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# The Effects of the Assorted Cross-Correlation Definitions

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**Abstract:** The literature offers varying and in general incompatible definitions of the *cross-correlation* function $\mathsf{R}_{xy}(n,m)$ and its *jointly wide-sense stationary* special case $\mathsf{R}_{xy}(m)$. The choice of definition has consequences for results involving the *cross-spectral density* function $\tilde{\mathsf{S}}_{xy}(\omega)$, and the more general Z-transform density $\check{\mathsf{S}}_{xy}(z)$. In some stochastic processing systems involving simple additive noise or even additive noise combined with non-linear operations, these varying definitions lead to identical results. In some other systems involving nonlinear and linear parallel operations, including those involving system identification problems, results differ.

## 1 Introduction

The literature offers varying and in general incompatible definitions of the *cross-correlation* function $\mathsf{R}_{xy}(n,m)$ and its *jointly wide-sense stationary* (Definition 6 page 20) special case $\mathsf{R}_{xy}(m)$. The choice of definition has consequences for results involving the *cross-spectral density* function $\tilde{\mathsf{S}}_{xy}(\omega)$, which is defined as the *Discrete-time Fourier Transform* of $\mathsf{R}_{xy}(m)$, as well as the more general $\check{\mathsf{S}}_{xy}(z)$, which is defined as the *Z-Transform* of $\mathsf{R}_{xy}(m)$:

**Definition 1.***[19, page 265], [12, page 52], [2, page 50], [4, page 118]*

$$\check{\mathsf{S}}_{xy}(z) \triangleq \underbrace{\mathrm{Z}\,\mathsf{R}_{xy}(m) \triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}_{xy}(m) z^{-m}}_{\text{Z-Transform } of\ \mathsf{R}_{xy}(m)} \qquad \tilde{\mathsf{S}}_{xy}(\omega) \triangleq \underbrace{\check{\mathrm{F}}\,\mathsf{R}_{xy}(m) \triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}_{xy}(m) e^{-i\omega m}}_{\text{Discrete-time Fourier Transform } of\ \mathsf{R}_{xy}(m)}$$

## 2 Definitions

Here is a very limited overview of definitions of $\mathsf{R}_{xy}(m)$ in the literature:

References that put the conjugate $*$ on y:

| | | | | |
|---|---|---|---|---|
| • Papoulis | [19, page 263] | $R_{xy}(m)$ | $\triangleq$ | $E\{\mathrm{x}(m)\,\mathrm{y}^*(0)\}$ |
| • Cadzow | [7, page 341] | $r_{xy}(m)$ | $\triangleq$ | $E[\mathrm{x}(m)\,\mathrm{y}^*(0)]$ |
| • MatLab | [16, 17] | $R_{xy}(m)$ | $\triangleq$ | $E\{x_{n+m}y_n^*\}$ |

References that put the conjugate $*$ on x:

| | | | | |
|---|---|---|---|---|
| • Kay | [12, page 52] | $r_{xy}[m]$ | $\triangleq$ | $\mathcal{E}\{x^*[0]y[m]\}$ |
| • Weisstein | [25, page 594][1] | $f \star g$ | $\triangleq$ | $\int_{-\infty}^{\infty} \bar{f}(\tau)g(t+\tau)\,\tau$ |

- Leuridan et al. [15, page 2](7) $GXY_1 \triangleq \sum X_1^* Y$

References that use no conjugate:

- Bendat and Piersol [4, page 111] $R_{xy}(m) \triangleq E[\mathrm{x}(0)\,\mathrm{y}(m)]$
- Helstrom [10, page 369] $\mathsf{R}_{\mathsf{xy}}(t_1, t_2) \triangleq E[\mathrm{x}(t_1)\,\mathrm{y}(t_2)]$
- Proakis and Manolakis [20, page A4] $\gamma_{xy}(t_1, t_2) \triangleq E(X_{t_1} Y_{t_2})$
- Shin and Hammond [22, page 280] $R_{xy}(\tau) \triangleq E[x(t)y(t+\tau)]$
- Bracewell [5, page 46][2] $g^* \star h \triangleq \int_{-\infty}^{\infty} g^*(u)h(u+x)\,\mathrm{u}$

In this paper, each *sequence* (Definition 7 page 20) mentioned hereafter is assumed to be an element of the *space of all absolutely square summable sequences* $\ell^2_{\mathbb{C}}$ (Definition 8 page 20). Furthermore, the random sequences $(\!|\mathrm{x}(n)|\!)_{n\in\mathbb{Z}}$ and $(\!|\mathrm{y}(n)|\!)_{n\in\mathbb{Z}}$ are assumed to be *jointly wide-sense stationary* (Definition 6 page 20).

In terms of the expectation operator $\mathrm{E}$ (Definition 4 page 19), there are a total of eight choices for defining the cross-correlation $\mathsf{R}_{\mathsf{xy}}(m)$ of *complex-valued jointly wide-sense stationary* (Definition 6 page 20) sequences $(\!|\mathrm{x}(n)|\!)_{n\in\mathbb{Z}}$ and $(\!|\mathrm{y}(n)|\!)_{n\in\mathbb{Z}}$. There are eight because each of the two sequences may be defined with or without the conjugate operator $*$, and one sequence may lead or lag the other ($2 \times 2 \times 2 = 8$). Definition 2 (next) provides a formalized list of the eight possible definitions.

**Definition 2.**

| | | | |
|---|---|---|---|
| (1). ***Papoulis:*** | $\mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}[\mathrm{x}\ (m)\mathrm{y}^*(0\ )]$ | (5). ***Bendat-Piersol:***[3] | $\mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}[\mathrm{x}\ (0)\ \mathrm{y}\ \ (m)]$ |
| (2). ***Kay:*** | $\mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}[\mathrm{x}^*(0)\ \mathrm{y}\ \ (m)]$ | (6). ***alt-BP:*** | $\mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}[\mathrm{x}\ (m)\mathrm{y}\ \ (0\ )]$ |
| (3). ***alt-Papoulis:*** | $\mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}[\mathrm{x}\ (0)\ \mathrm{y}^*(m)]$ | (7). ***BP-star:*** | $\mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}[\mathrm{x}^*(0)\ \mathrm{y}^*(m)]$ |
| (4). ***alt-Kay:*** | $\mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}[\mathrm{x}^*(m)\mathrm{y}\ \ (0\ )]$ | (8). ***alt-BP-star:*** | $\mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}[\mathrm{x}^*(m)\mathrm{y}^*(0\ )]$ |

## 3 Results

*Remark.*

The 8 definitions of $\mathsf{R}_{\mathsf{xy}}(m)$ listed in Definition 2 yield ...

- 2 relations on the pair $\left(\mathsf{R}_{\mathsf{yx}}(m), \mathsf{R}_{\mathsf{xy}}(m)\right)$ (Lemma 1 page 3)
- 2 relations on the pair $\left(\check{\mathsf{S}}_{\mathsf{xy}}(z),\ \check{\mathsf{S}}_{\mathsf{yx}}(z)\right)$ (Proposition 1 page 3)
- 4 relations on the triple $\left(\check{\mathsf{S}}_{\mathsf{xy}}(z),\ \check{\mathrm{H}}(z),\ \check{\mathsf{S}}_{\mathsf{xx}}(z)\right)$ (Proposition 2 page 4)
- 3 relations on the triple $\left(\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega),\ \tilde{\mathsf{H}}(\omega),\ \tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)\right)$ (Corollary 2 page 7)
- only 4 cases in which $\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)$ is guarenteed to be *real-valued* (Corollary 1 page 4)

*Remark.*

Moreover, if $(\!|\mathrm{x}(n)|\!)$ and $(\!|\mathrm{y}(n)|\!)$ are **real-valued**, the 8 definitions of $\mathsf{R}_{\mathsf{xy}}(m)$ yield ...

- 1 relation on $\left(\mathsf{R}_{\mathsf{xy}}(m), \mathsf{R}_{\mathsf{yx}}(m)\right)$ (Corollary 3 page 8)
- 1 relation on $\left(\check{\mathsf{S}}_{\mathsf{xy}}(z),\ \check{\mathsf{S}}_{\mathsf{yx}}(z)\right)$ (Proposition 3 page 8)
- 2 relations on $\left(\check{\mathsf{S}}_{\mathsf{xy}}(z),\ \check{\mathrm{H}}(z),\ \check{\mathsf{S}}_{\mathsf{xx}}(z)\right)$ (Proposition 4 page 9)
- 2 relations on $\left(\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega),\ \tilde{\mathsf{H}}(\omega),\ \tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)\right)$ (Corollary 4 page 10)
- 1 relation on $\left(\check{\mathsf{S}}_{\mathsf{yy}}(z),\ \check{\mathrm{H}}(z),\ \check{\mathsf{S}}_{\mathsf{xx}}(z)\right)$ (Proposition 4 page 9)
- 1 relation on $\left(\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega),\ \tilde{\mathsf{H}}(\omega),\ \tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)\right)$ (Corollary 4 page 10)

[1] Bracewell and Weisstein here use the *integral operator* $\int_{\mathbb{R}} \mathrm{x}$ rather than the *expectation operator* $\mathrm{E}$. That is, they use a *time average* rather than an *ensemble average*. But in essence, the two types of operators are "the same" because both types represent *inner products*. That is, $\int_{x\in\mathbb{R}} \mathrm{f}(x)\,\mathrm{g}^*(x)\,\mathrm{x} \triangleq \langle \mathrm{f}(x)\,|\,\mathrm{g}(x)\rangle_1$ and $\mathrm{E}\left[\mathrm{x}(t)\,\mathrm{y}^*(t)\right] \triangleq \langle \mathrm{x}(t)\,|\,\mathrm{y}(t)\rangle_2$ (both are inner products, but operate in perpendicular orientations across the ensemble plane).

[2] Note that Bracewell's "*Pentagram notation for cross correlation*" $g^* \star h = \int_{-\infty}^{\infty} g^*(u)h(u+x)\,\mathrm{u}$ implies $g \star h = \int_{-\infty}^{\infty} g(u)h(u+x)\,\mathrm{u}$ (and hence in the "References that use no conjugate" category).

[3] Note that Bendat and Piersol are well known and highly cited for their work related to random vibration testing. In this field, data samples are customarily collected using an analog-to-digital converter (ADC) and as such, for this application (in contrast to wireless communication applications involving phase discriminating PSK or QAM), are customarily *real-valued*. Therefore, it is very understandable that these authors would define $\mathsf{R}_{\mathsf{xy}}(m)$ *without* any conjugate operator.

**Lemma 1.** *Let (1)–(8) below correspond to the eight definitions of* $\mathsf{R}_{xy}(m)$ *in Definition 2.*

$$\begin{array}{ll}
(1),\ (2),\ (3),\ \text{or}\ (4) \implies \mathsf{R}_{yx}(m) = \mathsf{R}_{xy}{}^*(-m) \text{ and } \mathsf{R}_{xx}(-m) = \mathsf{R}_{xx}{}^*(m) & \text{(conjugate symmetric)}\\
(5),\ (6),\ (7),\ \text{or}\ (8) \implies \mathsf{R}_{yx}(m) = \mathsf{R}_{xy}\ (-m) \text{ and } \mathsf{R}_{xx}(-m) = \mathsf{R}_{xx}\ (m) & \text{(symmetric)}
\end{array}$$

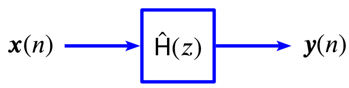
*Proof.*

$$\begin{array}{llll}
(1). & \mathsf{R}_{yx}(m) \triangleq \mathrm{E}\left[\mathrm{y}(m)\,\mathrm{x}^*(0)\right] & \text{by Papoulis' definition of } \mathsf{R}_{xy}(m) & \text{(Definition 2 page 2)}\\
 & = \left(\mathrm{E}\left[\mathrm{x}(0)\,\mathrm{y}^*(m)\right]\right)^* & \text{by \textit{antiautomorphic} property of *-algebras} & \text{(Definition 11 page 24)}\\
 & = \left(\mathrm{E}\left[\mathrm{x}(0-m)\,\mathrm{y}^*(m-m)\right]\right)^* & \text{by \textit{wide sense stationary} property} & \\
 & \triangleq \mathsf{R}_{xy}{}^*(-m) & \text{by Papoulis' definition of } \mathsf{R}_{xy}(m) & \text{(Definition 2 page 2)}
\end{array}$$

$$\mathsf{R}_{xx}(-m) \triangleq \mathsf{R}_{xy}(-m)\big|_{y=x} = \mathsf{R}_{yx}{}^*(m)\big|_{y=x} = \mathsf{R}_{xx}{}^*(m)$$

$$\begin{array}{lllll}
(2). & \mathsf{R}_{yx}(m) \triangleq \mathrm{E}\left[\mathrm{y}^*(0)\,\mathrm{x}(m)\right] & = \left(\mathrm{E}\left[\mathrm{x}^*(m)\,\mathrm{y}(0)\right]\right)^* & = \left(\mathrm{E}\left[\mathrm{x}^*(m-m)\,\mathrm{y}(0-m)\right]\right)^* & \triangleq \mathsf{R}_{xy}{}^*(-m)\\
(3). & \mathsf{R}_{yx}(m) \triangleq \mathrm{E}\left[\mathrm{y}(0)\,\mathrm{x}^*(m)\right] & = \left(\mathrm{E}\left[\mathrm{x}(m)\,\mathrm{y}^*(0)\right]\right)^* & = \left(\mathrm{E}\left[\mathrm{x}(m-m)\,\mathrm{y}^*(0-m)\right]\right)^* & \triangleq \mathsf{R}_{xy}{}^*(-m)\\
(4). & \mathsf{R}_{yx}(m) \triangleq \mathrm{E}\left[\mathrm{y}^*(m)\,\mathrm{x}(0)\right] & = \left(\mathrm{E}\left[\mathrm{x}^*(0)\,\mathrm{y}(m)\right]\right)^* & = \left(\mathrm{E}\left[\mathrm{x}^*(-m)\,\mathrm{y}(0)\right]\right)^* & \triangleq \mathsf{R}_{xy}{}^*(-m)\\
(5). & \mathsf{R}_{yx}(m) \triangleq \mathrm{E}\left[\mathrm{y}(0)\,\mathrm{x}(m)\right] & = \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{y}(0)\right] & = \mathrm{E}\left[\mathrm{x}(m-m)\,\mathrm{y}(0-m)\right] & \triangleq \mathsf{R}_{xy}(-m)\\
(6). & \mathsf{R}_{yx}(m) \triangleq \mathrm{E}\left[\mathrm{y}(m)\,\mathrm{x}(0)\right] & = \mathrm{E}\left[\mathrm{x}(0)\,\mathrm{y}(m)\right] & = \mathrm{E}\left[\mathrm{x}(0-m)\,\mathrm{y}(m-m)\right] & \triangleq \mathsf{R}_{xy}(-m)\\
(7). & \mathsf{R}_{yx}(m) \triangleq \mathrm{E}\left[\mathrm{y}^*(0)\,\mathrm{x}^*(m)\right] & = \mathrm{E}\left[\mathrm{x}^*(m)\,\mathrm{y}^*(0)\right] & = \mathrm{E}\left[\mathrm{x}^*(m-m)\,\mathrm{y}^*(0-m)\right] & \triangleq \mathsf{R}_{xy}(-m)\\
(8). & \mathsf{R}_{yx}(m) \triangleq \mathrm{E}\left[\mathrm{y}^*(m)\,\mathrm{x}^*(0)\right] & = \mathrm{E}\left[\mathrm{x}^*(0)\,\mathrm{y}^*(m)\right] & = \mathrm{E}\left[\mathrm{x}^*(0-m)\,\mathrm{y}^*(m-m)\right] & \triangleq \mathsf{R}_{xy}(-m)
\end{array}$$

**Proposition 1.** *Let (1)–(8) correspond to the eight definitions of* $\mathsf{R}_{xy}(m)$ *in Definition 2. Let* $\mathbf{H}$ *be a* linear time-invariant *(*LTI*) operator with* impulse response $(\!|\mathrm{h}(n)|\!)$ *on a* wide-sense stationary *sequence* $(\!|\mathrm{x}(n)|\!)$ *yielding a sequence* $(\!|\mathrm{y}(n)|\!) \triangleq (\!|\mathbf{H}\,\mathrm{x}(n)|\!)$*. Let* $\check{\mathrm{H}}(z)$ *be the* Z-Transform *of* $(\!|\mathrm{h}(n)|\!)$*.*




$$\begin{array}{l}
(1),\ (2),\ (3),\ \text{or}\ (4) \implies \check{\mathsf{S}}_{yx}(z) = \check{\mathsf{S}}_{xy}{}^*\left(\frac{1}{z^*}\right) \text{ and } \check{\mathsf{S}}_{xx}(z) = \check{\mathsf{S}}_{xx}{}^*\left(\frac{1}{z^*}\right)\\
(5),\ (6),\ (7),\ \text{or}\ (8) \implies \check{\mathsf{S}}_{yx}(z) = \check{\mathsf{S}}_{xy}\ \left(\frac{1}{z}\right) \text{ and } \check{\mathsf{S}}_{xx}(z) = \check{\mathsf{S}}_{xx}\ \left(\frac{1}{z}\right)
\end{array}$$

*Proof.*

$$\begin{array}{llll}
(1)-(4): & \check{\mathsf{S}}_{yx}(z) \triangleq \mathbf{Z}\,\mathsf{R}_{yx}(m) & \text{by definition of } \check{\mathsf{S}}_{xy}(z) & \\
 & \triangleq \displaystyle\sum_{m\in\mathbb{Z}} \mathsf{R}_{yx}(m) z^{-m} & \text{by definition of } \mathbf{Z} & \text{(Definition 10 page 21)}\\
 & = \displaystyle\sum_{m\in\mathbb{Z}} \mathsf{R}_{xy}{}^*(-m) z^{-m} & \text{by \textit{conjugate symmetry} property} & \text{(Lemma 1 page 3)}\\
 & = \left[\displaystyle\sum_{m\in\mathbb{Z}} \mathsf{R}_{xy}(-m)(z^*)^{-m}\right]^* & \text{by \textit{antiautomorphic} property of *-algebras} & \text{(Definition 11 page 24)}\\
 & = \left[\displaystyle\sum_{-p\in\mathbb{Z}} \mathsf{R}_{xy}(p)(z^*)^{p}\right]^* & \text{where } p \triangleq -m & \implies m = -p\\
 & = \left[\displaystyle\sum_{p\in\mathbb{Z}} \mathsf{R}_{xy}(p)(z^*)^{p}\right]^* & \text{because } (\!|\mathrm{x}(n)|\!), (\!|\mathrm{y}(n)|\!) \in \ell^2_{\mathbb{C}} & \text{(Definition 8 page 20)}\\
 & = \left[\displaystyle\sum_{p\in\mathbb{Z}} \mathsf{R}_{xy}(p)\left(\frac{1}{z^*}\right)^{-p}\right]^* & & \\
 & \triangleq \check{\mathsf{S}}_{xy}{}^*\left(\frac{1}{z^*}\right) & \text{by definition of } \check{\mathsf{S}}_{xy}(z) & \text{(Definition 1 page 1)}
\end{array}$$

$$
\begin{aligned}
&(1)-(4): \check{\mathsf{S}}_{\mathsf{xx}}{}^{*}(z) \triangleq \left[\check{\mathsf{S}}_{\mathsf{yx}}(z)\right]^{*}_{\mathrm{y=x}} && = \left[\check{\mathsf{S}}_{\mathsf{xy}}{}^{*}\left(\frac{1}{z^*}\right)\right]^{*}_{\mathrm{y=x}} && = \left[\check{\mathsf{S}}_{\mathsf{xx}}{}^{*}\left(\frac{1}{z^*}\right)\right]^{*} && = \check{\mathsf{S}}_{\mathsf{xx}}\left(\frac{1}{z^*}\right)\\
&(5)-(8): \check{\mathsf{S}}_{\mathsf{yx}}(z) = \sum_{m\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(-m)z^{-m} && = \sum_{-p\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(p)z^{p} && = \sum_{p\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(p)\left(\frac{1}{z}\right)^{-p} && \triangleq \check{\mathsf{S}}_{\mathsf{xy}}\left(\frac{1}{z}\right)\\
&(5)-(8): \check{\mathsf{S}}_{\mathsf{xx}}(z) = \check{\mathsf{S}}_{\mathsf{yx}}(z)\big|_{\mathrm{y=x}} && = \check{\mathsf{S}}_{\mathsf{xy}}\left(\frac{1}{z}\right)\bigg|_{\mathrm{y=x}} && = \check{\mathsf{S}}_{\mathsf{xx}}\left(\frac{1}{z}\right)\bigg| && = \check{\mathsf{S}}_{\mathsf{xx}}\left(\frac{1}{z}\right)
\end{aligned}
$$

