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Generalized symmetric bi-derivations in MV-algebras

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Abstract: In this study, we define the generalized symmetric bi-derivation and its trace on MV-algebras and give its examples. We investigate some properties of generalized symmetric bi-derivation's trace. We introduce isotone generalized symmetric bi-derivation and bi-additive generalized symmetric bi-derivation. We show that if A is a linearly ordered MV-algebra, Γ is a bi-additive generalized symmetric bi-derivation on A related to symmetric bi-derivation D and γ is a trace of Γ, then $\gamma = 0$ or $\gamma 1 = 1$

Keywords: MV-algebra, symmetric bi-derivation, generalized symmetric bi-derivation

1 Introduction

MV-algebra was introduced by C. C. Chang in 1958 ([3]). He used the concept of MV-algebra for making algebraic proof of the completeness theorem of infinite valued Lukasiewicz propositional calculus. Since MV-algebra is generalization of Boolean algebra, it has many applications and many researchers investigated different properties of MV-algebras. ([2], [4], [5], [7])

The concept of derivation was introduced by Posner in 1957 in ring. After this study, many authors examine properties of ring by using many kind of derivation such as generalized derivation, symmetric bi-derivation, permuting tri-derivation, reverse derivation multiplicative derivation, etc. For example, they investigated commutativity of ring by using derivation. In 1975, Szasz introduced the derivation on lattices. Xin et al. examined properties of derivation on lattices and gave some characterization of lattices. For example, they gave relation between modular and distributive lattices and ordered elements of lattices by using derivation. After then, many authors introduced kind of derivation such as ring, in lattices and examine some properties. Similarly, many kind of derivations moved on different algebraic structures and their properties were investigated using derivation.

In MV-algebras, the concept of derivation was defined by N. O. Alshehri in 2010. In 2013, H. Yazarli introduced symmetric bi-derivation on MV-algebras. In this study, we define the generalized symmetric bi-derivation and its trace on MV-algebras and give its examples. We investigate some properties of generalized symmetric bi-derivation's trace. We introduce isotone generalized symmetric bi-derivation and bi-additive generalized symmetric bi-derivation. We show that if A is a linearly ordered MV-algebra, Γ is a bi-additive generalized symmetric bi-derivation on A related to symmetric bi-derivation D and γ is a trace of Γ, then $\gamma = 0$ or $\gamma 1 = 1$. Also, we show that if A is a linearly ordered MV-algebra, Γ_1 and Γ_2 are bi-additive generalized symmetric bi-derivations on A related to D_1 and D_2 respectively and d_1, d_2 are traces of D_1, D_2, γ_1, γ_2 are traces of Γ_1 and Γ_2 respectively. If $\gamma_1\gamma_2 = 0$ where $(\gamma_1\gamma_2)(a) = \gamma_1(\gamma_2 a)$ for all $a \in A$, then $\gamma_1 = 0$ or $\gamma_2 = 0$.

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Definition 1. *[7] Let us define $+$ a binary operation, $'$ a unary operation on the set A and 0 be a constant in A. That the following axioms are satisfied for $a,b \in A$,*
(1) $(A,+,0)$ *is a commutative monoid,*
(2) $\left(a'\right)' = a$,
(3) $0' + a = 0'$,
(4) $\left(a' + b\right)' + b = \left(b' + a\right)' + a$,
then we say that $(A,+,',0)$ is MV-algebra.

Let us denote $1 = 0'$, $a \cdot b = \left(a' + b'\right)'$, $a \vee b = a + \left(b \cdot a'\right)$, $a \wedge b = a \cdot \left(b + a'\right)$ for all $a,b \in A$. Then we have that $(A,\cdot,1)$ is a commutative monoid, $(A,\vee,\wedge,0,1)$ is a bounded distributive lattice. Let A be an MV-algebra and $\emptyset \neq B \subseteq A$. We say that B is subalgebra of A if and only if $a+b \in B$ and $a' \in B$ for all $a,\, b \in B$. Let us define the relation on MV-algebra A as following can define a partial order $\leq$ by setting

$$a \leq b \;\Leftrightarrow a \wedge b = a \text{ for each } a,b \in A,$$

then $(A,\leq)$ is a partially ordered set. If the order relation $\leq$ on A is total then A is called linearly ordered.

