^{c}\mathscr{D}_{a^+}^{\alpha;\psi}h(t)=\frac{1}{\Gamma(n-r)}\int_a^t\psi^{'}(t)\left(\psi(t)-\psi(s)\right)^{n-\alpha-1}h^{(n)}(s)ds.$$

Here $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 24 *[3] A map $f:J\times E\to E$ is said to be Caratheodory if*

(i)$t\to f(t,u)$ is measurable for each $u\in E$,
(ii)$u\to F(t,u)$ is continuous for almost all $t\in J$.

The following theorem will play an important role in our analysis.

Theorem 25 *[9] Let D be a bounded, closed and convex subset of a Banach space such that $0\in D$, and let S be a continuous mapping of D into itself. If the implication*

$$V=\overline{conv}S(V)\quad or\quad V=S(V)\bigcup\{0\}\Rightarrow r(V)=0$$

holds for every subset V of D, then S has a fixed point.

Lemma 26 *[9] Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$, G a continuous function on $J\times J$ and f a function from $J\times E\to E$ which satisfies the Caratheodory conditions, and suppose there exists $p\in L^1(J,\mathbb{R}_+)$ such that, for each $t\in J$ and each bounded set $B\subset E$, we have*

$$\lim_{h\to 0^+}r\left(f(J_{t,h}\times B)\right)\leq p(t)r(B);$$

here $J_{t,h}=[t-h,t]\bigcap J$. If V is an equicontinuous subset of D, then

$$r\left(\left\{\int_J G(t,s)f(s,y(s))ds:y\in V\right\}\right)\leq\int_J\|G(t,s)\|\,p(s)r(V(s))ds.$$

3 Main results

First of all, we define what we mean by a solution of the problem (1).

Definition 31*A function $y\in AC^1(J,E)$ is said to be a solution of the problem (1) if y satisfies the equation $^{c}\mathscr{D}^{\alpha;\psi}y(t)=f(t,y(t))$ on J, and the conditions $y(0)=y_0$ and $y^{'}(0)=y_1$.*

Lemma 32*Let $1<\alpha<2$ and let $h:J\to E$ be continuous. A function y is said to be a solution of the fractional integral equation*

$$y(t)=y_0+y_1(\psi(t))+\frac{1}{\Gamma(\alpha)}\int_0^t\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}h(s)ds,\tag{2}$$

if and only if y is a solution of the ψ-type fractional initial value problem (1).

*Proof.*The problem (1) is reduced to an equivalent integral equation

$$\begin{aligned}y(t)&=\mathscr{I}^{\alpha;\psi}h(t)+c_0+c_1(\psi(t))\\&=c_0+c_1(\psi(t))+\frac{1}{\Gamma(\alpha)}\int_0^t\psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1}h(s)ds,\end{aligned}$$

for some constants $c_0,c_1\in E$. Initial conditions give

$$c_0=y_0,\quad c_1=y_1.$$

Hence, we get (2). Conversly, if y satisfies the equation (2), the equation (1) holds.

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Theorem 33 *Assume that conditions (H1)-(H3) hold. Let $p^* = \sup_{t\in J} p(t)$. If*

$$\frac{p^*(\psi(T))^\alpha}{\Gamma(\alpha+1)} < 1, \tag{3}$$

then the problem (1) has at least one solution.

Proof. Transform the problem (1) into a fixed point problem. Consider the operator $S : C(J,E) \to C(J,E)$ defined by

$$S(y)(t) = y_0 + y_1(\psi(t)) + \frac{1}{\Gamma(\alpha))}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f(s,y(s))ds.$$

Clearly, the fixed points of the operator S are solutions of the problem (1).
Let

$$r_0 \geq \frac{\|y_0\| + \|y_1\|(\psi(T))}{1 - \frac{p^*(\psi(T))^\alpha}{\Gamma(\alpha+1)}}, \tag{4}$$

and consider

$$D_{r_0} = \{y \in C(J,E) : \|y\|_\infty \leq r_0\}.$$

Clearly, the subset D_{r_0} is closed, bounded and convex. We shall show that S satisfies the hypotheses of Theorem 25. The proof will be split in three steps.