**Corollary 1.** *Let* $(\!|(1),(2),\cdots(8)|\!)$, H, $(\!|\mathrm{h}(n)|\!)$, $(\!|\mathrm{x}(n)|\!)$, *and* $(\!|\mathrm{y}(n)|\!)$ *be defined as in Proposition 1. Let* $\tilde{\mathsf{H}}(\omega)$ *be the* DTFT *(Definition 1 page 1) of* $(\!|\mathrm{h}(n)|\!)$.

$$
\begin{aligned}
&\{(1),\ (2),\ (3),\ \text{or}\ (4)\} \implies \{\tilde{\mathsf{S}}_{\mathsf{xx}}{}^{*}(\omega) = \tilde{\mathsf{S}}_{\mathsf{xx}}(\omega) \quad (\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)\ \textit{is}\ \text{real-valued})\ \}\\
&\{(1),\ (2),\ (3),\ \text{or}\ (4)\} \implies \{\tilde{\mathsf{S}}_{\mathsf{yx}}(\omega) = \tilde{\mathsf{S}}_{\mathsf{xy}}{}^{*}(\omega)\\
&\{(5),\ (6),\ (7),\ \text{or}\ (8)\} \implies \{\tilde{\mathsf{S}}_{\mathsf{yx}}(\omega) = \tilde{\mathsf{S}}_{\mathsf{xy}}(-\omega)
\end{aligned}
$$

*Proof.*

$$
\begin{aligned}
&(1)\text{–}(4)\quad \tilde{\mathsf{S}}_{\mathsf{xx}}{}^{*}(\omega) = \check{\mathsf{S}}_{\mathsf{xx}}{}^{*}(z)\Big|_{z=e^{i\omega}} && = \check{\mathsf{S}}_{\mathsf{xx}}{}^{*}\left(\frac{1}{z^*}\right)\bigg|_{z=e^{i\omega}} && \overbrace{= \check{\mathsf{S}}_{\mathsf{xx}}(z)\big|_{z=e^{i\omega}}}^{\text{by Proposition 1}} && = \tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)\\
&(1)\text{–}(4)\quad \tilde{\mathsf{S}}_{\mathsf{yx}}(\omega) = \check{\mathsf{S}}_{\mathsf{yx}}(z)\big|_{z=e^{i\omega}} && = \check{\mathsf{S}}_{\mathsf{xy}}{}^{*}\left(\frac{1}{z^*}\right)\bigg|_{z=e^{i\omega}} && = \check{\mathsf{S}}_{\mathsf{xy}}{}^{*}\left(e^{i\omega}\right) && = \tilde{\mathsf{S}}_{\mathsf{xy}}{}^{*}(\omega)\\
&(5)\text{–}(8)\quad \tilde{\mathsf{S}}_{\mathsf{yx}}(\omega) = \check{\mathsf{S}}_{\mathsf{yx}}(z)\big|_{z=e^{i\omega}} && \underbrace{= \check{\mathsf{S}}_{\mathsf{xy}}\left(\frac{1}{z}\right)\bigg|_{z=e^{i\omega}}}_{\text{by Proposition 1}} && = \check{\mathsf{S}}_{\mathsf{xy}}\left(e^{-i\omega}\right) && = \tilde{\mathsf{S}}_{\mathsf{xy}}(-\omega)
\end{aligned}
$$

**Proposition 2.** *Let (1)–(8) below correspond to the eight definitions of* $\mathsf{R}_{\mathsf{xy}}(m)$ *in Definition 2.*

$$
\begin{aligned}
&(1) \implies \check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}^{*}\left(\tfrac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\ \textit{and}\ \check{\mathsf{S}}_{\mathsf{yy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathrm{H}}^{*}\left(\tfrac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\\
&(2) \implies \check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xx}}(z)\ \textit{and}\ \check{\mathsf{S}}_{\mathsf{yy}}(z) = \check{\mathrm{H}}^{*}\left(\tfrac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathrm{H}}^{*}\left(\tfrac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\\
&(3) \implies \check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}^{*}(z^*)\,\check{\mathsf{S}}_{\mathsf{xx}}(z)\ \textit{and}\ \check{\mathsf{S}}_{\mathsf{yy}}(z) = \check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}^{*}(z^*)\check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\\
&(4) \implies \check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\ \textit{and}\ \check{\mathsf{S}}_{\mathsf{yy}}(z) = \check{\mathrm{H}}^{*}(z^*)\check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}^{*}(z^*)\check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\\
&(5) \implies \check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xx}}(z)\ \textit{and}\ \check{\mathsf{S}}_{\mathsf{yy}}(z) = \check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\\
&(6) \implies \check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\ \textit{and}\ \check{\mathsf{S}}_{\mathsf{yy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\\
&(7) \implies \check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}^{*}(z^*)\,\check{\mathsf{S}}_{\mathsf{xx}}(z)\ \textit{and}\ \check{\mathsf{S}}_{\mathsf{yy}}(z) = \check{\mathrm{H}}^{*}\left(\tfrac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}^{*}(z^*)\check{\mathrm{H}}^{*}\left(\tfrac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\\
&(8) \implies \check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\ \textit{and}\ \check{\mathsf{S}}_{\mathsf{yy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)
\end{aligned}
$$

*Proof.*

(1).

$$\begin{aligned}
\check{S}_{yt}(z) &\triangleq Z\,R_{yt}(m) && \text{by definition of } \check{S}_{ty}(z) && \text{(Definition 1 page 1)}\\
&\triangleq Z\,E\,[y(m)\,t^*(0)] && \text{by Papoulis' definition of } R_{ty}(m) && \text{(Definition 2 page 2)}\\
&= Z\,E\left[\left(\sum_{k\in\mathbb{Z}} h(k)\,x(m-k)\right)t^*(0)\right] && \text{by \textit{linear time-invariant} property of H} &&\\
&= Z\sum_{k\in\mathbb{Z}} h(k)\,E\,[x(m-k)\,t^*(0)] && \text{by \textit{linearity} of E} && \text{(Proposition 11 page 20)}\\
&\triangleq Z\sum_{k\in\mathbb{Z}} h(k)\,R_{xt}(m-k) && \text{by Papoulis' definition of } R_{ty}(m) && \text{(Definition 2 page 2)}\\
&\triangleq Z\,[h(m)\star R_{xt}(m)] && \text{by definition of \textit{convolution}} && \text{(Definition 9 page 20)}\\
&= [Z\,h(m)][Z\,R_{xt}(m)] && \text{by \textit{convolution theorem}} && \text{(Proposition 13 page 23)}\\
&\triangleq \check{H}(z)\,\check{S}_{xt}(z) && \text{by definitions of } \check{H}(z) \text{ and } \check{S}_{xt}(z) && \text{(Definition 1 page 1)}
\end{aligned}$$

$$\boxed{\check{S}_{xy}(z)} = \check{S}_{yx}^{\;*}\left(\frac{1}{z^*}\right) \quad \text{(by Proposition 1)} \quad \triangleq \check{S}_{yt}^{\;*}\left(\frac{1}{z^*}\right)\Bigg|_{t\triangleq x} = \check{H}^*\left(\frac{1}{z^*}\right)\check{S}_{xx}^{\;*}\left(\frac{1}{z^*}\right) = \boxed{\check{H}^*\left(\frac{1}{z^*}\right)\check{S}_{xx}(z)}$$

$$\boxed{\check{S}_{yy}(z)} \triangleq \check{S}_{yt}(z)\big|_{t\triangleq y} = \check{H}(z)\,\check{S}_{xt}(z)\big|_{t\triangleq y} = \boxed{\check{H}(z)\,\check{S}_{xy}(z)} = \boxed{\check{H}(z)\,\check{H}^*\left(\frac{1}{z^*}\right)\check{S}_{xx}(z)}$$

(2).

$$\begin{aligned}
\check{S}_{ty}(z) &\triangleq Z\,R_{ty}(m) \triangleq Z\,E\,[t^*(0)\,y(m)] = Z\,E\,[t^*(0)(h(m)\star x(m))]\\
&= Z\,E\left[t^*(0)\left(\sum_{k\in\mathbb{Z}} h(k)\,x(m-k)\right)\right] = Z\left(\sum_{k\in\mathbb{Z}} h(k)\,E\,[t^*(0)\,x(m-k)]\right) \triangleq Z\left(\sum_{k\in\mathbb{Z}} h(k)\,R_{tx}(m-k)\right)\\
&\triangleq Z\,(h(m)\star R_{tx}(m)) = [Z\,h(m)][Z\,R_{tx}(m)] \triangleq \check{H}(z)\,\check{S}_{tx}(z)
\end{aligned}$$

$$\boxed{\check{S}_{xy}(z)} \triangleq \check{S}_{ty}\big|_{t=x} = \check{H}(z)\,\check{S}_{tx}(z)\big|_{t=x} = \boxed{\check{H}(z)\,\check{S}_{xx}(z)}$$

$$\begin{aligned}
\boxed{\check{S}_{yy}(z)} &\triangleq \check{S}_{ty}(z)\big|_{t\triangleq y} = \check{S}_{yt}^{\;*}\left(\frac{1}{z^*}\right)\Bigg|_{t\triangleq y} = \check{H}(z)\,\check{S}_{tx}(z)\big|_{t\triangleq y} = \check{H}(z)\,\check{S}_{yx}(z) = \boxed{\check{H}(z)\,\check{S}_{xy}^{\;*}\left(\frac{1}{z^*}\right)}\\
&= \check{H}(z)\,\check{H}^*\left(\frac{1}{z^*}\right)\check{S}_{xx}^{\;*}\left(\frac{1}{z^*}\right) = \boxed{\check{H}(z)\,\check{H}^*\left(\frac{1}{z^*}\right)\check{S}_{xx}(z)} \quad \text{by Proposition 1}
\end{aligned}$$

(3).

$$\begin{aligned}
\check{S}_{ty}(z) &\triangleq Z\,R_{ty}(m) \triangleq Z\,E\,[t(0)\,y^*(m)] = Z\,E\left(t(0)\left[\sum_{k\in\mathbb{Z}} h(k)\,x(m-k)\right]^*\right)\\
&= Z\,E\left[t(0)\sum_{k\in\mathbb{Z}} h^*(k)\,x^*(m-k)\right] = Z\sum_{k\in\mathbb{Z}} h^*(k)\,E\,[t(0)\,x^*(m-k)] \triangleq Z\sum_{k\in\mathbb{Z}} h^*(k)\,R_{tx}(m-k)\\
&\triangleq Z\,[h^*(m)\star R_{tx}(m)] = [Z\,h^*(m)][Z\,R_{tx}(m)] = \check{H}^*(z^*)\,\check{S}_{tx}(z) \quad \text{by Proposition 12 page 21}
\end{aligned}$$

$$\boxed{\check{S}_{xy}(z)} \triangleq \check{S}_{ty}(z)\big|_{t=x} = \check{H}^*(z^*)\,\check{S}_{tx}(z)\big|_{t=x} = \boxed{\check{H}^*(z^*)\,\check{S}_{xx}(z)}$$

$$\begin{aligned}
\boxed{\check{S}_{yy}(z)} &\triangleq \check{S}_{ty}(z)\big|_{t=y} = \check{H}^*(z^*)\,\check{S}_{tx}(z)\big|_{t=y} = \check{H}^*(z^*)\,\check{S}_{yx}(z) = \boxed{\check{H}^*(z^*)\,\check{S}_{xy}^{\;*}\left(\frac{1}{z^*}\right)}\\
&= \check{H}^*(z^*)\,\check{H}\left(\frac{1}{z}\right)\check{S}_{xx}^{\;*}\left(\frac{1}{z^*}\right) = \boxed{\check{H}^*(z^*)\,\check{H}\left(\frac{1}{z}\right)\check{S}_{xx}(z)}
\end{aligned}$$

(4).
$$\check{\mathsf{S}}_{\mathsf{yt}}(z) \triangleq \mathrm{Z}\,\mathsf{R}_{\mathsf{yt}}(m) \triangleq \mathrm{Z}\,\mathrm{E}\,[\mathrm{y}^*(m)\,\mathrm{t}(0)] = \mathrm{Z}\,\mathrm{E}\left[\left[\sum_{k\in\mathbb{Z}} \mathrm{h}(k)\,\mathrm{x}(m-k)\right]^*\mathrm{t}(0)\right]$$
$$= \mathrm{Z}\sum_{k\in\mathbb{Z}} \mathrm{h}^*(k)\,\mathrm{E}\,[\mathrm{x}^*(m-k)\,\mathrm{t}(0)] = \mathrm{Z}\sum_{k\in\mathbb{Z}} \mathrm{h}^*(k)\,\mathsf{R}_{\mathsf{xt}}(m-k) \triangleq \mathrm{Z}\,[\mathrm{h}^*(m)\star\mathsf{R}_{\mathsf{xt}}(m)]$$
$$= [\mathrm{Z}\,\mathrm{h}^*(m)][\mathrm{Z}\,\mathsf{R}_{\mathsf{xt}}(m)] = \check{\mathrm{H}}^*(z^*)\,\check{\mathsf{S}}_{\mathsf{xt}}(z) \quad \text{by Proposition 12 page 21}$$
$$\boxed{\check{\mathsf{S}}_{\mathsf{xy}}(z)} = \check{\mathsf{S}}_{\mathsf{yx}}{}^*\left(\frac{1}{z^*}\right) \triangleq \check{\mathsf{S}}_{\mathsf{yt}}{}^*\left(\frac{1}{z^*}\right)\bigg|_{\mathrm{t=x}} = \check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xt}}{}^*\left(\frac{1}{z^*}\right)\bigg|_{\mathrm{t=x}}$$
$$\triangleq \check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}{}^*\left(\frac{1}{z^*}\right) = \boxed{\check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)} \quad \text{by Proposition 1}$$
$$\boxed{\check{\mathsf{S}}_{\mathsf{yy}}(z)} \triangleq \check{\mathsf{S}}_{\mathsf{yt}}(z)\big|_{\mathrm{t=y}} = \check{\mathrm{H}}^*(z^*)\,\check{\mathsf{S}}_{\mathsf{xt}}(z)\big|_{\mathrm{t=y}} = \boxed{\check{\mathrm{H}}^*(z^*)\,\check{\mathsf{S}}_{\mathsf{xy}}(z)} = \boxed{\check{\mathrm{H}}^*(z^*)\,\check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)}$$

(5).
$$\check{\mathsf{S}}_{\mathsf{ty}}(z) \triangleq \mathrm{Z}\,\mathsf{R}_{\mathsf{ty}}(m) \triangleq \mathrm{Z}\,\mathrm{E}\,[\mathrm{t}(0)\,\mathrm{y}(m)]$$
$$= \mathrm{Z}\,\mathrm{E}\left[\mathrm{t}(0)\left(\sum_{k\in\mathbb{Z}} \mathrm{h}(k)\,\mathrm{x}(m-k)\right)\right] = \mathrm{Z}\sum_{k\in\mathbb{Z}} \mathrm{h}(k)\,\mathrm{E}\,[\mathrm{t}(0)\,\mathrm{x}(m-k)] \triangleq \mathrm{Z}\sum_{k\in\mathbb{Z}} \mathrm{h}(k)\,\mathsf{R}_{\mathsf{tx}}(m-k)$$
$$\triangleq \mathrm{Z}\,[\mathrm{h}(m)\star\mathsf{R}_{\mathsf{tx}}(m)] \quad [\mathrm{Z}\,\mathrm{h}(m)][\mathrm{Z}\,\mathsf{R}_{\mathsf{tx}}(m)] = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{tx}}(z)$$
$$\boxed{\check{\mathsf{S}}_{\mathsf{xy}}(z)} \triangleq \check{\mathsf{S}}_{\mathsf{ty}}(z)\big|_{\mathrm{t=x}} = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{tx}}(z)\big|_{\mathrm{t=x}} = \boxed{\check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xx}}(z)}$$
$$\boxed{\check{\mathsf{S}}_{\mathsf{yy}}(z)} = \check{\mathsf{S}}_{\mathsf{yy}}\left(\frac{1}{z}\right) \triangleq \check{\mathsf{S}}_{\mathsf{ty}}\left(\frac{1}{z}\right)\bigg|_{\mathrm{t=y}} = \check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{tx}}\left(\frac{1}{z}\right)\bigg|_{\mathrm{t=y}} = \check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{yx}}\left(\frac{1}{z}\right)$$
$$= \boxed{\check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xy}}(z)} = \check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xx}}(z) = \boxed{\check{\mathrm{H}}(z)\,\check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)}$$

(6).
$$\check{\mathsf{S}}_{\mathsf{yt}}(z) \triangleq \mathrm{Z}\,\mathsf{R}_{\mathsf{yt}}(m) \triangleq \mathrm{Z}\,\mathrm{E}\,[\mathrm{y}(m)\,\mathrm{t}(0)] = \mathrm{Z}\,\mathrm{E}\left[\left(\sum_{k\in\mathbb{Z}} \mathrm{h}(k)\,\mathrm{x}(m-k)\right)\mathrm{t}(0)\right]$$
$$= \mathrm{Z}\sum_{k\in\mathbb{Z}} \mathrm{h}(k)\,\mathrm{E}\,[\mathrm{x}(m-k)\,\mathrm{t}(0)] \triangleq \mathrm{Z}\sum_{k\in\mathbb{Z}} \mathrm{h}(k)\,\mathsf{R}_{\mathsf{xt}}(m-k)$$
$$\triangleq \mathrm{Z}\,[\mathrm{h}(m)\star\mathsf{R}_{\mathsf{xt}}(m)] = \mathrm{Z}\,[\mathrm{h}(m)][\mathrm{Z}\,\mathsf{R}_{\mathsf{xt}}(m)] = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xt}}(z)$$
$$\boxed{\check{\mathsf{S}}_{\mathsf{xy}}(z)} = \check{\mathsf{S}}_{\mathsf{yx}}\left(\frac{1}{z}\right) \triangleq \check{\mathsf{S}}_{\mathsf{yt}}\left(\frac{1}{z}\right)\bigg|_{\mathrm{t=x}} = \check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xt}}\left(\frac{1}{z}\right)\bigg|_{\mathrm{t=x}} = \check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}\left(\frac{1}{z}\right) = \boxed{\check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)}$$
$$\boxed{\check{\mathsf{S}}_{\mathsf{yy}}(z)} \triangleq \check{\mathsf{S}}_{\mathsf{yt}}(z)\big|_{\mathrm{t=y}} = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xt}}(z)\big|_{\mathrm{t=y}} = \boxed{\check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xy}}(z)} = \boxed{\check{\mathrm{H}}(z)\,\check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)}$$