Let A be an MV-algebra and $B(A) = \{a \in A : a+a = a\} = \{a \in A : a \cdot a = a\}$. Then $(B(A),+,',0)$ is subalgebra of A.

If A is an MV-algebra, then we have the following situations for $a,b,c \in A$,
(1) $a+1 = 1$,
(2) $a+a' = 1$,
(3) $a \cdot a' = 0$,
(4) $a+b = 0$ implies $a = b = 0$,
(5) $a \cdot b = 1$ implies $a = b = 1$,
(6) $a \leq b$ implies $a \vee c \leq b \vee c$ and $a \wedge c \leq b \wedge c$,
(7) $a \leq b$ implies $a+c \leq b+c$ and $a \cdot c \leq b \cdot c$,
(8) $a \leq b \Leftrightarrow b' \leq a'$,
(9) $a+b = b \Leftrightarrow a \cdot b = a$.

Theorem 1. *[5] In an MV-algebra A, the following situations are equivalent for $a \in A$,*
(i) $a \in B(A)$,
(ii) $a \vee a' = 1$,
(iii) $a \wedge a' = 0$,
(iv) $a+a = a$,
(v) $a \cdot a = a$,
(vi) $a+b = a \vee b$, *for* $b \in A$,
(vii) $a \cdot b = a \wedge b$, *for* $b \in A$.

Theorem 2. *[3]In an MV-algebra A, the following situations are equivalent for all $a,b \in A$,*
(i) $a \leq b$,
(ii) $b+a' = 1$,
(iii) $a \cdot b' = 0$.

Definition 2. *[3]Let A be an MV-algebra and $\emptyset \neq X \subseteq A$. If the following situations are satisfied,*
(i) $0 \in X$,
(ii) $a+b \in X$ *for all* $a,b \in X$,

(iii) $a \in X$ and $b \leq a$ imply $b \in X$,
then X is called an ideal of A.

Proposition 1. *[3]In linearly ordered MV-algebra A, if $a+b=a+c$ and $a+c \neq 1$ then $b=c$.*

Definition 3. *[1]Let $d: A \to A$ be a function on MV-algebra A. We say that d is a derivation of A, if it satisfies for $a,b \in A$,*

$$d(a \cdot b) = (d(a) \cdot b) + (a \cdot d(b)).$$

Definition 4. *Let A be an MV-algebra. We say that a mapping $D: A \times A \to A$ is a symmetric if $D(a,b) = D(b,a)$ holds for all $a,b \in A$.*

Definition 5. *In MV-algebra A, a mapping $d: A \to A$ defined by $d(a) = D(a,a)$ is called trace of D, where $D: A \times A \to A$ is a symmetric mapping.*

We denote $d(a) = da$.

Definition 6. *Let $D: A \times A \to A$ be a symmetric mapping in MV-algebra A. We say that D is a symmetric bi-derivation on A, if it satisfies the following situation,*

$$D(a \cdot b, c) = (D(a,c) \cdot b) + (a \cdot D(b,c))$$

for all $a,b,c \in A$.

Obviously, a symmetric bi-derivation D on A satisfies the relation $D(a, b \cdot c) = (D(a,b) \cdot c) + (b \cdot D(a,c))$ for all $a,b,c \in A$.

Proposition 2. *Let A be an MV -algebra, D be a symmetric bi-derivation on A and d be a trace of D. Then, for all $x \in A$,*
(i) $d0 = 0$,
(ii) $da \cdot a' = a \cdot da' = 0$,
(iii) $da = da + (a \cdot D(a,1))$,
(iv) $da \leq a$,
(v) If X is an ideal of an MV-algebra, $d(X) \subseteq X$.

2 Generalized symmetric bi-derivation of MV-algebras

Throughout this article, A is an MV-algebra.