Claim 1. S is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,E)$. Then for each $t \in J$,

$$\begin{aligned}\|S(y_n)(t) - S(y)(t)\| &\leq \left\| \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}[f(s,y_n(s)) - f(s,y(s))]ds \right\| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|f(s,y_n(s)) - f(s,y(s))\|ds.\end{aligned}$$

Since f is of Caratheodory type, then by the Lebesgue dominated converegence theorem, we have

$$\|S(y_n) - S(y)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Claim 2. S maps D_{r_0} into itself. For each $y \in D_{r_0}$, by (H2) and (4), we have, for each $t \in J$,

$$\begin{aligned}\|S(y)(t)\| &\leq \|y_0 + y_1(\psi(t))\| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|f(s,y(s))\|ds \\ &\leq \|y_0\| + \|y_1\|(\psi(T)) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} p(s)\|y(s)\|ds \\ &\leq \|y_0\| + \|y_1\|(\psi(T)) + \frac{r_0}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} p(s)ds \\ &\leq \|y_0\| + \|y_1\|(\psi(T)) + \frac{r_0 p^*(\psi(T))^\alpha}{\Gamma(\alpha+1)} \\ &\leq r_0.\end{aligned}$$

Claim 3. $S(D_{r_0})$ is bounded and equicontinuous.

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By Claim 2, it is obvious that $S(D_{r_0}) \subset C(J,E)$ is bounded. For the equicontinuity of $S(D_{r_0})$, let $t_1,t_2 \in J$, $t_1 < t_2$ and $y \in D_{r_0}$. Then

$$\begin{aligned}
&\|S(y)(t_2) - S(y)(t_1)\| \\
&\leq \|y_1(\psi(t_2)) - y_1(\psi(t_1))\| \\
&+ \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_2} \psi^{'}(s)\left(\psi(t_2)-\psi(s)\right)^{\alpha-1} f(s,y(s)) - \int_0^{t_1} \psi^{'}(s)\left(\psi(t_1)-\psi(s)\right)^{\alpha-1} f(s,y(s)) \right\| ds \\
&\leq \|y_1\| \left((\psi(t_2)) - (\psi(t_1))\right) + \frac{r_0}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi^{'}(s)\left(\psi(t_2)-\psi(s)\right)^{\alpha-1} p(s)ds \\
&+ \frac{r_0}{\Gamma(\alpha)} \int_0^{t_1} \left[\psi^{'}(s)((\psi(t_2)-\psi(s))^{\alpha-1} - \psi^{'}(s)(\psi(t_1)-\psi(s))^{\alpha-1}\right] p(s)ds.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero.

Now let V be a subset of D_{r_0} such that $V \subset \overline{conv}\left(S(V) \bigcup \{0\}\right)$.

From Claim 3, the subset V is bounded and equicontinuous and therefore the function $v \to v(t) = r(V(t))$ is continuous on J. Since the function $t \to y_0 + y_1(\psi(t))$ is continuous on J, the set $\overline{\{y_0 + y_1(\psi(t)), t \in J\}} \subset E$ is compact. Using this fact, (H3), Lemma 26 and properties of the measure r, we have, for each $t \in J$,

$$\begin{aligned}
v(t) &\leq r\left(S(V)(t) \bigcup \{0\}\right) \\
&\leq r\left(S(V)(t)\right) \\
&\leq \frac{1}{\Gamma(\alpha))} \int_0^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} p(s) r\left(S(V)(s)\right) ds \\
&\leq \frac{1}{\Gamma(\alpha))} \int_0^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} p(s) v(s) ds \\
&\leq \|v\|_\infty \frac{1}{\Gamma(\alpha))} \int_0^t \psi^{'}(s)\left(\psi(t)-\psi(s)\right)^{\alpha-1} p(s) ds \\
&\leq \|v\|_\infty \frac{p^*(\psi(T))^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

This means that

$$\|v\|_\infty \leq \|v\|_\infty \frac{p^*(\psi(T))^\alpha}{\Gamma(\alpha+1)}.$$

By (3) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem (see [3]), V is relatively compact in D_{r_0}. Applying now Theorem 25, we conclude that S has a fixed point which is a solution of the problem (1).

4 Conclusion

In this paper, the necessary conditions have been established for the solution of the fractional differential equations with initial conditions. The concepts of fractional theory, semigroup, the Monch fixed point theorem, and measure of noncompactness have been effectively used in the existing fractional-orderndérivatives.

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Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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