(7).
$$\check{\mathsf{S}}_{\mathsf{ty}}(z) \triangleq \mathrm{Z}\,\mathsf{R}_{\mathsf{ty}}(m) \triangleq \mathrm{Z}\,\mathrm{E}\,[\mathrm{t}^*(0)\,\mathrm{y}^*(m)] = \mathrm{Z}\,\mathrm{E}\left[\mathrm{t}^*(0)\left(\sum_{k\in\mathbb{Z}} \mathrm{h}(k)\,\mathrm{x}(m-k)\right)^*\right]$$
$$= \mathrm{Z}\sum_{k\in\mathbb{Z}} \mathrm{h}^*(k)\,\mathrm{E}\,[\mathrm{t}^*(0)\,\mathrm{x}^*(m-k)] \triangleq \mathrm{Z}\sum_{k\in\mathbb{Z}} \mathrm{h}^*(k)\,\mathsf{R}_{\mathsf{tx}}(m-k)$$
$$\triangleq \mathrm{Z}\,[\mathrm{h}^*(m)\star\mathsf{R}_{\mathsf{tx}}(m)] = [\mathrm{Z}\,\mathrm{h}^*(m)][\mathrm{Z}\,\mathsf{R}_{\mathsf{tx}}(m)] = \check{\mathrm{H}}^*(z^*)\,\check{\mathsf{S}}_{\mathsf{tx}}(z)$$
$$\boxed{\check{\mathsf{S}}_{\mathsf{xy}}(z)} \triangleq \check{\mathsf{S}}_{\mathsf{ty}}(z)\big|_{\mathrm{t=x}} = \check{\mathrm{H}}^*(z^*)\,\check{\mathsf{S}}_{\mathsf{tx}}(z)\big|_{\mathrm{t=x}} = \boxed{\check{\mathrm{H}}^*(z^*)\,\check{\mathsf{S}}_{\mathsf{xx}}(z)}$$
$$\boxed{\check{\mathsf{S}}_{\mathsf{yy}}(z)} = \check{\mathsf{S}}_{\mathsf{yy}}\left(\frac{1}{z}\right) \triangleq \check{\mathsf{S}}_{\mathsf{ty}}\left(\frac{1}{z}\right)\bigg|_{\mathrm{t=y}} = \check{\mathrm{H}}^*\left(\frac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{tx}}\left(\frac{1}{z}\right)\bigg|_{\mathrm{t=y}} = \check{\mathrm{H}}^*\left(\frac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{yx}}\left(\frac{1}{z}\right)$$
$$= \boxed{\check{\mathrm{H}}^*\left(\frac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{xy}}(z)} = \check{\mathrm{H}}^*\left(\frac{1}{z^*}\right)\check{\mathrm{H}}^*(z^*)\,\check{\mathsf{S}}_{\mathsf{xx}}(z) = \boxed{\check{\mathrm{H}}^*(z^*)\,\check{\mathrm{H}}^*\left(\frac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)}$$

$$
\begin{aligned}
(8).\quad \check{\mathsf{S}}_{yt}(z) &\triangleq \mathrm{Z}\,\mathsf{R}_{yt}(m) &&\triangleq \mathrm{Z}\,\mathrm{E}\left[y^*(m)\,t^*(0)\right] &&= \mathrm{Z}\,\mathrm{E}\left[\left(\sum_{k\in\mathbb{Z}} h^*(k)\,x^*(m-k)\right) t^*(0)\right] \\
&= \mathrm{Z}\sum_{k\in\mathbb{Z}} h^*(k)\,\mathrm{E}\left[x^*(m-k)\,t^*(0)\right] &&\triangleq \mathrm{Z}\sum_{k\in\mathbb{Z}} h^*(k)\,\mathsf{R}_{xt}(m-k) \\
&\triangleq \mathrm{Z}\left[h(m)\star \mathsf{R}_{xt}(m)\right] &&= \mathrm{Z}\left[h(m)\star \mathsf{R}_{xt}(m)\right] &&= [\mathrm{Z}\,h(m)][\mathrm{Z}\,\mathsf{R}_{xt}(m)] \\
&= \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{xt}(z) \\
\boxed{\check{\mathsf{S}}_{xy}(z)} &= \check{\mathsf{S}}_{yx}\left(\frac{1}{z}\right) \triangleq \check{\mathsf{S}}_{yt}\left(\frac{1}{z}\right)\Big|_{t=x} &&= \check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{xt}\left(\frac{1}{z}\right)\Big|_{t=x} = \check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{xx}\left(\frac{1}{z}\right) &&= \boxed{\check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{xx}(z)} \\
\boxed{\check{\mathsf{S}}_{yy}(z)} &\triangleq \check{\mathsf{S}}_{yt}(z)\big|_{t=y} &&= \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{xt}(z)\big|_{t=y} = \boxed{\check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{xy}(z)} &&= \boxed{\check{\mathrm{H}}(z)\,\check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{xx}(z)}
\end{aligned}
$$

*Remark.* Note that in several cases, the results listed in Proposition 2 can be "simplified" (as measured by the number of glyphs required to render it on a page) by the use of Proposition 1. For example, (1) in Proposition 2 can be simplified from

$$\check{\mathsf{S}}_{xy}(z) = \check{\mathrm{H}}^*\left(\tfrac{1}{z^*}\right)\check{\mathsf{S}}_{xx}(z) \qquad \text{to} \qquad \check{\mathsf{S}}_{yx}(z) = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{xx}(z).$$

However, such simplification arguably obfuscates the relations comparisons listed in Remark 3.

**Corollary 2.** *Let (1)–(8) below correspond to the eight definitions of* $\mathsf{R}_{xy}(m)$ *in Definition 2.*

$$
\begin{array}{lllll}
(1) \implies \tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathrm{H}}^*(\ \ \omega)\tilde{\mathsf{S}}_{xx}(\omega) & \textit{and} & \tilde{\mathsf{S}}_{yy}(\omega) = \tilde{\mathrm{H}}\ (\ \ \omega)\tilde{\mathsf{S}}_{xy}(\omega) & = |\tilde{\mathrm{H}}\ (\ \ \omega)|^2 & \tilde{\mathsf{S}}_{xx}(\omega) \\
(2) \implies \tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathrm{H}}\ (\ \ \omega)\tilde{\mathsf{S}}_{xx}(\omega) & \textit{and} & \tilde{\mathsf{S}}_{yy}(\omega) = \tilde{\mathrm{H}}^*(\ \ \omega)\tilde{\mathsf{S}}_{xy}(\omega) & = |\tilde{\mathrm{H}}\ (\ \ \omega)|^2 & \tilde{\mathsf{S}}_{xx}(\omega) \\
(3) \implies \tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathrm{H}}^*(-\omega)\tilde{\mathsf{S}}_{xx}(\omega) & \textit{and} & \tilde{\mathsf{S}}_{yy}(\omega) = \tilde{\mathrm{H}}\ (-\omega)\tilde{\mathsf{S}}_{xy}(\omega) & = |\tilde{\mathrm{H}}\ (-\omega)|^2 & \tilde{\mathsf{S}}_{xx}(\omega) \\
(4) \implies \tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathrm{H}}\ (-\omega)\tilde{\mathsf{S}}_{xx}(\omega) & \textit{and} & \tilde{\mathsf{S}}_{yy}(\omega) = \tilde{\mathrm{H}}^*(-\omega)\tilde{\mathsf{S}}_{xy}(\omega) & = |\tilde{\mathrm{H}}\ (-\omega)|^2 & \tilde{\mathsf{S}}_{xx}(\omega) \\
(5) \implies \tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathrm{H}}\ (\ \ \omega)\tilde{\mathsf{S}}_{xx}(\omega) & \textit{and} & \tilde{\mathsf{S}}_{yy}(\omega) = \tilde{\mathrm{H}}\ (-\omega)\tilde{\mathsf{S}}_{xy}(\omega) & = \ \tilde{\mathrm{H}}\ (\ \ \omega)\,\tilde{\mathrm{H}}\ (-\omega) & \tilde{\mathsf{S}}_{xx}(\omega) \\
(6) \implies \tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathrm{H}}\ (-\omega)\tilde{\mathsf{S}}_{xx}(\omega) & \textit{and} & \tilde{\mathsf{S}}_{yy}(\omega) = \tilde{\mathrm{H}}\ (\ \ \omega)\tilde{\mathsf{S}}_{xy}(\omega) & = \ \tilde{\mathrm{H}}\ (\ \ \omega)\,\tilde{\mathrm{H}}\ (-\omega) & \tilde{\mathsf{S}}_{xx}(\omega) \\
(7) \implies \tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathrm{H}}^*(-\omega)\tilde{\mathsf{S}}_{xx}(\omega) & \textit{and} & \tilde{\mathsf{S}}_{yy}(\omega) = \tilde{\mathrm{H}}^*(\ \ \omega)\tilde{\mathsf{S}}_{xy}(\omega) & = \ \tilde{\mathrm{H}}^*(\ \ \omega)\,\tilde{\mathrm{H}}^*(-\omega) & \tilde{\mathsf{S}}_{xx}(\omega) \\
(8) \implies \tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathrm{H}}\ (-\omega)\tilde{\mathsf{S}}_{xx}(\omega) & \textit{and} & \tilde{\mathsf{S}}_{yy}(\omega) = \tilde{\mathrm{H}}\ (\ \ \omega)\tilde{\mathsf{S}}_{xy}(\omega) & = \ \tilde{\mathrm{H}}\ (\ \ \omega)\,\tilde{\mathrm{H}}\ (-\omega) & \tilde{\mathsf{S}}_{xx}(\omega)
\end{array}
$$

*Proof.*

$$\text{(1).}\quad \boxed{\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)} = \check{\mathsf{S}}_{\mathsf{xy}}(z)\big|_{z=e^{i\omega}}$$

$$= \check{\mathsf{H}}^*\left(\frac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\Big|_{z=e^{i\omega}} \qquad \text{by } \check{\mathsf{S}}_{\mathsf{xy}}(z) \text{ result} \qquad \text{(Proposition 2 page 4)}$$

$$= \check{\mathsf{H}}^*\left(e^{i\omega}\right)\check{\mathsf{S}}_{\mathsf{xx}}\left(e^{i\omega}\right) \qquad \text{(evaluation around unit circle in } z\text{-plane)}$$

$$= \boxed{\tilde{\mathsf{H}}^*(\omega)\,\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} \qquad \text{by definition of } DTFT \qquad \text{(Definition 1 page 1)}$$

$$\boxed{\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)} = \check{\mathsf{S}}_{\mathsf{yy}}(z)\big|_{z=e^{i\omega}}$$

$$= \check{\mathsf{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xy}}(z)\big|_{z=e^{i\omega}} \qquad \text{by } \check{\mathsf{S}}_{\mathsf{xy}}(z) \text{ result} \qquad \text{(Proposition 2 page 4)}$$

$$= \check{\mathsf{H}}\left(e^{i\omega}\right)\check{\mathsf{S}}_{\mathsf{xy}}\left(e^{i\omega}\right) \qquad \text{(evaluation around unit circle in } z\text{-plane)}$$

$$= \boxed{\tilde{\mathsf{H}}(\omega)\,\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega)} \qquad \text{by definition of } DTFT \qquad \text{(Definition 1 page 1)}$$

$$\boxed{\tilde{\mathsf{S}}_{\mathsf{yy}}(\omega)} = \check{\mathsf{S}}_{\mathsf{yy}}(z)\big|_{z=e^{i\omega}}$$

$$= \check{\mathsf{H}}(z)\,\check{\mathsf{H}}^*\left(\frac{1}{z^*}\right)\check{\mathsf{S}}_{\mathsf{xx}}{}^*\left(\frac{1}{z^*}\right)\Big|_{z=e^{i\omega}} \qquad \text{by } \check{\mathsf{S}}_{\mathsf{xy}}(z) \text{ result} \qquad \text{(Proposition 2 page 4)}$$

$$= \check{\mathsf{H}}\left(e^{i\omega}\right)\check{\mathsf{H}}^*\left(\frac{1}{e^{-i\omega}}\right)\check{\mathsf{S}}_{\mathsf{xx}}{}^*\left(\frac{1}{e^{-i\omega}}\right)$$

$$= \check{\mathsf{H}}\left(e^{i\omega}\right)\check{\mathsf{H}}^*\left(e^{i\omega}\right)\check{\mathsf{S}}_{\mathsf{xx}}{}^*\left(e^{i\omega}\right) \qquad = \tilde{\mathsf{H}}(\omega)\,\tilde{\mathsf{H}}^*(\omega)\,\tilde{\mathsf{S}}_{\mathsf{xx}}{}^*(\omega) \qquad = \left|\tilde{\mathsf{H}}(\omega)\right|^2\tilde{\mathsf{S}}_{\mathsf{xx}}{}^*(\omega)$$

$$= \boxed{\left|\tilde{\mathsf{H}}(\omega)\right|^2\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} \qquad \text{because } \tilde{\mathsf{S}}_{\mathsf{xx}}(\omega) \text{ is } \textit{real-valued} \qquad \text{(Corollary 1 page 4)}$$

The other seven sets of proofs follow in like manner.

## 4 Real-valued x(n) and y(n)

**Corollary 3.** *Let (1)–(8) below correspond to the eight definitions of* $\mathsf{R}_{\mathsf{xy}}(m)$ *in Definition 2.*

*If* $(\!|\mathrm{x}(n)|\!)$ *and* $(\!|\mathrm{y}(n)|\!)$ *are* ***real-valued****, then*

$$\left\{\begin{array}{lllll} \text{each of (1)–(8)} & \Longrightarrow \mathsf{R}_{\mathsf{xy}}{}^*(m) = \mathsf{R}_{\mathsf{xy}}(m) & \text{and } \mathsf{R}_{\mathsf{xx}}{}^*(m) = \mathsf{R}_{\mathsf{xx}}(m) & \text{(real-valued) and} \\ \text{each of (1)–(8)} & \Longrightarrow \mathsf{R}_{\mathsf{yx}}(m) = \mathsf{R}_{\mathsf{xy}}(-m) & \text{and } \mathsf{R}_{\mathsf{xx}}(-m) = \mathsf{R}_{\mathsf{xx}}(m) & \text{(symmetric)}\ . \end{array}\right\}$$

*Proof.* This follows directly from the *real-valued* hypothesis, Definition 2, and Lemma 1

**Proposition 3.** *Let (1)–(8) below correspond to the eight definitions of* $\mathsf{R}_{\mathsf{xy}}(m)$ *in Definition 2.*

*If* $(\!|\mathrm{x}(n)|\!)$ *and* $(\!|\mathrm{y}(n)|\!)$ *are* ***real-valued****, then*

$$\text{each of (1)–(8)} \Longrightarrow \check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathsf{S}}_{\mathsf{xy}}{}^*(z^*) = \check{\mathsf{S}}_{\mathsf{yx}}\left(\tfrac{1}{z}\right) \text{ and } \check{\mathsf{S}}_{\mathsf{xx}}(z) = \check{\mathsf{S}}_{\mathsf{xx}}{}^*(z^*) = \check{\mathsf{S}}_{\mathsf{xx}}\left(\tfrac{1}{z}\right)\ .$$

*Proof.*

$$
\begin{aligned}
\boxed{\check{\mathsf{S}}_{\mathsf{xy}}{}^{*}\left(z^{*}\right)} &\triangleq \left[\sum_{m\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(m)(z^{*})^{-m}\right]^{*} && \text{by definition of } \check{\mathsf{S}}_{\mathsf{xy}} && \text{(Definition 1 page 1)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}{}^{*}(m) z^{-m} && \text{by \textit{antiautomorphic} property of *-algebras} && \text{(Definition 11 page 24)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(m) z^{-m} && \text{by \textit{real-valued} hypothesis} &&\\
&\triangleq \boxed{\check{\mathsf{S}}_{\mathsf{xy}}(z)} && \text{by definition of } \check{\mathsf{S}}_{\mathsf{xy}}(z) && \text{(Definition 1 page 1)}\\
&\triangleq \sum_{m\in\mathbb{Z}} \mathsf{R}_{\mathsf{xy}}(m) z^{-m} && \text{by definition of } \check{\mathsf{S}}_{\mathsf{xy}}(z) && \text{(Definition 1 page 1)}\\
&= \sum_{m\in\mathbb{Z}} \mathsf{R}_{\mathsf{yx}}(-m) z^{-m} && \text{by \textit{real-valued} hypothesis and Lemma 1 page 3} &&\\
&= \sum_{-p\in\mathbb{Z}} \mathsf{R}_{\mathsf{yx}}(p) z^{p} && \text{where } p \triangleq -m &&\\
&= \sum_{p\in\mathbb{Z}} \mathsf{R}_{\mathsf{yx}}(p) z^{p} && \text{because } (\!|\mathrm{x}(n)|\!), (\!|\mathrm{y}(n)|\!) \in \ell^2_{\mathbb{C}} && \text{(Definition 8 page 20)}\\
&= \sum_{p\in\mathbb{Z}} \mathsf{R}_{\mathsf{yx}}(p) \left(\frac{1}{z}\right)^{-p} && &&\\
&\triangleq \boxed{\check{\mathsf{S}}_{\mathsf{yx}}\left(\frac{1}{z}\right)} && \text{by definition of } \check{\mathsf{S}}_{\mathsf{yx}}(z) && \text{(Definition 1 page 1)}
\end{aligned}
$$

**Proposition 4.** *Let (1)–(8) below correspond to the eight definitions of $\mathsf{R}_{\mathsf{xy}}(m)$ in Definition 2.*

*If $(\!|\mathrm{x}(n)|\!)$ and $(\!|\mathrm{y}(n)|\!)$ are* ***real-valued****, then*

$$
\begin{aligned}
&\text{(1),(4),(6), or (8)} \implies \check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z) \text{ and } \check{\mathsf{S}}_{\mathsf{yy}}(z) = \check{\mathrm{H}}\left(z\right)\check{\mathsf{S}}_{\mathsf{xy}}(z) \quad \text{and}\\
&\text{(2),(3),(5), or (7)} \implies \check{\mathsf{S}}_{\mathsf{xy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xx}}(z) \text{ and } \check{\mathsf{S}}_{\mathsf{yy}}(z) = \check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xy}}(z) \quad \text{and}\\
&\text{each of (1)–(8)} \implies \check{\mathsf{S}}_{\mathsf{yy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathrm{H}}\left(\tfrac{1}{z}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z) = \check{\mathrm{H}}(z)\,\check{\mathrm{H}}^{*}\left(\tfrac{1}{z^{*}}\right)\check{\mathsf{S}}_{\mathsf{xx}}(z)\,.
\end{aligned}
$$