Definition 7. *Let $D: A \times A \to A$ be a symmetric bi-derivation and $\Gamma: A \times A \to A$ be a symmetric mapping. We say that Γ is a generalized symmetric bi-derivation related to D, if it satisfies*

$$\Gamma(a \cdot b, c) = (\Gamma(a,c) \cdot b) + (a \cdot D(b,c))$$

for all $a,b,c \in A$. The mapping $\delta: A \to A$ defined by $\gamma(a) = \Gamma(a,a)$ is called the trace of generalized symmetric bi-derivation Γ.

Example 1. Let $A = \{0, x, y, 1\}$. Consider the following tables,

+	0	x	y	1
0	0	x	y	1
x	x	x	1	1
y	y	1	y	1
1	1	1	1	1

'	0	x	y	1
	1	y	x	0

Then $(A, +, ', 0)$ is an MV-algebra. Let us define a map $D : A \times A \to A$ by,

$$D(a,b) = \begin{cases} y, & (a,b) \in \{(y,y), (y,1), (1,y)\} \\ 0, & \text{otherwise} \end{cases}.$$

Then D is a symmetric bi-derivation of A. Therefore the mapping Γ defined by

$$\Gamma(a,b) = \begin{cases} x, & (a,b) \in \{(x,x), (x,1), (1,x)\} \\ y, & (a,b) \in \{(y,y), (y,1), (1,y)\} \\ 0, & a = 0 \text{ or } b = 0 \\ 0, & (a,b) \in \{(x,y), (y,x)\} \\ 1, & (a,b) = (1,1) \end{cases}$$

is generalized symmetric bi-derivation related to D.

Example 2. Let $A = [0,1]$ be the real unit interval. For $a, b \in A$, if we define $a \oplus b = \min\{1, a+b\}$, $a \cdot b = \max\{0, a+b-1\}$ and $a' = 1 - a$, then $(A, \oplus, ', 0)$ is an MV-algebra. For all $n \geq 2$, $n \in \mathbb{Z}$, $A_n = \left\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}$ is a linearly ordered MV-algebra. Let us take $A_3 = \left\{0, \frac{1}{2}, 1\right\}$ and define mappings

$$D(a,b) = \begin{cases} \frac{1}{2}, & a, b \in \left\{\frac{1}{2}\right\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\Gamma(a,b) = \begin{cases} 1, & a, b \in \{1\} \\ \frac{1}{2}, & (a,b) \in \left\{\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 1\right), \left(1, \frac{1}{2}\right)\right\} \\ 0, & \text{otherwise} \end{cases}.$$

Γ is a generalized symmetric bi-derivation with D. But Γ is not a symmetric bi-derivation. Because $\Gamma\left(\frac{1}{2} \cdot 1, 1\right) = \Gamma\left(\frac{1}{2}, 1\right) = \frac{1}{2}$ and $\left(\Gamma\left(\frac{1}{2}, 1\right) \cdot 1\right) + \left(\frac{1}{2} \cdot \Gamma(1,1)\right) = \frac{1}{2} + \frac{1}{2} = 1$. And so, $\Gamma\left(\frac{1}{2} \cdot 1, 1\right) \neq \left(\Gamma\left(\frac{1}{2}, 1\right) \cdot 1\right) + \left(\frac{1}{2} \cdot \Gamma(1,1)\right)$.

Proposition 3. *Let Γ be a generalized symmetric bi-derivation on A related to symmetric bi-derivation D with its trace d and γ be a trace of Γ. Then, for all $a \in A$,*
(i) $\gamma 0 = 0$,
(ii) $\gamma a \cdot a' = a \cdot da' = 0$,

(iii) *$\gamma a = \gamma a + (a \cdot D(a,1)) = (\Gamma(1,a) \cdot a) + da$,*
(iv) *$\gamma a \leq a$,*
(v) *$\gamma(X) \subseteq X$ where X is an ideal of A.*

Proof. (i) For all $a \in A$,

$$\Gamma(a,0) = \Gamma(a,0\cdot 0) = (\Gamma(a,0)\cdot 0) + (0 \cdot D(a,0)) = 0+0 = 0.$$

Since γ is the trace of Γ,

$$\gamma 0 = \Gamma(0,0) = \Gamma(0\cdot 0,0) = (\Gamma(0,0)\cdot 0) + (0\cdot D(0,0)) = 0+0 = 0.$$