*Proof.*

$$
\begin{array}{llll}
(1). & \boxed{\check{S}_{xy}(z)} & = \check{S}_{xy}{}^{*}(z^{*}) & \text{by Proposition 3} \\
 & & = \check{H}\left(\frac{1}{z}\right)\check{S}_{xx}{}^{*}(z^{*}) & \text{by Proposition 2 page 4} \\
 & & = \boxed{\check{H}\left(\frac{1}{z}\right)\check{S}_{xx}(z)} & \text{by Proposition 3} \\
 & \boxed{\check{S}_{yy}(z)} & = \boxed{\check{H}(z)\,\check{S}_{xy}(z)} & \text{by Proposition 2 page 4} \\
 & & = \boxed{\check{H}(z)\,\check{H}\left(\frac{1}{z}\right)\check{S}_{xx}(z)} & \text{by } \check{S}_{xy}(z) \text{ result} \\
 & & = \boxed{\check{H}(z)\,\check{H}^{*}\left(\frac{1}{z^{*}}\right)\check{S}_{xx}(z)} & \text{by Lemma 5 page 23} \\
(2). & \boxed{\check{S}_{xy}(z)} & = \boxed{\check{H}(z)\,\check{S}_{xx}(z)} & \text{by Proposition 2 page 4} \\
 & \boxed{\check{S}_{yy}(z)} & = \check{S}_{yy}{}^{*}(z^{*}) & \text{by Proposition 3} \\
 & & = \check{H}\left(\frac{1}{z}\right)\check{S}_{xy}{}^{*}(z^{*}) & \text{by Proposition 2 page 4} \\
 & & = \boxed{\check{H}\left(\frac{1}{z}\right)\check{S}_{xy}(z)} & \text{by Proposition 3} \\
 & & = \boxed{\check{H}(z)\,\check{H}\left(\frac{1}{z}\right)\check{S}_{xx}(z)} & \text{by } \check{S}_{xy}(z) \text{ result} \\
 & \boxed{\check{S}_{yy}(z)} & = \check{S}_{yy}{}^{*}(z^{*}) & \text{by Proposition 3} \\
 & & = \check{H}^{*}(z^{*})\,\check{H}^{*}\left(\frac{1}{z^{*}}\right)\check{S}_{xx}{}^{*}(z^{*}) & \text{by Proposition 3} \\
 & & = \check{H}^{*}(z^{*})\,\check{H}^{*}\left(\frac{1}{z^{*}}\right)\check{S}_{xx}(z) & \text{by Proposition 3} \\
 & & = \boxed{\check{H}(z)\,\check{H}^{*}\left(\frac{1}{z^{*}}\right)\check{S}_{xx}(z)} & \text{by Lemma 5 page 23}
\end{array}
$$

**Corollary 4.** *Let (1)–(8) below correspond to the eight definitions of $R_{xy}(m)$ in Definition 2.*

*If $(\!(x(n))\!)$ and $(\!(y(n))\!)$ are **real-valued**, then*

$$
\begin{array}{lll}
(1),(4),(6), \text{ or } (8) & \Longrightarrow \tilde{S}_{xy}(\omega) = \tilde{H}^{*}(\omega)\tilde{S}_{xx}(\omega) \text{ and } \tilde{S}_{yy}(\omega) = \tilde{H}\ (\omega)\tilde{S}_{xy}(\omega) & \text{and} \\
(2),(3),(5), \text{ or } (7) & \Longrightarrow \tilde{S}_{xy}(\omega) = \tilde{H}\ (\omega)\tilde{S}_{xx}(\omega) \text{ and } \tilde{S}_{yy}(\omega) = \tilde{H}^{*}(\omega)\tilde{S}_{xy}(\omega) & \text{and} \\
\text{each of } (1)\text{–}(8) & \Longrightarrow \tilde{S}_{yy}(\omega) = \left|\tilde{H}(\omega)\right|^{2}\tilde{S}_{xx}(\omega) & .
\end{array}
$$

*Proof.*

$$
\begin{aligned}
(1).\quad \tilde{\mathsf{S}}_{xy}(\omega) &= \check{\mathsf{S}}_{xy}(z)\Big|_{z=e^{i\omega}} && \text{by definition of } DTFT && \text{(Definition 1 page 1)}\\
&= \check{\mathrm{H}}\left(\frac{1}{z}\right)\check{\mathsf{S}}_{xx}(z)\Big|_{z=e^{i\omega}} && \text{by Proposition 4}\\
&= \check{\mathrm{H}}\left(e^{-i\omega}\right)\check{\mathsf{S}}_{xx}\left(e^{i\omega}\right)\\
&= \tilde{\mathsf{H}}(-\omega)\,\tilde{\mathsf{S}}_{xx}(\omega) && \text{by definition of } DTFT && \text{(Definition 1 page 1)}\\
&= \tilde{\mathsf{H}}^{*}(\omega)\,\tilde{\mathsf{S}}_{xx}(\omega) && \text{by Lemma 6 page 24}
\end{aligned}
$$

The remainder of the proof for Corollary 4 follows in similar fashion.

## 5 Case studies

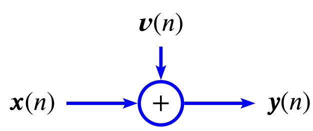
It has been suggested by the giants that the usefulness of a mathematical idea can be measured by

- how useful it is in applications[4] and
- how well it connects and is connected to the larger web of mathematical ideas.[5]

As such, this section which presents applications, may prove useful in gauging the usefulness of the preceding sections.

### *5.1 Case study: Additive noise*

**Proposition 5.** *Let* S *be the* system *illustrated to the right, where* T *is an operator that is* not necessarily linear.




$$
\left\{\begin{array}{lll}
\text{(A). } \mathrm{x}(n) \text{ is} & \textit{WSS} & \text{and}\\
\text{(B). } \mathrm{x}(n) \text{ and } \mathrm{v}(n) \text{ are} & \textit{uncorrelated} & \text{and}\\
\text{(C). } \mathrm{v}(n) \text{ is} & \textit{zero-mean} &
\end{array}\right\}
\implies
\left\{\begin{array}{ll}
\mathsf{R}_{xy}(m) = \mathsf{R}_{xx}(m) & \text{and}\\
\check{\mathsf{S}}_{xy}(z) = \check{\mathsf{S}}_{xx}(z) & \text{and}\\
\tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathsf{S}}_{xx}(\omega) & \text{and}\\
\mathsf{R}_{yy}(m) = \mathsf{R}_{xx}(m) + \check{\mathsf{S}}_{vv}(z) & \text{and}\\
\check{\mathsf{S}}_{yy}(z) = \check{\mathsf{S}}_{xx}(z) + \check{\mathsf{S}}_{vv}(z) & \text{and}\\
\tilde{\mathsf{S}}_{yy}(\omega) = \tilde{\mathsf{S}}_{xx}(\omega) + \tilde{\mathsf{S}}_{vv}(\omega) &
\end{array}\right\}
\begin{array}{l}\text{for all}\\ \text{(1)–(8)}\end{array}
$$

*Proof.*

$$
\begin{aligned}
(1).\quad \mathsf{R}_{xy}(m) &\triangleq \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{y}^{*}(0)\right] && \text{by (A) and Papoulis' definition of } \mathsf{R}_{xy} && \text{(Definition 2 page 2)}\\
&\triangleq \mathrm{E}\left(\mathrm{x}(m)[\mathrm{x}(0)+\mathrm{v}(0)]^{*}\right) && \text{by definition of y}\\
&= \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{x}^{*}(0)\right] + \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{v}^{*}(0)\right] && \text{by } \textit{linearity} \text{ of E} && \text{(Proposition 11 page 20)}\\
&= \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{x}^{*}(0)\right] + \mathrm{E}\left[\mathrm{x}(m)\right]\mathrm{E}\left[\mathrm{v}^{*}(0)\right] && \text{by } \textit{uncorrelated} \text{ hypothesis} && \text{(B)}\\
&= \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{x}^{*}(0)\right] + \mathrm{E}\left[\mathrm{x}(m)\right]\cancelto{0}{\mathrm{E}\left[\mathrm{v}^{*}(0)\right]} && \text{by } \textit{zero-mean} \text{ hypothesis} && \text{(C)}\\
&= \mathsf{R}_{xx}(m) && \text{by definition of } \mathsf{R}_{xx} && \text{(Definition 2 page 2)}
\end{aligned}
$$

---

[4] "I regard as quite useless the reading of large treatises of pure analysis: too large a number of methods pass at once before the eyes. It is in the works of applications that one must study them; one judges their ability there and one apprises the manner of making use of them." —Joseph Louis Lagrange (1736–1813). [23, page xi]

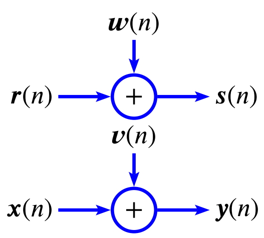
[5] "The "seriousness" of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the *significance* of the mathematical ideas which it connects. We may say, roughly, that a mathematical idea is "significant" if it can be connected, in a natural illuminating way, with a large complex of other mathematical ideas." —G.H. Hardy (1877–1947). [9]

$$
\begin{aligned}
\mathsf{R}_{yy}(m) &\triangleq \mathrm{E}\left[\mathrm{y}(m)\,\mathrm{y}^*(0)\right] && \text{by (A) and Papoulis' definition of } \mathsf{R}_{yy}\\
&\triangleq \mathrm{E}\left[(\mathrm{x}(m)+\mathrm{v}(m))(\mathrm{x}(0)+\mathrm{v}(0))^*\right] && \text{by definition of y}\\
&= \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{x}^*(0)\right]+\mathrm{E}\left[\mathrm{x}(m)\,\mathrm{v}^*(0)\right]+\mathrm{E}\left[\mathrm{v}(m)\,\mathrm{x}^*(0)\right]+\mathrm{E}\left[\mathrm{v}(m)\,\mathrm{v}^*(0)\right] && \\
&= \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{x}^*(0)\right]+\mathrm{E}\,\mathrm{x}(m)\,\mathrm{E}\,\mathrm{v}^*(0)+\mathrm{E}\,\mathrm{v}(m)\,\mathrm{E}\,\mathrm{x}^*(0)+\mathrm{E}\left[\mathrm{v}(m)\,\mathrm{v}^*(0)\right] && \text{by \textit{uncorrelated} hypothesis (B)}\\
&= \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{x}^*(0)\right]+\mathrm{E}\,\mathrm{x}(m)\cancelto{0}{\mathrm{E}\,\mathrm{v}^*(0)}+\cancelto{0}{\mathrm{E}\,\mathrm{v}(m)}\,\mathrm{E}\,\mathrm{x}^*(0)+\mathrm{E}\left[\mathrm{v}(m)\,\mathrm{v}^*(0)\right] && \text{by \textit{zero-mean} hypothesis (C)}\\
&= \mathsf{R}_{xx}(m)+\mathsf{R}_{vv}(m) && \text{by Papoulis' definition of } \mathsf{R}_{xx}
\end{aligned}
$$

| | | | | |
|---|---|---|---|---|
| (2). | $\mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}^*(0)\,\mathrm{y}(m)\right]$ | $\triangleq \mathrm{E}\left(\mathrm{x}^*(0)[\mathrm{x}(m)+\mathrm{v}(m)]\right)$ | $= \mathrm{E}\left[\mathrm{x}^*(0)\,\mathrm{x}(m)\right]+\mathrm{E}\left[\mathrm{x}^*(0)\,\mathrm{v}(m)\right]$ | $= \mathsf{R}_{xx}(m)$ |
| (3). | $\mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}(0)\,\mathrm{y}^*(m)\right]$ | $\triangleq \mathrm{E}\left(\mathrm{x}(0)[\mathrm{x}(m)+\mathrm{v}(m)]^*\right)$ | $= \mathrm{E}\left[\mathrm{x}(0)\,\mathrm{x}^*(m)\right]+\mathrm{E}\left[\mathrm{x}(0)\,\mathrm{v}^*(m)\right]$ | $= \mathsf{R}_{xx}(m)$ |
| (4). | $\mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}^*(m)\,\mathrm{y}(0)\right]$ | $\triangleq \mathrm{E}\left(\mathrm{x}^*(m)[\mathrm{x}(0)+\mathrm{v}(0)]\right)$ | $= \mathrm{E}\left[\mathrm{x}^*(m)\,\mathrm{x}(0)\right]+\mathrm{E}\left[\mathrm{x}^*(m)\,\mathrm{v}(0)\right]$ | $= \mathsf{R}_{xx}(m)$ |
| (5). | $\mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}(0)\,\mathrm{y}(m)\right]$ | $\triangleq \mathrm{E}\left(\mathrm{x}(0)[\mathrm{x}(m)+\mathrm{v}(m)]\right)$ | $= \mathrm{E}\left[\mathrm{x}(0)\,\mathrm{x}(m)\right]+\mathrm{E}\left[\mathrm{x}(0)\,\mathrm{v}(m)\right]$ | $= \mathsf{R}_{xx}(m)$ |
| (6). | $\mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{y}(0)\right]$ | $\triangleq \mathrm{E}\left(\mathrm{x}(m)[\mathrm{x}(0)+\mathrm{v}(0)]\right)$ | $= \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{x}(0)\right]+\mathrm{E}\left[\mathrm{x}(m)\,\mathrm{v}(0)\right]$ | $= \mathsf{R}_{xx}(0)$ |
| (7). | $\mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}^*(0)\,\mathrm{y}^*(m)\right]$ | $\triangleq \mathrm{E}\left(\mathrm{x}^*(0)[\mathrm{x}^*(m)+\mathrm{v}^*(m)]\right)$ | $= \mathrm{E}\left[\mathrm{x}^*(0)\,\mathrm{x}(m)\right]+\mathrm{E}\left[\mathrm{x}^*(0)\,\mathrm{v}^*(m)\right]$ | $= \mathsf{R}_{xx}(m)$ |
| (8). | $\mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}^*(m)\,\mathrm{y}^*(0)\right]$ | $\triangleq \mathrm{E}\left(\mathrm{x}^*(m)[\mathrm{x}^*(0)+\mathrm{v}^*(0)]\right)$ | $= \mathrm{E}\left[\mathrm{x}^*(m)\,\mathrm{x}(0)\right]+\mathrm{E}\left[\mathrm{x}^*(m)\,\mathrm{v}^*(0)\right]$ | $= \mathsf{R}_{xx}(m)$ |

| | | | | |
|---|---|---|---|---|
| (2). | $\mathsf{R}_{yy}(m) \triangleq \mathrm{E}\left[\mathrm{y}^*(0)\,\mathrm{y}(m)\right]$ | $\triangleq \mathrm{E}\left([\mathrm{x}(0)+\mathrm{v}(0)]^*[\mathrm{x}(m)+\mathrm{v}(m)]\right)$ | $=\mathsf{R}_{xx}(m)+\cancelto{0}{\mathsf{R}_{xv}(m)}+\cancelto{0}{\mathsf{R}_{vx}(m)}+\mathsf{R}_{vv}(m)$ | $= \mathsf{R}_{xx}(m)+\mathsf{R}_{vv}(m)$ |
| (3). | $\mathsf{R}_{yy}(m) \triangleq \mathrm{E}\left[\mathrm{y}(0)\,\mathrm{y}^*(m)\right]$ | $\triangleq \mathrm{E}\left([\mathrm{x}(0)+\mathrm{v}(0)][\mathrm{x}(m)+\mathrm{v}(m)]^*\right)$ | $=\mathsf{R}_{xx}(m)+\cancelto{0}{\mathsf{R}_{xv}(m)}+\cancelto{0}{\mathsf{R}_{vx}(m)}+\mathsf{R}_{vv}(m)$ | $= \mathsf{R}_{xx}(m)+\mathsf{R}_{vv}(m)$ |
| (4). | $\mathsf{R}_{yy}(m) \triangleq \mathrm{E}\left[\mathrm{y}^*(m)\,\mathrm{y}(0)\right]$ | $\triangleq \mathrm{E}\left([\mathrm{x}(m)+\mathrm{v}(m)]^*[\mathrm{x}(0)+\mathrm{v}(0)]\right)$ | $=\mathsf{R}_{xx}(m)+\cancelto{0}{\mathsf{R}_{xv}(m)}+\cancelto{0}{\mathsf{R}_{vx}(m)}+\mathsf{R}_{vv}(m)$ | $= \mathsf{R}_{xx}(m)+\mathsf{R}_{vv}(m)$ |
| (5). | $\mathsf{R}_{yy}(m) \triangleq \mathrm{E}\left[\mathrm{y}(0)\,\mathrm{y}(m)\right]$ | $\triangleq \mathrm{E}\left([\mathrm{x}(0)+\mathrm{v}(0)][\mathrm{x}(m)+\mathrm{v}(m)]\right)$ | $=\mathsf{R}_{xx}(m)+\cancelto{0}{\mathsf{R}_{xv}(m)}+\cancelto{0}{\mathsf{R}_{vx}(m)}+\mathsf{R}_{vv}(m)$ | $= \mathsf{R}_{xx}(m)+\mathsf{R}_{vv}(m)$ |
| (6). | $\mathsf{R}_{yy}(m) \triangleq \mathrm{E}\left[\mathrm{y}(m)\,\mathrm{y}(0)\right]$ | $\triangleq \mathrm{E}\left([\mathrm{x}(m)+\mathrm{v}(m)][\mathrm{x}(0)+\mathrm{v}(0)]\right)$ | $=\mathsf{R}_{xx}(m)+\cancelto{0}{\mathsf{R}_{xv}(m)}+\cancelto{0}{\mathsf{R}_{vx}(m)}+\mathsf{R}_{vv}(m)$ | $= \mathsf{R}_{xx}(m)+\mathsf{R}_{vv}(m)$ |
| (7). | $\mathsf{R}_{yy}(m) \triangleq \mathrm{E}\left[\mathrm{y}^*(0)\,\mathrm{y}^*(m)\right]$ | $\triangleq \mathrm{E}\left([\mathrm{x}(0)+\mathrm{v}(0)]^*[\mathrm{x}(m)+\mathrm{v}(m)]^*\right)$ | $=\mathsf{R}_{xx}(m)+\cancelto{0}{\mathsf{R}_{xv}(m)}+\cancelto{0}{\mathsf{R}_{vx}(m)}+\mathsf{R}_{vv}(m)$ | $= \mathsf{R}_{xx}(m)+\mathsf{R}_{vv}(m)$ |
| (8). | $\mathsf{R}_{yy}(m) \triangleq \mathrm{E}\left[\mathrm{y}^*(m)\,\mathrm{y}^*(0)\right]$ | $\triangleq \mathrm{E}\left([\mathrm{x}(m)+\mathrm{v}(m)]^*[\mathrm{x}(0)+\mathrm{v}(0)]^*\right)$ | $=\mathsf{R}_{xx}(m)+\cancelto{0}{\mathsf{R}_{xv}(m)}+\cancelto{0}{\mathsf{R}_{vx}(m)}+\mathsf{R}_{vv}(m)$ | $= \mathsf{R}_{xx}(m)+\mathsf{R}_{vv}(m)$ |

### *5.2 Case study: Dual additive noise*




**Proposition 6.** *Let* S *be the* system *illustrated to the right.*

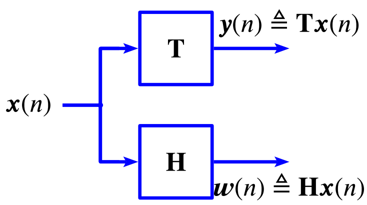
$$
\left\{
\begin{array}{ll}
\text{(A). } \mathrm{x}(n) \text{ and } \mathrm{t}(n) \text{ are \textit{wide sense stationary}} & \text{and}\\
\text{(B). } \mathrm{x}(n) \text{ and } \mathrm{t}(n) \text{ are \textit{uncorrelated}} & \text{and}\\
\text{(C). } \mathrm{t}(n) \text{ and } \mathrm{v}(n) \text{ are \textit{uncorrelated}} & \text{and}\\
\text{(D). } \mathrm{t}(n) \text{ and } \mathrm{v}(n) \text{ are \textit{uncorrelated}} & \text{and}\\
\text{(E). } \mathrm{v}(n) \text{ and } \mathrm{t}(n) \text{ are \textit{zero-mean}} &
\end{array}
\right\}
\implies
\left\{
\begin{array}{ll}
\mathsf{R}_{sy}(m) = \mathsf{R}_{rx}(m) & \text{and}\\
\check{\mathsf{S}}_{sy}(z) = \check{\mathsf{S}}_{rx}(z) & \text{and}\\
\tilde{\mathsf{S}}_{sy}(\omega) = \tilde{\mathsf{S}}_{rx}(\omega) &
\end{array}
\right\}
\quad \text{for all (1)–(8)}
$$

*Proof.*

$$
\begin{aligned}
(1).\quad \boxed{\mathsf{R}_{sy}(m)} &\triangleq \mathrm{E}\left[\mathrm{s}(m)\,\mathrm{y}^*(0)\right] && \text{by Papoulis' definition of } \mathsf{R}_{xy} && \text{(Definition 2 page 2)}\\
&\triangleq \mathrm{E}\left([\mathrm{t}(m)+\mathrm{t}(m)][\mathrm{x}(0)+\mathrm{v}(0)]^*\right) && \text{by definition of S}\\
&= \mathrm{E}\left[\mathrm{t}(m)\,\mathrm{x}^*(0)\right]+\mathrm{E}\left[\mathrm{t}(m)\,\mathrm{v}^*(0)\right]+\mathrm{E}\left[\mathrm{t}(m)\,\mathrm{x}^*(0)\right]+\mathrm{E}\left[\mathrm{t}(m)\,\mathrm{v}^*(0)\right]\\
&= \mathrm{E}\left[\mathrm{t}(m)\,\mathrm{x}^*(0)\right]+\mathrm{E}\,\mathrm{t}(m)\,\mathrm{E}\,\mathrm{v}^*(0)\\
&\quad +\mathrm{E}\,\mathrm{t}(m)\,\mathrm{E}\,\mathrm{x}^*(0)+\mathrm{E}\,\mathrm{t}(m)\,\mathrm{E}\,\mathrm{v}^*(0) && \text{by \textit{uncorrelated} hypotheses} && \text{(B), (C), and (D)}\\
&= \mathrm{E}\left[\mathrm{t}(m)\,\mathrm{x}^*(0)\right]+\mathrm{E}\,\mathrm{t}(m)\cancelto{0}{\mathrm{E}\,\mathrm{v}^*(0)}\\
&\quad +\cancelto{0}{\mathrm{E}\,\mathrm{t}(m)}\,\mathrm{E}\,\mathrm{x}^*(0)+\cancelto{0}{\mathrm{E}\,\mathrm{t}(m)}\,\mathrm{E}\,\mathrm{v}^*(0) && \text{by \textit{zero-mean} hypothesis} && \text{(E)}\\
&\triangleq \boxed{\mathsf{R}_{rx}(m)} && \text{by definition of } \mathsf{R}_{rx} && \text{(Definition 2 page 2)}
\end{aligned}
$$

$$
\begin{aligned}
(2).\quad \mathsf{R}_{sy}(m) &\triangleq \mathrm{E}\left[\mathrm{s}^*(0)\,\mathrm{y}(m)\right] &&= \mathrm{E}\left([\mathrm{t}(0)+\mathrm{t}(0)]^*[\mathrm{x}(m)+\mathrm{v}(m)]\right) &&= \mathsf{R}_{rx}(m)+\cancelto{0}{\mathsf{R}_{rv}(m)}+\cancelto{0}{\mathsf{R}_{wx}(m)}+\cancelto{0}{\mathsf{R}_{wv}(m)} &&\triangleq \mathsf{R}_{rx}(m)\\
(3).\quad \mathsf{R}_{sy}(m) &\triangleq \mathrm{E}\left[\mathrm{s}(0)\,\mathrm{y}^*(m)\right] &&= \mathrm{E}\left([\mathrm{t}(0)+\mathrm{t}(0)][\mathrm{x}(m)+\mathrm{v}(m)]^*\right) &&= \mathsf{R}_{rx}(m)+\cancelto{0}{\mathsf{R}_{rv}(m)}+\cancelto{0}{\mathsf{R}_{wx}(m)}+\cancelto{0}{\mathsf{R}_{wv}(m)} &&\triangleq \mathsf{R}_{rx}(m)\\
(4).\quad \mathsf{R}_{sy}(m) &\triangleq \mathrm{E}\left[\mathrm{s}^*(m)\,\mathrm{y}(0)\right] &&= \mathrm{E}\left([\mathrm{t}(m)+\mathrm{t}(m)]^*[\mathrm{x}(0)+\mathrm{v}(0)]\right) &&= \mathsf{R}_{rx}(m)+\cancelto{0}{\mathsf{R}_{rv}(m)}+\cancelto{0}{\mathsf{R}_{wx}(m)}+\cancelto{0}{\mathsf{R}_{wv}(m)} &&\triangleq \mathsf{R}_{rx}(m)\\
(5).\quad \mathsf{R}_{sy}(m) &\triangleq \mathrm{E}\left[\mathrm{s}(0)\,\mathrm{y}(m)\right] &&= \mathrm{E}\left([\mathrm{t}(0)+\mathrm{t}(0)][\mathrm{x}(m)+\mathrm{v}(m)]\right) &&= \mathsf{R}_{rx}(m)+\cancelto{0}{\mathsf{R}_{rv}(m)}+\cancelto{0}{\mathsf{R}_{wx}(m)}+\cancelto{0}{\mathsf{R}_{wv}(m)} &&\triangleq \mathsf{R}_{rx}(m)\\
(6).\quad \mathsf{R}_{sy}(m) &\triangleq \mathrm{E}\left[\mathrm{s}(m)\,\mathrm{y}(0)\right] &&= \mathrm{E}\left([\mathrm{t}(m)+\mathrm{t}(m)][\mathrm{x}(0)+\mathrm{v}(0)]\right) &&= \mathsf{R}_{rx}(m)+\cancelto{0}{\mathsf{R}_{rv}(m)}+\cancelto{0}{\mathsf{R}_{wx}(m)}+\cancelto{0}{\mathsf{R}_{wv}(m)} &&\triangleq \mathsf{R}_{rx}(m)\\
(7).\quad \mathsf{R}_{sy}(m) &\triangleq \mathrm{E}\left[\mathrm{s}^*(0)\,\mathrm{y}^*(m)\right] &&= \mathrm{E}\left([\mathrm{t}(0)+\mathrm{t}(0)]^*[\mathrm{x}(m)+\mathrm{v}(m)]^*\right) &&= \mathsf{R}_{rx}(m)+\cancelto{0}{\mathsf{R}_{rv}(m)}+\cancelto{0}{\mathsf{R}_{wx}(m)}+\cancelto{0}{\mathsf{R}_{wv}(m)} &&\triangleq \mathsf{R}_{rx}(m)\\
(8).\quad \mathsf{R}_{sy}(m) &\triangleq \mathrm{E}\left[\mathrm{s}^*(m)\,\mathrm{y}^*(0)\right] &&= \mathrm{E}\left([\mathrm{t}(m)+\mathrm{t}(m)]^*[\mathrm{x}(0)+\mathrm{v}(0)]^*\right) &&= \mathsf{R}_{rx}(m)+\cancelto{0}{\mathsf{R}_{rv}(m)}+\cancelto{0}{\mathsf{R}_{wx}(m)}+\cancelto{0}{\mathsf{R}_{wv}(m)} &&\triangleq \mathsf{R}_{rx}(m)
\end{aligned}
$$

## *5.3 Case study: Parallel operators*




**Proposition 7.** *Let* S *be the* system *illustrated to the right, where* T *is* not necessarily linear. *Let*

$$(\!|\mathrm{h}(n)|\!) \triangleq \mathrm{H}\,\bar{\delta}(n) \triangleq \sum_{m\in\mathbb{Z}} \mathrm{h}(m)\bar{\delta}(n-m)$$

*be the* impulse response *of* H.

$$
\left\{\begin{array}{ll}\text{(A). } \mathrm{x}(n) \text{ is } \textit{WSS} \text{ and}\\ \text{(B). } \mathrm{H} \text{ is } \quad \textit{LTI}\end{array}\right\} \implies \left\{\begin{array}{l}
\check{\mathsf{S}}_{wy}(z) = \check{\mathrm{H}}(z) \quad \check{\mathsf{S}}_{xy}(z) \text{ for (1),(3),(5),(6) and}\\
\tilde{\mathsf{S}}_{wy}(\omega) = \tilde{\mathrm{H}}(\omega) \quad \tilde{\mathsf{S}}_{xy}(\omega) \text{ for (1),(3),(5),(6) and}\\
\hline
\check{\mathsf{S}}_{wy}(z) = \check{\mathrm{H}}^*(z^*)\,\check{\mathsf{S}}_{xy}(z) \text{ for (2),(4),(7),(8) and}\\
\tilde{\mathsf{S}}_{wy}(\omega) = \tilde{\mathrm{H}}^*(-\omega)\tilde{\mathsf{S}}_{xy}(\omega) \text{ for (2),(4),(7),(8)}
\end{array}\right\}
$$

*Proof.*

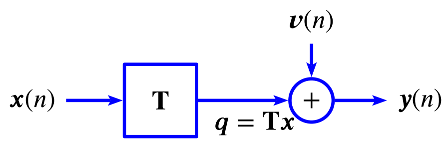
$$
\begin{aligned}
(1).\quad \mathsf{R}_{wy}(m) &\triangleq \mathrm{E}\left[\mathrm{t}(m)\,\mathrm{y}^*(0)\right] && \text{by (A) and Papoulis' definition of } \mathsf{R}_{wy} && \text{(Definition 2 page 2)}\\
&\triangleq \mathrm{E}\left([\mathrm{H}\,\mathrm{x}](m)\,\mathrm{y}^*(0)\right) && \text{by definition of S}\\
&= \mathrm{H}\,\mathrm{E}\left(\mathrm{x}(m)\,\mathrm{y}^*(0)\right) && \text{by \textit{LTI} hypothesis} && \text{(B)}\\
&\triangleq \mathrm{H}\,\mathsf{R}_{xy}(m) && \text{by Papoulis' definition of } \mathsf{R}_{wy} && \text{(Definition 2 page 2)}\\
&= \sum_{n\in\mathbb{Z}} \mathrm{h}(n)\,\mathsf{R}_{xy}(m-n) && \text{by definition of \textit{impulse response} } (\!|\mathrm{h}(n)|\!)\\
&= [\mathrm{h}\star\mathsf{R}_{xy}](m) && \text{by definition of \textit{convolution}} && \text{(Definition 9 page 20)}
\end{aligned}
$$

$$\begin{array}{llllll}
(2). & \mathsf{R}_{wy}(m) \triangleq \mathrm{E}\left[t^*(0)\,y(m)\right] & \triangleq \mathrm{E}\left([\mathrm{H}\,x]^*(0)\,y(m)\right) & = \mathrm{H}^*\left(\mathrm{E}\left[x^*(0)\,y(m)\right]\right) & \triangleq \mathrm{H}^*\,\mathsf{R}_{xy}(m) & = [h^* \star \mathsf{R}_{xy}](m)\\
(3). & \mathsf{R}_{wy}(m) \triangleq \mathrm{E}\left[t(0)\,y^*(m)\right] & \triangleq \mathrm{E}\left([\mathrm{H}\,x](0)\,y^*(m)\right) & = \mathrm{H}\left(\mathrm{E}\left[x(0)\,y^*(m)\right]\right) & \triangleq \mathrm{H}\,\mathsf{R}_{xy}(m) & = [h \star \mathsf{R}_{xy}](m)\\
(4). & \mathsf{R}_{wy}(m) \triangleq \mathrm{E}\left[t^*(m)\,y(0)\right] & \triangleq \mathrm{E}\left([\mathrm{H}\,x]^*(m)\,y(0)\right) & = \mathrm{H}^*\left(\mathrm{E}\left[x^*(m)\,y(0)\right]\right) & \triangleq \mathrm{H}^*\,\mathsf{R}_{xy}(m) & = [h^* \star \mathsf{R}_{xy}](m)\\
(5). & \mathsf{R}_{wy}(m) \triangleq \mathrm{E}\left[t(0)\,y(m)\right] & \triangleq \mathrm{E}\left([\mathrm{H}\,x](0)\,y(m)\right) & = \mathrm{H}\left(\mathrm{E}\left[x(0)\,y(m)\right]\right) & \triangleq \mathrm{H}\,\mathsf{R}_{xy}(m) & = [h \star \mathsf{R}_{xy}](m)\\
(6). & \mathsf{R}_{wy}(m) \triangleq \mathrm{E}\left[t(m)\,y(0)\right] & \triangleq \mathrm{E}\left([\mathrm{H}\,x](m)\,y(0)\right) & = \mathrm{H}\left(\mathrm{E}\left[x(m)\,y(0)\right]\right) & \triangleq \mathrm{H}\,\mathsf{R}_{xy}(m) & = [h \star \mathsf{R}_{xy}](m)\\
(7). & \mathsf{R}_{wy}(m) \triangleq \mathrm{E}\left[t^*(0)\,y^*(m)\right] & \triangleq \mathrm{E}\left([\mathrm{H}\,x]^*(0)\,y^*(m)\right) & = \mathrm{H}^*\left(\mathrm{E}\left[x^*(0)\,y^*(m)\right]\right) & \triangleq \mathrm{H}^*\,\mathsf{R}_{xy}(m) & = [h^* \star \mathsf{R}_{xy}](m)\\
(8). & \mathsf{R}_{wy}(m) \triangleq \mathrm{E}\left[t^*(m)\,y^*(0)\right] & \triangleq \mathrm{E}\left([\mathrm{H}\,x]^*(m)\,y^*(0)\right) & = \mathrm{H}^*\left(\mathrm{E}\left[x^*(m)\,y^*(0)\right]\right) & \triangleq \mathrm{H}^*\,\mathsf{R}_{xy}(m) & = [h^* \star \mathsf{R}_{xy}](m)
\end{array}$$

$$\begin{array}{lllll}
(1),(3),(5),(6). & \check{\mathsf{S}}_{wy}(z) \triangleq \mathrm{Z}\,\mathsf{R}_{wy}(m) & = \mathrm{Z}\,[h \star \mathsf{R}_{xy}](m) & = \check{\mathsf{H}}(z)\,\check{\mathsf{S}}_{xy}(z) & \\
(2),(4),(7),(8). & \check{\mathsf{S}}_{wy}(z) \triangleq \mathrm{Z}\,\mathsf{R}_{wy}(m) & = \mathrm{Z}\,[h^* \star \mathsf{R}_{xy}](m) & = \check{\mathsf{H}}^*\left(z^*\right)\check{\mathsf{S}}_{xy}(z) & \text{by Proposition 12 page 21}\\
(1),(3),(5),(6). & \tilde{\mathsf{S}}_{wy}(\omega) \triangleq \check{\mathrm{F}}\,\mathsf{R}_{wy}(m) & = \check{\mathrm{F}}\,[h \star \mathsf{R}_{xy}](m) & = \tilde{\mathsf{H}}(\omega)\,\tilde{\mathsf{S}}_{xy}(\omega) & \\
(2),(4),(7),(8). & \tilde{\mathsf{S}}_{wy}(\omega) \triangleq \check{\mathrm{F}}\,\mathsf{R}_{wy}(m) & = \check{\mathrm{F}}\,[h^* \star \mathsf{R}_{xy}](m) & = \tilde{\mathsf{H}}^*(-\omega)\,\tilde{\mathsf{S}}_{xy}(\omega) & \text{by Proposition 12 page 21}
\end{array}$$

### *5.4 Case study: Operator with measurement noise*

**Lemma 2.** *Let* S *be the* system *illustrated to the right, where* T *is* not necessarily linear.



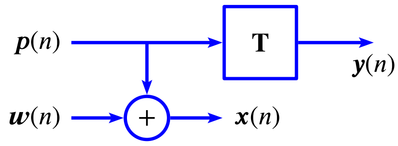

$$\left\{\begin{array}{lll}\text{(A). } x(n) \text{ is} & \textit{wide sense stationary} & \text{and}\\ \text{(B). } x(n) \text{ and } v(n) \text{ are} & \textit{uncorrelated} & \text{and}\\ \text{(C). } v(n) \text{ is} & \textit{zero-mean} & \end{array}\right\} \implies \left\{\begin{array}{ll}\mathsf{R}_{xy}(m) = \mathsf{R}_{xq}(m) & \text{and}\\ \check{\mathsf{S}}_{xy}(z) = \check{\mathsf{S}}_{xq}(z) & \text{and}\\ \tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathsf{S}}_{xq}(\omega) & \end{array}\right\} \quad \text{for all (1)–(8)}$$

*Proof.*

$$\begin{array}{rlll}
\mathsf{R}_{xy}(m) & \triangleq \mathrm{E}\left[x(m)\,y^*(0)\right] & \text{by definition of } \mathsf{R}_{xy} & \text{(Definition 2 page 2)}\\
& \triangleq \mathrm{E}\left[x(m)(q(0)+v(0))^*\right] & \text{by definition of S} & \\
& = \mathrm{E}\left[x(m)\,q^*(0) + p(m)\,v^*(0)\right] & \text{by } \textit{distributive} \text{ property of } (\mathbb{C},+,\cdot,0,1) & \\
& = \mathrm{E}\left[x(m)\,q^*(0)\right] + \mathrm{E}\left[x(m)\,v^*(0)\right] & \text{by } \textit{linearity} \text{ of E} & \text{(Proposition 11 page 20)}\\
& = \mathrm{E}\left[x(m)\,q^*(0)\right] + \left[\mathrm{E}\,x(m)\right]\left[\mathrm{E}\,v^*(0)\right] & \text{by } \textit{uncorrelated} \text{ hypothesis} & \text{(B)}\\
& = \mathrm{E}\left[x(m)\,q^*(0)\right] + \mathrm{E}\left[p(m)\right]\cancelto{0}{\mathrm{E}\left[v^*(0)\right]} & \text{by } \textit{zero-mean} \text{ hypothesis} & \text{(C)}\\
& = \mathsf{R}_{xq}(m) & \text{by definition of } \mathsf{R}_{xq} & \text{(Definition 2 page 2)}
\end{array}$$

$$\begin{array}{lllll}
(2). & \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[x^*(0)\,y(m)\right] & \triangleq \mathrm{E}\left(x^*(0)[q(m)+v(m)]\right) & \triangleq \mathrm{E}\left[x^*(0)\,q(m)\right] + \mathrm{E}\left[x^*(0)\right]\cancelto{0}{\left[\mathrm{E}\,v(m)\right]} & = \mathsf{R}_{xq}(m)\\
(3). & \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[x(0)\,y^*(m)\right] & \triangleq \mathrm{E}\left(x(0)[q(m)+v(m)]^*\right) & \triangleq \mathrm{E}\left[x(0)\,q^*(m)\right] + \mathrm{E}\left[x^*(0)\right]\cancelto{0}{\left[\mathrm{E}\,v^*(m)\right]} & = \mathsf{R}_{xq}(m)\\
(4). & \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[x^*(m)\,y(0)\right] & \triangleq \mathrm{E}\left(x^*(m)[q(0)+v(0)]\right) & \triangleq \mathrm{E}\left[x^*(m)\,q(0)\right] + \mathrm{E}\left[x^*(m)\right]\cancelto{0}{\left[\mathrm{E}\,v(0)\right]} & = \mathsf{R}_{xq}(m)\\
(5). & \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[x(0)\,y(m)\right] & \triangleq \mathrm{E}\left(x(0)[q(m)+v(m)]\right) & \triangleq \mathrm{E}\left[x(0)\,q(m)\right] + \mathrm{E}\left[x(0)\right]\cancelto{0}{\left[\mathrm{E}\,v(m)\right]} & = \mathsf{R}_{xq}(m)\\
(6). & \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[x(m)\,y(0)\right] & \triangleq \mathrm{E}\left(x(m)[q(0)+v(0)]\right) & \triangleq \mathrm{E}\left[x(m)\,q(0)\right] + \mathrm{E}\left[x(m)\right]\cancelto{0}{\left[\mathrm{E}\,v(0)\right]} & = \mathsf{R}_{xq}(m)\\
(7). & \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[x^*(0)\,y^*(m)\right] & \triangleq \mathrm{E}\left(x^*(0)[q(m)+v(m)]^*\right) & \triangleq \mathrm{E}\left[x^*(0)\,q^*(m)\right] + \mathrm{E}\left[x^*(0)\right]\cancelto{0}{\left[\mathrm{E}\,v^*(m)\right]} & = \mathsf{R}_{xq}(m)\\
(8). & \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[x^*(m)\,y^*(0)\right] & \triangleq \mathrm{E}\left(x^*(m)[q(0)+v(0)]^*\right) & \triangleq \mathrm{E}\left[x^*(m)\,q^*(0)\right] + \mathrm{E}\left[x^*(m)\right]\cancelto{0}{\left[\mathrm{E}\,v^*(0)\right]} & = \mathsf{R}_{xq}(m)
\end{array}$$