(ii) For all $a \in A$,

$$0 = \Gamma(a,0) = \Gamma\left(a, a\cdot a'\right) = \left(\Gamma(a,a)\cdot a'\right) + \left(a \cdot D\left(a,a'\right)\right)$$

and so, $\gamma a \cdot a' = 0$ and $a \cdot D\left(a,a'\right) = 0$. On the other hand, we obtain

$$0 = \Gamma\left(0,a'\right) = \Gamma\left(a\cdot a', a'\right) = \left(\Gamma\left(a,a'\right)\cdot a'\right) + \left(a\cdot D\left(a',a'\right)\right).$$

That is, $a \cdot da' = 0$ and $\Gamma\left(a,a'\right)\cdot a' = 0$ for all $a \in A$.
(iii) For all $a \in A$,

$$\gamma a = \Gamma(a,a) = \Gamma(a,a\cdot 1) = (\Gamma(a,a)\cdot 1) + (a\cdot D(a,1)) = \gamma a + (a \cdot D(a,1))$$

and

$$\gamma a = \Gamma(a,a) = \Gamma(1\cdot a,a) = (\Gamma(1,a)\cdot a) + (1\cdot D(a,a)) = (\Gamma(1,a)\cdot a) + da.$$

(iv) For all $a \in A$,

$$1 = 0' = \left(\gamma a \cdot a'\right)' = \left[\left((\gamma a)' + \left(a'\right)'\right)'\right]' = (\gamma a)' + a$$

By Theorem 2, $\gamma a \leq a$, $a \in A$.
(v) If $b \in \gamma(X)$, then $\gamma(a) = b$ for some $a \in X$. From (iv), $\gamma(a) \leq a$ and so $b \in X$, since X is an ideal of A. Hence $\gamma(X) \subseteq X$.

Corollary 1. *For all $a \in A$, $D\left(a,a'\right) \leq \left(\Gamma\left(a,a'\right)\right)'$.*

Proof. For all $a \in A$, we get $\Gamma\left(a,a'\right) \leq a'$ from $a\cdot\Gamma\left(a,a'\right) = 0$. Since $a \leq \left(D\left(a,a'\right)\right)'$, $\Gamma\left(a,a'\right) \leq \left(D\left(a,a'\right)\right)'$. Therefore $D\left(a,a'\right) \leq \left(\Gamma\left(a,a'\right)\right)'$.

Corollary 2. *For all $a,b \in A$, $\Gamma(a,b) \leq a$ and $D\left(a',b\right) \leq a'$.*

*Proof.*For all $a,b \in A$, since

$$0 = \Gamma\left(a \cdot a^{'}, b\right) = \left(\Gamma(a,b) \cdot a^{'}\right) + \left(a \cdot D\left(a^{'}, b\right)\right)$$

we get $\Gamma(a,b) \leq a$ and $D\left(a^{'}, b\right) \leq a^{'}$.

Proposition 4. *Let Γ be a generalized symmetric bi-derivation on A related to symmetric bi-derivation D with its trace d and γ be a trace of Γ. If $a \leq b$ for $a,b \in A$, then the following situations hold:*
(i) $\gamma\left(a \cdot b^{'}\right) = 0$,
(ii) $\gamma b^{'} \leq a^{'}$,
(iii) $\gamma a \cdot \gamma b^{'} = 0$.

Proof. (i) Let $a \leq b$, for $a,b \in A$. From Theorem 2, we have $a \cdot b^{'} = 0$. Since $\gamma 0 = 0$, we have $\gamma\left(a \cdot b^{'}\right) = 0$.

(ii) Let $a \leq b$, for $a,b \in A$. Since $a \cdot \gamma b^{'} \leq b \cdot \gamma b^{'} \leq b \cdot b^{'} = 0$, we get $a \cdot \gamma b^{'} = 0$ and so $\gamma b^{'} \leq a^{'}$.

(iii) Since $a \leq b$, we get $\gamma a \leq b$ and so $\gamma a \cdot \gamma b^{'} \leq b \cdot \gamma b^{'} \leq b \cdot b^{'} = 0$. Hence $\gamma a \cdot \gamma b^{'} = 0$.