**Lemma 3.** *Let* S *be the* system *illustrated to the right, where* T *is* not necessarily linear.




$$\left\{\begin{array}{lll}\text{(A). } \mathrm{x}(n) \text{ is} & \textit{WSS} & \text{and}\\ \text{(B). } \mathrm{u}(n) \text{ is} & \textit{zero-mean} & \text{and}\\ \text{(C). } \mathrm{x}(n) \text{ and } \mathrm{u}(n) \text{ are } \textit{uncorrelated} & & \end{array}\right\} \implies \left\{\begin{array}{ll}\mathsf{R}_{xy}(m) = \mathsf{R}_{py}(m) & \text{and}\\ \check{\mathsf{S}}_{xy}(z) = \check{\mathsf{S}}_{py}(z) & \text{and}\\ \tilde{\mathsf{S}}_{xy}(\omega) = \tilde{\mathsf{S}}_{py}(\omega) & \end{array}\right\} \quad \text{for all (1)–(8)}$$

*Proof.*

| | | |
|---|---|---|
| $\mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{y}^*(0)\right]$ | by definition of $\mathsf{R}_{py}$ | (Definition 2 page 2) |
| $\triangleq \mathrm{E}\left([\mathrm{p}(m)+\mathrm{u}(m)]\,\mathrm{y}^*(0)\right)$ | by definition of S | |
| $= \mathrm{E}\left[\mathrm{p}(m)\,\mathrm{y}^*(0)+\mathrm{u}(m)\,\mathrm{y}^*(0)\right]$ | by *distributive* property of $(\mathbb{C},+,\cdot,0,1)$ | |
| $= \mathrm{E}\left[\mathrm{p}(m)\,\mathrm{y}^*(0)\right]+\mathrm{E}\left[\mathrm{u}(m)\,\mathrm{y}^*(0)\right]$ | because E is a *linear operator* | (Proposition 11 page 20) |
| $= \mathrm{E}\left[\mathrm{p}(m)\,\mathrm{y}^*(0)\right]+\mathrm{E}\left[\mathrm{u}(m)\right]\mathrm{E}\left[\mathrm{y}^*(0)\right]$ | by *uncorrelated* hypothesis | (C) |
| $= \mathrm{E}\left[\mathrm{p}(m)\,\mathrm{y}^*(0)\right]+\mathrm{E}\left[\mathrm{u}(m)\right]\cancelto{0}{\mathrm{E}\left[\mathrm{y}^*(0)\right]}$ | by *zero-mean* hypothesis | (B) |
| $\triangleq \mathsf{R}_{py}(m)$ | by definition of $\mathsf{R}_{xy}$ | (Definition 2 page 2) |

$$\text{(2).}\quad \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}^*(0)\,\mathrm{y}(m)\right] \triangleq \mathrm{E}\left([\mathrm{p}(0)+\mathrm{u}(0)]^*\,\mathrm{y}(m)\right) = \mathrm{E}\left[\mathrm{p}^*(0)\,\mathrm{y}(m)\right]+\cancelto{0}{\mathrm{E}\left[\mathrm{u}^*(0)\right]}\mathrm{E}\left[\mathrm{y}(m)\right] = \mathsf{R}_{py}(m)$$

$$\text{(3).}\quad \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}(0)\,\mathrm{y}^*(m)\right] \triangleq \mathrm{E}\left([\mathrm{p}(0)+\mathrm{u}(0)]\,\mathrm{y}^*(m)\right) = \mathrm{E}\left[\mathrm{p}(0)\,\mathrm{y}^*(m)\right]+\cancelto{0}{\mathrm{E}\left[\mathrm{u}^*(0)\right]}\mathrm{E}\left[\mathrm{y}^*(m)\right] = \mathsf{R}_{py}(m)$$

$$\text{(4).}\quad \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}^*(m)\,\mathrm{y}(0)\right] \triangleq \mathrm{E}\left([\mathrm{p}(m)+\mathrm{u}(m)]^*\,\mathrm{y}(0)\right) = \mathrm{E}\left[\mathrm{p}^*(m)\,\mathrm{y}(0)\right]+\cancelto{0}{\mathrm{E}\left[\mathrm{u}^*(m)\right]}\mathrm{E}\left[\mathrm{y}(0)\right] = \mathsf{R}_{py}(m)$$

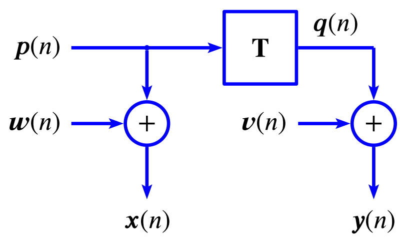
$$\text{(5).}\quad \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}(0)\,\mathrm{y}(m)\right] \triangleq \mathrm{E}\left([\mathrm{p}(0)+\mathrm{u}(0)]\,\mathrm{y}(m)\right) = \mathrm{E}\left[\mathrm{p}(0)\,\mathrm{y}(m)\right]+\cancelto{0}{\mathrm{E}\left[\mathrm{u}(0)\right]}\mathrm{E}\left[\mathrm{y}(m)\right] = \mathsf{R}_{py}(m)$$

$$\text{(6).}\quad \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}(m)\,\mathrm{y}(0)\right] \triangleq \mathrm{E}\left([\mathrm{p}(m)+\mathrm{u}(m)]\,\mathrm{y}(0)\right) = \mathrm{E}\left[\mathrm{p}(m)\,\mathrm{y}(0)\right]+\cancelto{0}{\mathrm{E}\left[\mathrm{u}(m)\right]}\mathrm{E}\left[\mathrm{y}(0)\right] = \mathsf{R}_{py}(m)$$

$$\text{(7).}\quad \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}^*(0)\,\mathrm{y}^*(m)\right] \triangleq \mathrm{E}\left([\mathrm{p}(0)+\mathrm{u}(0)]^*\,\mathrm{y}^*(m)\right) = \mathrm{E}\left[\mathrm{p}^*(0)\,\mathrm{y}^*(m)\right]+\cancelto{0}{\mathrm{E}\left[\mathrm{u}^*(0)\right]}\mathrm{E}\left[\mathrm{y}^*(m)\right] = \mathsf{R}_{py}(m)$$

$$\text{(8).}\quad \mathsf{R}_{xy}(m) \triangleq \mathrm{E}\left[\mathrm{x}^*(m)\,\mathrm{y}^*(0)\right] \triangleq \mathrm{E}\left([\mathrm{p}(m)+\mathrm{u}(m)]^*\,\mathrm{y}^*(0)\right) = \mathrm{E}\left[\mathrm{p}^*(m)\,\mathrm{y}^*(0)\right]+\cancelto{0}{\mathrm{E}\left[\mathrm{u}^*(m)\right]}\mathrm{E}\left[\mathrm{y}^*(0)\right] = \mathsf{R}_{py}(m)$$

**Proposition 8** (***measurement additive noise cross-correlation***)**.** *Let* S *be the* system *illustrated to the right, where* T *is* not necessarily linear.




$$\left\{\begin{array}{ll}\text{(A). } \mathrm{x}(n) \text{ is } \textit{WSS} & \text{and}\\ \text{(B). } \mathrm{u}(n) \text{ is } \textit{zero-mean} & \text{and}\\ \text{(C). } \mathrm{v}(n) \text{ is } \textit{zero-mean} & \text{and}\\ \text{(D). } \mathrm{x}(n),\ \mathrm{t}(n),\ \mathrm{v}(n) \text{ are } \textit{uncorrelated} & \end{array}\right\} \implies \left\{\begin{array}{ll}\mathsf{R}_{pq}(m)=\mathsf{R}_{py}(m)=\mathsf{R}_{xq}(m)=\mathsf{R}_{xy}(m) & \text{and}\\ \check{\mathsf{S}}_{pq}(z)=\check{\mathsf{S}}_{py}(z)=\check{\mathsf{S}}_{xq}(z)=\check{\mathsf{S}}_{xy}(z) & \text{and}\\ \tilde{\mathsf{S}}_{pq}(\omega)=\tilde{\mathsf{S}}_{py}(\omega)=\tilde{\mathsf{S}}_{xq}(\omega)=\tilde{\mathsf{S}}_{xy}(\omega) & \end{array}\right\} \quad \text{for all (1)–(8)}$$

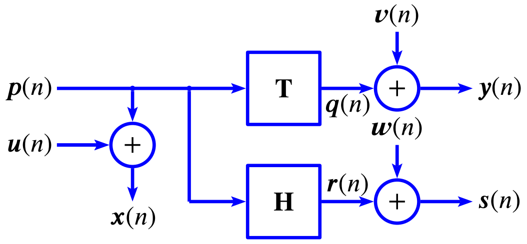
*Proof.*

$$
\begin{array}{lll}
\mathsf{R}_{\mathsf{pq}}(m) = \mathsf{R}_{\mathsf{py}}(m) & \text{by Lemma 2 page 14} & \\
\mathsf{R}_{\mathsf{pq}}(m) = \mathsf{R}_{\mathsf{xq}}(m) & \text{by Lemma 3 page 15} & \\
\mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}\,[\mathrm{x}(m)\,\mathrm{y}^*(0)] & \text{by definition } \mathsf{R}_{\mathsf{xy}} & \text{(Definition 2 page 2)} \\
\qquad \triangleq \mathrm{E}\,([\mathrm{p}(m)+\mathrm{t}(m)]\,\mathrm{y}^*(0)) & \text{by definition S} & \\
\qquad = \mathrm{E}\,[\mathrm{p}(m)\,\mathrm{y}^*(0)+\mathrm{t}(m)\,\mathrm{y}^*(0)] & & \\
\qquad = \mathrm{E}\,[\mathrm{p}(m)\,\mathrm{y}^*(0)] + \mathrm{E}\,[\mathrm{t}(m)\,\mathrm{y}^*(0)] & \text{by } \textit{linearity} \text{ of E} & \text{(Proposition 11 page 20)} \\
\qquad = \mathrm{E}\,[\mathrm{p}(m)\,\mathrm{y}^*(0)] + \cancelto{0}{\mathrm{E}\,[\mathrm{t}(m)\,\mathrm{y}^*(0)]} & \text{by } \textit{uncorrelated} \text{ hypothesis} & \text{(D)} \\
\qquad = \mathsf{R}_{\mathsf{py}}(m) & \text{by definition of } \mathsf{R}_{\mathsf{py}} & \text{(Definition 2 page 2)}
\end{array}
$$

$$
\begin{array}{llllll}
(2). & \mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}\,[\mathrm{x}^*(0)\,\mathrm{y}(m)] & \triangleq \mathrm{E}\left([\mathrm{p}(0)+\mathrm{t}(0)]^*\,\mathrm{y}(m)\right) & = \mathrm{E}\,[\mathrm{p}^*(0)\,\mathrm{y}(m)] + \cancelto{0}{[\mathrm{E}\,\mathrm{t}^*(0)]}[\mathrm{E}\,\mathrm{y}(m)] & = \mathsf{R}_{\mathsf{py}}(m) \\
(3). & \mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}\,[\mathrm{x}(0)\,\mathrm{y}^*(m)] & \triangleq \mathrm{E}\,([\mathrm{p}(0)+\mathrm{t}(0)]\,\mathrm{y}^*(m)) & = \mathrm{E}\,[\mathrm{p}(0)\,\mathrm{y}^*(m)] + \cancelto{0}{[\mathrm{E}\,\mathrm{t}(0)]}[\mathrm{E}\,\mathrm{y}^*(m)] & = \mathsf{R}_{\mathsf{py}}(m) \\
(4). & \mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}\,[\mathrm{x}^*(m)\,\mathrm{y}(0)] & \triangleq \mathrm{E}\left([\mathrm{p}(m)+\mathrm{t}(m)]^*\,\mathrm{y}(0)\right) & = \mathrm{E}\,[\mathrm{p}^*(m)\,\mathrm{y}(0)] + \cancelto{0}{[\mathrm{E}\,\mathrm{t}^*(m)]}[\mathrm{E}\,\mathrm{y}(0)] & = \mathsf{R}_{\mathsf{py}}(m) \\
(5). & \mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}\,[\mathrm{x}(0)\,\mathrm{y}(m)] & \triangleq \mathrm{E}\,([\mathrm{p}(0)+\mathrm{t}(0)]\,\mathrm{y}(m)) & = \mathrm{E}\,[\mathrm{p}(0)\,\mathrm{y}(m)] + \cancelto{0}{[\mathrm{E}\,\mathrm{t}(0)]}[\mathrm{E}\,\mathrm{y}(m)] & = \mathsf{R}_{\mathsf{py}}(m) \\
(6). & \mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}\,[\mathrm{x}(m)\,\mathrm{y}(0)] & \triangleq \mathrm{E}\,([\mathrm{p}(m)+\mathrm{t}(m)]\,\mathrm{y}(0)) & = \mathrm{E}\,[\mathrm{p}(m)\,\mathrm{y}(0)] + \cancelto{0}{[\mathrm{E}\,\mathrm{t}(m)]}[\mathrm{E}\,\mathrm{y}(0)] & = \mathsf{R}_{\mathsf{py}}(m) \\
(7). & \mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}\,[\mathrm{x}^*(0)\,\mathrm{y}^*(m)] & \triangleq \mathrm{E}\left([\mathrm{p}(0)+\mathrm{t}(0)]^*\,\mathrm{y}^*(m)\right) & = \mathrm{E}\,[\mathrm{p}^*(0)\,\mathrm{y}^*(m)] + \cancelto{0}{[\mathrm{E}\,\mathrm{t}^*(0)]}[\mathrm{E}\,\mathrm{y}^*(m)] & = \mathsf{R}_{\mathsf{py}}(m) \\
(8). & \mathsf{R}_{\mathsf{xy}}(m) \triangleq \mathrm{E}\,[\mathrm{x}^*(m)\,\mathrm{y}^*(0)] & \triangleq \mathrm{E}\left([\mathrm{p}(m)+\mathrm{t}(m)]^*\,\mathrm{y}^*(0)\right) & = \mathrm{E}\,[\mathrm{p}^*(m)\,\mathrm{y}^*(0)] + \cancelto{0}{[\mathrm{E}\,\mathrm{t}^*(m)]}[\mathrm{E}\,\mathrm{y}^*(0)] & = \mathsf{R}_{\mathsf{py}}(m)
\end{array}
$$

### *5.5 Case study: Parallel operators with measurement noise*

**Proposition 9.** *Let* S *be the* system *illustrated to the right, where* T *is an operator that is* not necessarily linear.




$$
\left\{\begin{array}{lll}
\text{(A). H is} & \textit{LTI} & \text{and} \\
\text{(B). } \mathrm{x}(n) \text{ is} & \textit{WSS} & \text{and} \\
\text{(C). u and v are} & \textit{zero-mean} & \text{and} \\
\text{(D). p, u, v are} & \textit{uncorrelated} &
\end{array}\right\}
\implies
\left\{\begin{array}{ll}
\check{\mathsf{S}}_{\mathsf{sy}}(z) = \check{\mathrm{H}}(z)\,\check{\mathsf{S}}_{\mathsf{xy}}(z) & \text{for (1),(3),(5),(6) and} \\
\tilde{\mathsf{S}}_{\mathsf{sy}}(\omega) = \tilde{\mathrm{H}}(\omega)\,\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega) & \text{for (1),(3),(5),(6) and} \\
\hline
\check{\mathsf{S}}_{\mathsf{sy}}(z) = \check{\mathrm{H}}^*(z^*)\,\check{\mathsf{S}}_{\mathsf{xy}}(z) & \text{for (2),(4),(7),(8) and} \\
\tilde{\mathsf{S}}_{\mathsf{sy}}(\omega) = \tilde{\mathrm{H}}^*(-\omega)\,\tilde{\mathsf{S}}_{\mathsf{xy}}(\omega) & \text{for (2),(4),(7),(8)}
\end{array}\right\}
$$

*Proof.*

| | | | |
|---|---|---|---|
| (1), (3), (5), (6). | $\check{S}_{sy}(z) = \check{S}_{rq}(z)$ | by Proposition 6 page 12 | and (B), (C) and (D) |
| | $= \check{H}(z)\,\check{S}_{pq}(z)$ | by Proposition 7 page 13 | and (A) |
| | $= \check{H}(z)\,\check{S}_{xq}(z)$ | by Lemma 3 page 15 | |
| | $= \check{H}(z)\,\check{S}_{xy}(z)$ | by Lemma 2 page 14 | |
| (1), (3), (5), (6). | $\tilde{S}_{sy}(\omega) = \check{S}_{sy}(z)\big\|_{z=e^{i\omega}}$ | by definition of Z | (Definition 10 page 21) |
| | $= \check{H}(z)\,\check{S}_{xy}(z)\big\|_{z=e^{i\omega}}$ | by previous result | (1) |
| | $= \tilde{H}(\omega)\,\tilde{S}_{xy}(\omega)$ | | |
| (2), (4), (7), (8). | $\check{S}_{sy}(z) = \check{S}_{rq}(z)$ | by Proposition 6 page 12 | and (B), (C) and (D) |
| | $= \check{H}^*(z^*)\,\check{S}_{pq}(z)$ | by Proposition 7 page 13 | and (A) |
| | $= \check{H}^*(z^*)\,\check{S}_{xq}(z)$ | by Lemma 3 page 15 | |
| | $= \check{H}^*(z^*)\,\check{S}_{xy}(z)$ | by Lemma 2 page 14 | |
| (2), (4), (7), (8). | $\tilde{S}_{sy}(\omega) = \check{S}_{sy}(z)\big\|_{z=e^{i\omega}}$ | by definition of Z | (Definition 10 page 21) |
| | $= \check{H}^*(z^*)\,\check{S}_{xy}(z)\big\|_{z=e^{i\omega}}$ | by previous result | (1) |
| | $= \tilde{H}^*(-\omega)\,\tilde{S}_{xy}(\omega)$ | by Proposition 12 page 21 | |

### 5.6 Case study: Non-linear system identification

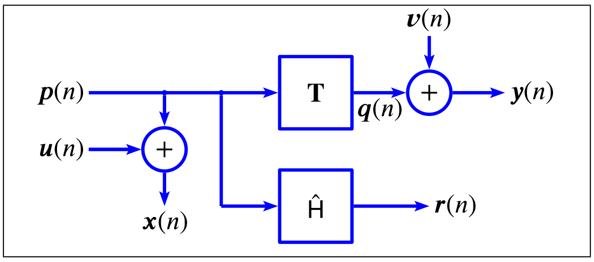



**Fig. 1** Least Square estimation (Proposition 10 page 18)

*Remark.*The defintion of "best" or "optimal" is given by a *cost function* $C(\hat{H})$. There are several possible cost functions. One possibility is to define an error $p(n) \triangleq q(n) - t(n)$. We note that if $\hat{H}$ is closely tuned to match T, then not only should $p(n)$ be close to 0 for all $n \in \mathbb{Z}$, but the *auto-correlation* $\hat{R}_{ee}(m)$ of $p(n)$ should also be close to 0 for all $m \in \mathbb{Z}$. Moreover by extension, the *auto-spectral density* $\tilde{S}_{vv}(\omega) \triangleq \breve{F}\,\hat{R}_{ee}(m)$ should also be close to 0. As such, we can define an arguably relevant *cost function* for the system S of Figure 1 (page 17) in terms of $\tilde{S}_{xx}$, $\tilde{S}_{yy}$ and $\tilde{S}_{xy}$. In the case of Papoulis's $R_{xy}(m)$, the development of such a cost function $C(\hat{H})$ might look something like this:

(1). $C_{rq}\left(\hat{H}\right)$

| | | |
|---|---|---|
| $\triangleq \breve{F}\,E\left([t(n) - q(n)][t(0) - q(0)]^*\right)$ | by definition of $C_{rq}$ | (Definition 3 page 18) |
| $= \tilde{F}\,[E\,[t(n)\,t^*(0)] - E\,[t(n)\,q^*(0)] - E\,[q(n)\,t^*(0)] + E\,[q(n)\,q^*(0)]]$ | by *linearity* of E | (Proposition 11 page 20) |
| $\triangleq \tilde{F}\,[R_{rr}(m) - R_{rq}(m) - R_{qr}(m) + R_{qq}(m)]$ | by definition of $R_{xy}$ | (Definition 2 page 2) |
| $\triangleq \boxed{\tilde{S}_{rr}(\omega) - \tilde{S}_{rq}(\omega) - \tilde{S}_{qr}(\omega) + \tilde{S}_{qq}(\omega)}$ | by definition of $\tilde{S}_{xy}$ | (Definition 1 page 1) |

Taking cue from the result of Remark 5.6, we arrive at *a* definition of cost:

**Definition 3.** *Let* S *be a system defined as in Figure 1 (page 17). Define the following* cost functions *for spectral* least-squares *estimates:*

$$\boxed{\mathrm{C}_{\mathrm{rq}}(\hat{\mathsf{H}}) \triangleq \tilde{\mathsf{S}}_{\mathsf{rr}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{qr}}(\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)}$$

*Remark.* Note that by Corollary 1 (page 4), $\tilde{\mathsf{S}}_{\mathsf{qr}} = \tilde{\mathsf{S}}_{\mathsf{rq}}{}^{*}$ for (1)–(4)… and thus the cost function C for (1)–(4) is *real-valued.* This in general is *not* true for (5)–(8). This in itself provides an argument, however weak that argument may be, for *not* selecting any of (5)–(8) as a standard for the definition of $\mathsf{R}_{\mathsf{xy}}(m)$.

Now for each of the eight $\mathsf{R}_{\mathsf{xy}}(m)$ definitions, we can transform the expression of $\mathrm{C}_{\mathrm{rq}}(\hat{\mathsf{H}})$ as given by Definition 3 into expressions involving $\hat{\mathsf{H}}$ (next lemma). In doing so, one might hope to be in a good position to take partial derivatives of the real and imaginary parts of $\hat{\mathsf{H}}$ to find an optimal *least-squares-like* solution for $\hat{\mathsf{H}}$.

**Lemma 4.** *Let* $\mathrm{C}_{\mathrm{rq}}(\hat{\mathsf{H}})$ *be defined as in Definition 3. Let (1)–(8) below correspond to the eight definitions of* $\mathsf{R}_{\mathsf{xy}}(m)$ *in Definition 2.*

$$
\begin{aligned}
(1).\quad \mathrm{C}_{\mathrm{rq}}\left(\hat{\mathsf{H}}\right) &= \overbrace{\tilde{\mathsf{S}}_{\mathsf{rr}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}{}^{*}(\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)}^{\text{by Definition 3 and Corollary 1 page 4}} &&= \overbrace{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)\left|\hat{\mathsf{H}}(\omega)\right|^{2} - \tilde{\mathsf{S}}_{\mathsf{py}}(\omega)\,\hat{\mathsf{H}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}{}^{*}(\omega)\,\hat{\mathsf{H}}^{*}(\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)}^{\text{by Corollary 2 page 7 and Proposition 9 page 16}}\\
(2).\quad \mathrm{C}_{\mathrm{rq}}\left(\hat{\mathsf{H}}\right) &= \tilde{\mathsf{S}}_{\mathsf{rr}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}{}^{*}(\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega) &&= \tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)\left|\hat{\mathsf{H}}(\omega)\right|^{2} - \tilde{\mathsf{S}}_{\mathsf{py}}(\omega)\,\hat{\mathsf{H}}^{*}(-\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}{}^{*}(\omega)\,\hat{\mathsf{H}}(-\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)\\
(3).\quad \mathrm{C}_{\mathrm{rq}}\left(\hat{\mathsf{H}}\right) &= \tilde{\mathsf{S}}_{\mathsf{rr}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}{}^{*}(\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega) &&= \tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)\left|\hat{\mathsf{H}}(-\omega)\right|^{2} - \tilde{\mathsf{S}}_{\mathsf{py}}(\omega)\,\hat{\mathsf{H}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}{}^{*}(\omega)\,\hat{\mathsf{H}}^{*}(\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)\\
(4).\quad \mathrm{C}_{\mathrm{rq}}\left(\hat{\mathsf{H}}\right) &= \tilde{\mathsf{S}}_{\mathsf{rr}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}{}^{*}(\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega) &&= \tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)\left|\hat{\mathsf{H}}(-\omega)\right|^{2} - \tilde{\mathsf{S}}_{\mathsf{py}}(\omega)\,\hat{\mathsf{H}}^{*}(-\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}{}^{*}(\omega)\,\hat{\mathsf{H}}(-\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)\\
(5).\quad \mathrm{C}_{\mathrm{rq}}\left(\hat{\mathsf{H}}\right) &= \tilde{\mathsf{S}}_{\mathsf{rr}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(-\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega) &&= \tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)\,\hat{\mathsf{H}}(\omega)\,\hat{\mathsf{H}}(-\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}(\omega)\,\hat{\mathsf{H}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}(-\omega)\,\hat{\mathsf{H}}(-\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)\\
(6).\quad \mathrm{C}_{\mathrm{rq}}\left(\hat{\mathsf{H}}\right) &= \tilde{\mathsf{S}}_{\mathsf{rr}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(-\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega) &&= \tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)\,\hat{\mathsf{H}}(\omega)\,\hat{\mathsf{H}}(-\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}(\omega)\,\hat{\mathsf{H}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}(-\omega)\,\hat{\mathsf{H}}(-\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)\\
(7).\quad \mathrm{C}_{\mathrm{rq}}\left(\hat{\mathsf{H}}\right) &= \tilde{\mathsf{S}}_{\mathsf{rr}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(-\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega) &&= \tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)\,\hat{\mathsf{H}}^{*}(\omega)\,\hat{\mathsf{H}}^{*}(-\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}(\omega)\,\hat{\mathsf{H}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}(-\omega)\,\hat{\mathsf{H}}^{*}(\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)\\
(8).\quad \mathrm{C}_{\mathrm{rq}}\left(\hat{\mathsf{H}}\right) &= \tilde{\mathsf{S}}_{\mathsf{rr}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{rq}}(-\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega) &&= \tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)\,\hat{\mathsf{H}}(\omega)\,\hat{\mathsf{H}}(-\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}(\omega)\,\hat{\mathsf{H}}^{*}(-\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}(-\omega)\,\hat{\mathsf{H}}(-\omega) + \tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)
\end{aligned}
$$

For the Papoulis $\mathsf{R}_{\mathsf{xy}}(m)$ definition (1), the $\mathrm{C}_{\mathrm{rq}}$ expression demonstrated in Lemma 4 is very useful. In particular, we can set the partial derivatives $\frac{\partial}{\partial\hat{\mathsf{H}}_R}\hat{\mathsf{H}}(\omega)$ and $\frac{\partial}{\partial\hat{\mathsf{H}}_I}\hat{\mathsf{H}}(\omega)$ of the real and imaginary parts of $\hat{\mathsf{H}}(\omega)$ to zero and solve the resulting two equations to find an optimal $\hat{\mathsf{H}}$ (as in Proposition 10 page 18).

However, this becomes troublesome in the case when encountering $\hat{\mathsf{H}}(-\omega)$ and the *impulse response* of $\hat{\mathsf{H}}$ is *complex-valued*—in which case in general $\hat{\mathsf{H}}(-\omega) \neq \hat{\mathsf{H}}^{*}(\omega)$.

Note that except for (1), *all* of the expressions demonstrated in Lemma 4 contain an $\hat{\mathsf{H}}(-\omega)$ and/or $\hat{\mathsf{H}}^{*}(-\omega)$.

This trouble provides an argument, however a weak one it might be, for choosing (1) as the standard definition of $\mathsf{R}_{\mathsf{xy}}(m)$.

**Proposition 10.** *Let* S *be the* system *illustrated in Figure 1 page 17.*

$$\boxed{\left\{\begin{array}{lll}\text{(A). x, u, and v are WSS} & & \text{and}\\ \text{(B). x, u, and v are uncorrelated} & & \text{and}\\ \text{(C). u and v are} & \text{zero-mean} & \text{and}\\ \text{(D). } \hat{\mathsf{H}} \text{ is} & \text{LTI} & \end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\displaystyle\arg\min_{\hat{\mathsf{H}}} \mathrm{C}_{\mathrm{rq}}(\hat{\mathsf{H}}) = \frac{\tilde{\mathsf{S}}_{\mathsf{xy}}{}^{*}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{uu}}(\omega)}\\ \text{for (1)}\end{array}\right\}}$$

*Proof.*

$$
\begin{aligned}
(1).\quad 0 &= \frac{\partial}{\partial\hat{\mathsf{H}}_R}\mathrm{C}_{\mathrm{rq}}\left[\hat{\mathsf{H}}(\omega)\right] = 2\,\hat{\mathsf{H}}_R(\omega)\,\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{py}}{}^{*}(\omega) + \cancelto{0}{\frac{\partial}{\partial\hat{\mathsf{H}}_R}\tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)} \Longrightarrow \hat{\mathsf{H}}_R(\omega) = \frac{\mathrm{R_e}\,\tilde{\mathsf{S}}_{\mathsf{py}}{}^{*}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)}\\
0 &= \frac{\partial}{\partial\hat{\mathsf{H}}_I}\mathrm{C}_{\mathrm{rq}}\left[\hat{\mathsf{H}}(\omega)\right] = 2\,\hat{\mathsf{H}}_I(\omega)\,\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega) - i\,\tilde{\mathsf{S}}_{\mathsf{py}}(\omega) + i\,\tilde{\mathsf{S}}_{\mathsf{py}}{}^{*}(\omega) + \cancelto{0}{\frac{\partial}{\partial\hat{\mathsf{H}}_R}\tilde{\mathsf{S}}_{\mathsf{qq}}(\omega)} \Longrightarrow \hat{\mathsf{H}}_I(\omega) = \frac{\mathrm{I_m}\,\tilde{\mathsf{S}}_{\mathsf{py}}{}^{*}(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)}
\end{aligned}
$$

$$\implies \hat{\mathsf{H}}(\omega) \triangleq \hat{\mathsf{H}}_R(\omega) + i\,\hat{\mathsf{H}}_I(\omega)\frac{\mathrm{R_e}\,\tilde{\mathsf{S}}_{\mathsf{py}}{}^*(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)} + \frac{i\,\mathrm{I_m}\,\tilde{\mathsf{S}}_{\mathsf{py}}{}^*(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)}$$

$$= \frac{\tilde{\mathsf{S}}_{\mathsf{py}}{}^*(\omega)}{\tilde{\mathsf{S}}_{\mathsf{pp}}(\omega)}$$

$$= \frac{\tilde{\mathsf{S}}_{\mathsf{xy}}{}^*(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega) - \tilde{\mathsf{S}}_{\mathsf{uu}}(\omega)} \qquad \text{by Proposition 5 page 11}$$

It follows immediately from Proposition 10 that, for (1) and in the special case of no input noise ($\mathrm{u}(n) = 0$), the standard estimate[6] $\hat{\mathsf{H}}_1$ is the optimal least-squares estimate of $\tilde{\mathsf{H}}$ (next).

**Corollary 5.** *Let* S *be the* system *illustrated in Figure 1 page 17.*

$$\left\{\begin{array}{l} (1).\ \text{hypotheses of Proposition 10 and} \\ (2).\ \mathrm{u}(n) = 0 \end{array}\right\} \implies \left\{ \hat{\mathsf{H}}(\omega) = \hat{\mathsf{H}}_1(\omega) \triangleq \frac{\tilde{\mathsf{S}}_{\mathsf{xy}}{}^*(\omega)}{\tilde{\mathsf{S}}_{\mathsf{xx}}(\omega)} \right\} \qquad \text{for (1)}$$

## 6 Which one?

Which definition of $\mathsf{R}_{\mathsf{xy}}(m)$ should we use? Any one of them is perfectly acceptable—as long as a clear definition is provided and that definition is used consistently. That being said, note the following:

1. The *expectation* operator $\mathrm{E}(\mathrm{X}\,\mathrm{Y}^*)$ is an *inner product*. As such, it would seem the most natural to follow the convention of other inner product definitions and thus put the conjugate $*$ on y (i.e. follow Papoulis):
   - $\langle \mathrm{x}(t)\,|\,\mathrm{y}(t)\rangle \triangleq \int_{t\in\mathbb{R}} \mathrm{x}(t)\,\mathrm{y}^*(t)\,\mathrm{t}$
   - $\langle \mathrm{x}(n)\,|\,\mathrm{y}(n)\rangle \triangleq \sum_{n\in\mathbb{Z}} \mathrm{x}(n)\,\mathrm{y}^*(n)$
   - $\langle \mathrm{X}\,|\,\mathrm{Y}\rangle \triangleq \mathrm{E}(\mathrm{X}\,\mathrm{Y}^*)$
2. If we view $\mathsf{R}_{\mathsf{xy}}(m)$ as an *analysis* of y in terms of x (or as a *projection* of y onto x), then it would seem more natural to put the conjugate on x (i.e. follow Kay). This is what is done in Fourier analysis when projecting a function $\mathrm{f}(t)$ onto the set of basis functions $\{e^{i\omega n} | \omega \in \mathbb{R}\}$, as in

$$\breve{\mathrm{F}}\,[\mathrm{y}(n)](\omega) \triangleq \langle \mathrm{y}(n)\,|\,e^{i\omega n}\rangle \qquad (\textit{project}\ \mathrm{y}(n)\ \text{onto}\ e^{i\omega n}\ \text{for some}\ \omega\in\mathbb{R})$$

$$\triangleq \sum_{n\in\mathbb{Z}} \mathrm{y}(n)\left[e^{+i\omega n}\right]^*$$

$$\triangleq \sum_{n\in\mathbb{Z}} \mathrm{y}(n)e^{-i\omega n}$$

   But arguably, a "projection of y onto x" would better be served by the use of $\mathsf{R}_{\mathsf{yx}}(m)$ rather than $\mathsf{R}_{\mathsf{xy}}(m)$.
3. As demonstrated in Section 5.6 (page 17), the Papoulis definition (1) is arguably more convenient for performing least-squares-like optimization.

# Appendix A Random Sequences

**Definition 4.** *[19, page 104], [2, page 30], [4, page 49] Let* $(\Omega, \mathbb{E}, \mathsf{P})$ *be a* probability space *and* X *a* random variable *on* $(\Omega, \mathbb{E}, \mathsf{P})$ *with* probability density function $p_x$.

*The* expectation operator $\mathrm{E_x}$ *on* X *is defined as* $\qquad \mathrm{E_x}\,\mathrm{X} \triangleq \int_{x\in\mathbb{F}} x p_X(x)\,\mathrm{x}.$

[6] [2, pages 98–100], [3, pages 106–109], [4, pages 187–190]

**Proposition 11(*Linearity of* E).** *[19, page 107], [2, page 30], Let* X *be a* random variable *on a* probability space $(\Omega, \mathbb{E}, \mathsf{P})$.

$$\mathrm{E}_{\mathsf{x}}(a\,\mathrm{X} + b\,\mathrm{Y} + c) = (a\,\mathrm{E}_{\mathsf{x}}\,\mathrm{X}) + (b\,\mathrm{E}_{\mathsf{x}}\,\mathrm{Y}) + c \quad \forall a,b,c \in \mathbb{R} \quad (\text{linear})$$

*Proof.*

$$\begin{aligned}
\mathrm{E}_{\mathsf{xy}}(a\,\mathrm{X} + b\,\mathrm{Y} + c) &\triangleq \int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} [ax + by + c] p_{\mathrm{x\,y}}(x,y)\,\mathrm{y}\,\mathrm{x} \qquad \text{by definition of E (Definition 4 page 19)} \\
&= \int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} ax p_{\mathrm{x\,y}}(x,y)\,\mathrm{y}\,\mathrm{x} + \int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} by p_{\mathrm{x\,y}}(x,y)\,\mathrm{y}\,\mathrm{x} + \int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} c p_{\mathrm{x\,y}}(x,y)\,\mathrm{y}\,\mathrm{x} \\
&= \int_{x\in\mathbb{R}} ax \underbrace{\int_{y\in\mathbb{R}} p_{\mathrm{x\,y}}(x,y)\,\mathrm{y}}_{p_{\mathrm{x}}(x)}\,\mathrm{x} + \int_{y\in\mathbb{R}} by \underbrace{\int_{x\in\mathbb{R}} p_{\mathrm{x\,y}}(x,y)\,\mathrm{x}}_{p_{\mathrm{y}}(y)}\,\mathrm{y} + c\underbrace{\int_{y\in\mathbb{R}}\int_{x\in\mathbb{R}} p_{\mathrm{x\,y}}(x,y)\,\mathrm{x}\,\mathrm{y}}_{1} \\
&= a\underbrace{\int_{x\in\mathbb{R}} x p_{\mathrm{x}}(x)\,\mathrm{x}}_{\mathrm{E\,X}} + b\underbrace{\int_{y\in\mathbb{R}} y p_{\mathrm{y}}(y)\,\mathrm{y}}_{\mathrm{E\,Y}} + c \\
&= (a\,\mathrm{E}_{\mathsf{x}}\,\mathrm{X}) + (b\,\mathrm{E}_{\mathsf{y}}\,\mathrm{Y}) + c
\end{aligned}$$

**Definition 5.** *[19, page 220], [4, pages 109–111], [2, page 3] Let* $(\!(\mathrm{x}(n))\!)_{n\in\mathbb{Z}}$ *be a* random sequence.

$(\!(\mathrm{x}(n))\!)$ *is* ***wide sense stationary (WSS)*** *if*

$$\begin{array}{lll}
(1). & \mathrm{E}\,\mathrm{x}(n) = \mathrm{E}\,\mathrm{x}(k) & \forall n,k\in\mathbb{Z} \quad \textit{and} \\
(2). & \mathrm{E}\left[\mathrm{x}(n+m)\,\mathrm{x}(n)\right] = \mathrm{E}\left[\mathrm{x}(k+m)\,\mathrm{x}(k)\right] & \forall n,k,m\in\mathbb{Z}
\end{array}$$

**Definition 6.** *[19, page 221] Let* $(\!(\mathrm{x}(n))\!)_{n\in\mathbb{Z}}$ *and* $(\!(\mathrm{y}(n))\!)_{n\in\mathbb{Z}}$ *be* random sequences.

$(\!(\mathrm{x}(n))\!)$ *and* $(\!(\mathrm{y}(n))\!)$ *are* ***jointly wide sense stationary (J-WSS)*** *if*

$$\begin{array}{lll}
(1). & (\!(\mathrm{x}(n))\!) \textit{ is } \text{wide sense stationary} & \textit{Definition 5 and} \\
(2). & (\!(\mathrm{y}(n))\!) \textit{ is } \text{wide sense stationary} & \textit{Definition 5 and} \\
(3). & \mathrm{E}\left[\mathrm{x}(n+m)\,\mathrm{y}(n)\right] = \mathrm{E}\left[\mathrm{x}(k+m)\,\mathrm{y}(k)\right] & \forall n,k,m\in\mathbb{Z}
\end{array}$$

## Appendix B Operations on Sequences

### *B.1 Convolution operation*

**Definition 7.** *[6, page 1], [24, page 23], [11, page 31] Let* $X^Y$ *be the set of all functions from a set* $Y$ *to a set* $X$*. Let* $\mathbb{Z}$ *be the set of integers.*

*A function* f *in* $X^Y$ *is a* ***sequence*** *over* $X$ *if* $Y = \mathbb{Z}$.
*A sequence may be denoted in the form* $(\!(x_n)\!)_{n\in\mathbb{Z}}$ *or simply as* $(\!(x_n)\!)$.

**Definition 8.** *[13, page 347] Let* $(\mathbb{C}, +, \cdot, 0, 1)$ *be the* field of complex numbers.

*The* ***space of all absolutely square summable sequences*** $\ell^2_{\mathbb{C}}$ *over* $\mathbb{C}$ *is defined as*

$$\ell^2_{\mathbb{C}} \triangleq \left\{ (\!(x_n)\!)_{n\in\mathbb{Z}} \,\middle|\, \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty \right\}$$

**Definition 9.**

*The* convolution *operation* $\star$ *is defined as*

$$(\!(x_n)\!) \star (\!(y_n)\!) \triangleq (\!(\sum_{m\in\mathbb{Z}} x_m y_{n-m})\!)_{n\in\mathbb{Z}} \qquad \forall (\!(x_n)\!)_{n\in\mathbb{Z}}, (\!(y_n)\!)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{C}}$$

## *B.2 Z-transform*

**Definition 10.**[7] *Let* $(\!|\mathrm{x}(n)|\!)_{n\in\mathbb{Z}}$ *be a sequence.*

*The* z-transform Z *of* $(\!|\mathrm{x}(n)|\!)$ *is defined as*

$$[\mathrm{Z}(\!|\mathrm{x}(n)|\!)](z) \triangleq \sum_{n\in\mathbb{Z}} \mathrm{x}(n) z^{-n} \qquad \forall (\!|\mathrm{x}(n)|\!)\in\ell^2_{\mathbb{C}}$$

**Proposition 12.** *Let* $X(z) \triangleq \mathrm{Z}\,\mathrm{x}(n)$ *be the* z-transform *of* $\mathrm{x}(n)$.

$$\underbrace{\left\{\check{\mathrm{x}}(z) \triangleq \mathrm{Z}(\!|\mathrm{x}(n)|\!)\right\}}_{\text{(Definition 10 page 21)}} \implies \left\{\begin{array}{llll} (1). & \mathrm{Z}(\!|\alpha\,\mathrm{x}(n)|\!) & = \alpha\,\check{\mathrm{x}}(z) & \forall (\!|x_n|\!)\in\ell^2_{\mathbb{C}} \text{ and} \\ (2). & \mathrm{Z}(\!|\mathrm{x}[n-k]|\!) & = z^{-k}\,\check{\mathrm{x}}(z) & \forall (\!|x_n|\!)\in\ell^2_{\mathbb{C}} \text{ and} \\ (3). & \mathrm{Z}(\!|\mathrm{x}(-n)|\!) & = \check{\mathrm{x}}\left(\frac{1}{z}\right) & \forall (\!|x_n|\!)\in\ell^2_{\mathbb{C}} \text{ and} \\ (4). & \mathrm{Z}(\!|\mathrm{x}^*(n)|\!) & = \check{\mathrm{x}}^*\left(z^*\right) & \forall (\!|x_n|\!)\in\ell^2_{\mathbb{C}} \text{ and} \\ (5). & \mathrm{Z}(\!|\mathrm{x}^*(-n)|\!) & = \check{\mathrm{x}}^*\left(\frac{1}{z^*}\right) & \forall (\!|x_n|\!)\in\ell^2_{\mathbb{C}} \end{array}\right\}$$

[7] *Laurent series*: [1, page 49]

*Proof.*

$$
\begin{aligned}
\alpha \mathbb{Z}\,\check{\mathrm{x}}(z) &\triangleq \alpha\,\mathrm{Z}⦇\mathrm{x}(n)⦈ && \text{by definition of } \check{\mathrm{x}}(z) \\
&\triangleq \alpha \sum_{n\in\mathbb{Z}} \mathrm{x}(n) z^{-n} && \text{by definition of Z operator} \\
&\triangleq \sum_{n\in\mathbb{Z}} (\alpha\,\mathrm{x}(n)) z^{-n} && \text{by \textit{distributive} property} \\
&\triangleq \mathrm{Z}⦇\alpha\,\mathrm{x}(n)⦈ && \text{by definition of Z operator} \\
z^{-k}\,\check{\mathrm{x}}(z) &= z^{-k}\,\mathrm{Z}⦇\mathrm{x}(n)⦈ && \text{by definition of } \check{\mathrm{x}}(z) && \text{(left hypothesis)} \\
&\triangleq z^{-k} \sum_{n=-\infty}^{n=+\infty} \mathrm{x}(n) z^{-n} && \text{by definition of Z} && \text{(Definition 10 page 21)} \\
&= \sum_{n=-\infty}^{n=+\infty} \mathrm{x}(n) z^{-n-k} \\
&= \sum_{m-k=-\infty}^{m-k=+\infty} \mathrm{x}[m-k] z^{-m} && \text{where } m \triangleq n+k && \implies n = m-k \\
&= \sum_{m=-\infty}^{m=+\infty} \mathrm{x}[m-k] z^{-m} \\
&= \sum_{n=-\infty}^{n=+\infty} \mathrm{x}[n-k] z^{-n} && \text{where } n \triangleq m \\
&\triangleq \mathrm{Z}⦇\mathrm{x}[n-k]⦈ && \text{by definition of Z} && \text{(Definition 10 page 21)} \\
\mathrm{Z}⦇\mathrm{x}^*(n)⦈ &\triangleq \sum_{n\in\mathbb{Z}} \mathrm{x}^*(n) z^{-n} && \text{by definition of Z} && \text{(Definition 10 page 21)} \\
&\triangleq \left( \sum_{n\in\mathbb{Z}} \mathrm{x}(n) (z^*)^{-n} \right)^* && \text{by definition of Z} && \text{(Definition 10 page 21)} \\
&\triangleq \check{\mathrm{x}}^*(z^*) && \text{by definition of Z} && \text{(Definition 10 page 21)} \\
\mathrm{Z}⦇\mathrm{x}(-n)⦈ &\triangleq \sum_{n\in\mathbb{Z}} \mathrm{x}(-n) z^{-n} && \text{by definition of Z} && \text{(Definition 10 page 21)} \\
&= \sum_{-m\in\mathbb{Z}} \mathrm{x}[m] z^{m} && \text{where } m \triangleq -n && \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} \mathrm{x}[m] z^{m} && \text{because } ⦇\mathrm{x}(n)⦈, ⦇z^n⦈ \in \ell^2_{\mathbb{C}} && \text{(Definition 8 page 20)} \\
&= \sum_{m\in\mathbb{Z}} \mathrm{x}[m] \left(\frac{1}{z}\right)^{-m} \\
&\triangleq \check{\mathrm{x}}\left(\frac{1}{z}\right) && \text{by definition of Z} && \text{(Definition 10 page 21)} \\
\mathrm{Z}⦇\mathrm{x}^*(-n)⦈ &\triangleq \sum_{n\in\mathbb{Z}} \mathrm{x}^*(-n) z^{-n} && \text{by definition of Z} && \text{(Definition 10 page 21)} \\
&= \sum_{-m\in\mathbb{Z}} \mathrm{x}^*[m] z^{m} && \text{where } m \triangleq -n && \implies n = -m \\
&= \sum_{m\in\mathbb{Z}} \mathrm{x}^*[m] z^{m} && \text{because } ⦇\mathrm{x}(n)⦈, ⦇z^n⦈ \in \ell^2_{\mathbb{C}} && \text{(Definition 8 page 20)} \\
&= \sum_{m\in\mathbb{Z}} \mathrm{x}^*[m] \left(\frac{1}{z}\right)^{-m} \\
&= \left( \sum_{m\in\mathbb{Z}} \mathrm{x}[m] \left(\frac{1}{z^*}\right)^{-m} \right)^* \\
&\triangleq \check{\mathrm{x}}^*\left(\frac{1}{z^*}\right) && \text{by definition of Z} && \text{(Definition 10 page 21)}
\end{aligned}
$$

**Proposition 13(Convolution Theorem).** *Let $\star$ be the convolution operator (Definition 9 page 20).*

$$\mathrm{Z}\underbrace{\left(\llparenthesis x_n\rrparenthesis \star \llparenthesis y_n\rrparenthesis\right)}_{\textit{sequence convolution}} = \underbrace{\left(\mathrm{Z}\llparenthesis x_n\rrparenthesis\right)\left(\mathrm{Z}\llparenthesis y_n\rrparenthesis\right)}_{\textit{series multiplication}} \qquad \forall \llparenthesis x_n\rrparenthesis_{n\in\mathbb{Z}}, \llparenthesis y_n\rrparenthesis_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{C}}$$

*Proof.*

$$\begin{aligned}
[\mathrm{Z}(x\star y)](z) &\triangleq \mathrm{Z}\left(\sum_{m\in\mathbb{Z}} x_m y_{n-m}\right) && \text{by Definition 9 page 20}\\
&\triangleq \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n} && \text{by Definition 10 page 21}\\
&= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} x_m y_{n-m} z^{-n}\\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} x_m y_{n-m} z^{-n}\\
&= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} x_m y_k z^{-(m+k)} && \text{where } k = n-m \iff n = m+k\\
&= \left[\sum_{m\in\mathbb{Z}} x_m z^{-m}\right]\left[\sum_{k\in\mathbb{Z}} y_k z^{-k}\right]\\
&\triangleq \left(\mathrm{Z}\llparenthesis x_n\rrparenthesis\right)\left(\mathrm{Z}\llparenthesis y_n\rrparenthesis\right) && \text{by Definition 10 page 21}
\end{aligned}$$

**Lemma 5.** *Let* $\mathrm{H}$ *be a* linear time-invariant *operator with* impulse response $\llparenthesis \mathrm{h}(n)\rrparenthesis$. *Let* $\llparenthesis \mathrm{y}(n)\rrparenthesis \triangleq \llparenthesis \mathrm{H}\,\mathrm{x}(n)\rrparenthesis$.

$$\left\{\begin{array}{lll}
\text{(A). } \llparenthesis \mathrm{x}(n)\rrparenthesis \text{ and } \llparenthesis \mathrm{y}(n)\rrparenthesis \text{ are real-valued} & \text{and}\\
\text{(B). } \llparenthesis \mathrm{x}(n)\rrparenthesis \text{ and } \llparenthesis \mathrm{h}(n)\rrparenthesis \text{ are in } \ell^2_{\mathbb{C}} & \text{and}\\
\text{(C). } \llparenthesis \mathrm{x}(n)\rrparenthesis \neq \llparenthesis\cdots,0,0,0,\cdots\rrparenthesis & \text{and}\\
\text{(D). } \llparenthesis \mathrm{h}(n)\rrparenthesis \text{ is linear time-invariant}
\end{array}\right\} \implies \left\{\begin{array}{ll}
\text{(1). } \llparenthesis \mathrm{h}(n)\rrparenthesis \text{ is real-valued} & \text{and}\\
\text{(2). } \check{\mathrm{H}}(z) = \check{\mathrm{H}}^*(z^*)
\end{array}\right\}$$

*Proof.* 1.Let $\mathrm{h}_R(n)$ and $\mathrm{h}_I(n)$ be the *real-part* and *imaginary-part*, respectively, of $\mathrm{h}(n)$.
2.lemma: $\sum_{m\in\mathbb{Z}} \mathrm{h}_I(m)\,\mathrm{x}(n-m) = 0$

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} \mathrm{h}_R(m)\,\mathrm{x}(n-m) + i\sum_{m\in\mathbb{Z}} \mathrm{h}_I(m)\,\mathrm{x}(n-m)\\
&= \sum_{m\in\mathbb{Z}} \mathrm{h}(m)\,\mathrm{x}(n-m) && \text{by definitions of } \mathrm{h}_R \text{ and } \mathrm{h}_I && \text{item (1)}\\
&= \mathrm{y}(n) && \text{because H is } \textit{LTI} && \text{hypothesis (D)}\\
&= \mathrm{y}^*(n) && \text{because y is } \textit{real-valued} && \text{hypothesis (A)}\\
&= \left(\sum_{m\in\mathbb{Z}} \mathrm{h}(m)\,\mathrm{x}(n-m)\right)^* && \text{because H is } \textit{LTI} && \text{hypothesis (D)}\\
&= \sum_{m\in\mathbb{Z}} \mathrm{h}^*(m)\,\mathrm{x}^*(n-m) && \text{by } \textit{antiautomorphic} \text{ property} && \text{(Definition 11 page 24)}\\
&= \sum_{m\in\mathbb{Z}} \mathrm{h}^*(m)\,\mathrm{x}(n-m) && \text{because y is } \textit{real-valued} && \text{hypothesis (A)}\\
&= \sum_{m\in\mathbb{Z}} \mathrm{h}_R(m)\,\mathrm{x}(n-m) - i\sum_{m\in\mathbb{Z}} \mathrm{h}_I(m)\,\mathrm{x}(n-m) && \text{by definitions of } \mathrm{h}_R \text{ and } \mathrm{h}_I && \text{item (1)}\\
\implies & \boxed{\sum_{m\in\mathbb{Z}} \mathrm{h}_I(m)\,\mathrm{x}(n-m) = 0}
\end{aligned}$$

3.Notes:
(a)Without hypothesis (C), it is trivial to satisfy (2) lemma.
(b)Without hypothesis (B), it is simple to satisfy (2) lemma with
$\mathrm{h}(n) = \llparenthesis\cdots,0,0,0,i,-i,0,0,0,\cdots\rrparenthesis$ and $\mathrm{x}(n) = \llparenthesis\cdots,1,1,1,\cdots\rrparenthesis$

(c)Without hypothesis (D), it is trivial to satisfy (2) lemma with $\mathrm{H}\,\mathrm{x}(n) \triangleq \mathrm{R_e}\left(\sum_{m\in\mathbb{Z}} \mathrm{h}(m)\,\mathrm{x}(n-m)\right)$

4.Proof that $\mathrm{h}(n)$ is *real-valued*:

$$\begin{aligned}
(2)\text{ lemma} &\implies \check{\mathrm{H}}_I(z)\,\check{\mathrm{X}}(z) = 0 && \text{by } \textit{Convolution Theorem} && \text{(Proposition 13 page 23)}\\
&\implies \check{\mathrm{H}}_I(z) = 0 && \text{because } \mathrm{x}(n) \neq (\!|\cdots,0,0,0,\cdots|\!) && \text{hypothesis (C)}\\
&\implies \mathrm{h}_I(n) = (\!|\cdots,0,0,0,\cdots|\!)\\
&\implies \mathrm{h}(n) \triangleq \mathrm{h}_R(n) + i\,\mathrm{h}_I(n) \text{ is } \textit{real-valued}
\end{aligned}$$

5.Proof that $\check{\mathrm{H}}(z) = \check{\mathrm{H}}^*(z^*)$:

$$\begin{aligned}
\check{\mathrm{H}}(z) &\triangleq \mathrm{Z}(\!|\mathrm{x}(n)|\!) && \text{by definition of } \check{\mathrm{H}}(z)\\
&= \mathrm{Z}(\!|\mathrm{x}^*(n)|\!) && \text{because } \mathrm{x}(n) \text{ is } \textit{real-valued} && \text{(item (4) page 24)}\\
&= \check{\mathrm{H}}^*(z^*) && \text{by Proposition 12}
\end{aligned}$$

**Lemma 6.** *Let $\tilde{\mathsf{H}}(\omega)$ be the DTFT (Definition 1 page 1) of a sequence* $\mathrm{h}(n)$.

$$\left\{\mathrm{h}(n) \textit{ is } \text{real-valued}\right\} \implies \left\{\tilde{\mathsf{H}}(-\omega) = \tilde{\mathsf{H}}^*(\omega) \ \text{(conjugate symmetric)}\right\}$$

*Proof.*

$$\begin{aligned}
\tilde{\mathsf{H}}(-\omega) &\triangleq \sum_{n\in\mathbb{Z}} \mathrm{h}(n)e^{-i(-\omega)n} && \text{by definition of } \tilde{\mathsf{H}}(\omega) && \text{(Definition 1 page 1)}\\
&= \sum_{n\in\mathbb{Z}} \mathrm{h}(n)e^{i\omega n}\\
&= \left[\sum_{n\in\mathbb{Z}} \mathrm{h}^*(n)e^{i\omega n}\right]^* && \text{by } \textit{antiautomorphic} \text{ property of *-algebras} && \text{(Definition 11 page 24)}\\
&= \left[\sum_{n\in\mathbb{Z}} \mathrm{h}(n)e^{-i\omega n}\right]^* && \text{by } \textit{real-valued} \text{ hypothesis}\\
&\triangleq \tilde{\mathsf{H}}^*(\omega) && \text{by definition of } \tilde{\mathsf{H}}(\omega) && \text{(Definition 1 page 1)}
\end{aligned}$$

## Appendix C Normed Algebras

**Definition 11.** *[21, page 178], [8, page 241 ] xsym]∗ Let $\mathbb{A}$ be an* algebra.

*The pair $(\mathbb{A}, *)$ is a* ***$*$-algebra****, or “****star-algebra****”, if*
1. $(x+y)^* = x^* + y^*$ $\forall x,y\in\mathbb{A}$ (distributive) *and*
2. $(\alpha x)^* = \bar{\alpha}x^*$ $\forall x\in\mathbb{A}, \alpha\in\mathbb{C}$ (conjugate linear) *and*
3. $(xy)^* = y^*x^*$ $\forall x,y\in\mathbb{A}$ (antiautomorphic) *and*
4. $x^{**} = x$ $\forall x\in\mathbb{A}$ (involutory)

*The operator $*$ is called an* involution *on the algebra $\mathbb{A}$.*

**Definition 12(Hermitian components).** *[18, page 430], [21, page 179], [8, page 242 ] Let $(X, \|\cdot\|)$ be a $*$-*algebra *(Definition 11 page 24).*

*For $x \in X$, the* real part *of $x$ is defined as* $\mathrm{R_e}\,x \triangleq \frac{1}{2}\left(x + x^*\right)$

*For $x \in X$, the* imaginary part *of $x$ is defined as* $\mathrm{I_m}\,x \triangleq \frac{1}{2i}\left(x - x^*\right)$

**Example 1.**[14, pages 106–107] Let $\mathbb{C}$ be the *set of complex numbers* and $* : \mathbb{C} \to \mathbb{C}$ the *conjugate operator*. The pair $(\mathbb{C}, *)$ is an $*$-*algebra*.

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