Proposition 5. *Let Γ be a generalized symmetric bi-derivation on A related to symmetric bi-derivation D with its trace d and γ be a trace of Γ. The the following situations hold:*
(i) $\gamma a \cdot \gamma a^{'} = 0$,
(ii) $\gamma a^{'} = (\gamma a)^{'}$ *if and only if γ is the identity on A.*

Proof.(i) We know that $a \leq a$. From Proposition 4 (iii), we get $\gamma a \cdot \gamma a^{'} = 0$.

(ii) Since $a \cdot \gamma a^{'} = 0$ for $a \in A$, we get $a \cdot \gamma a^{'} = a \cdot (\gamma a)^{'} = 0$. Since $a \leq \gamma a$ and $\gamma a \leq a$, we have $a = \gamma a$. Hence γ is the identity on A.

If γ is the identity on A, $\gamma a^{'} = (\gamma a)^{'}$ for all $a \in A$.

Definition 8. *Let Γ be a generalized symmetric bi-derivation on A related to symmetric bi-derivation D with its trace d and γ be a trace of Γ. If $a \leq b$ implies $\Gamma(a,c) \leq \Gamma(b,c)$ for $a,b,c \in A$, Γ is called an isotone.*

If Γ is an isotone and $a \leq b$ then $\gamma a \leq \gamma b$ for all $a,b \in A$.

Example 3. Let A be an MV-algebra and D be symmetric bi-derivation as in Example 1. Also, D is a generalized symmetric bi-derivation related to D. In A, $b \leq 1$, $D(b,1) = b$, $D(1,1) = 0$, but $0 \leq b$. From here, D is not isotone.

Example 4. In Example 2, Γ is an isotone generalized symmetric bi-derivation related to symmetric bi-derivation D.

Proposition 6. *Let Γ be a generalized symmetric bi-derivation on A related to symmetric bi-derivation D and γ be a trace of Γ. If $\gamma a^{'} = \gamma a$ for all $a \in A$, then the following situations hold:*
(i) $\gamma 1 = 0$,
(ii) $\gamma a \cdot \gamma a = 0$,
(iii) *If Γ is an isotone on A, then $\gamma = 0$.*

Proof. (i) Since $\gamma a^{'} = \gamma a$ for all $a \in A$, we get $\gamma 1 = \gamma 1^{'} = \gamma 0 = 0$.
(ii) For all $a \in A$, $\gamma a \cdot \gamma a = \gamma a \cdot \gamma a^{'} = 0$ from Proposition 5.
(iii) Let Γ be an isotone on A. For all $a \in A$, since $\gamma a \leq \gamma 1 = 0$, we get $\gamma a = 0$. Thus $\gamma = 0$.

Definition 9. *Let Γ be a generalized symmetric bi-derivation on A related to symmetric bi-derivation D. If $\Gamma(a+b,c) = \Gamma(a,c) + \Gamma(b,c)$ for all $a,b,c \in A$, Γ is called bi-additive mapping.*

Theorem 3. *Let Γ be a bi-additive generalized symmetric bi-derivation on A related to symmetric bi-derivation D and γ be a trace of Γ. Then $\gamma(B(A)) \subseteq B(A)$.*

Proof. Let $b \in \gamma(B(A))$. Thus $b = \gamma(a)$ for some $a \in B(A)$. Then

$$\begin{aligned} b+b = \gamma a + \gamma a &= \Gamma(a,a) + \Gamma(a,a) = \Gamma(a+a,a) \\ &= \Gamma(a,a) = b. \end{aligned}$$

Hence $b \in B(A)$. That is, $\gamma(B(A)) \subseteq B(A)$.

Theorem 4. *Let A be a linearly ordered MV-algebra, Γ be a bi-additive generalized symmetric bi-derivation on A related to symmetric bi-derivation D and γ be a trace of Γ. Then $\gamma = 0$ or $\gamma 1 = 1$.*

Proof. Since $a + a' = 1$ and $a + 1 = 1$ for all $a \in A$,

$$\gamma 1 = \Gamma(1,1) = \Gamma\left(a + a', 1\right) = \Gamma(a,1) + \Gamma\left(a', 1\right)$$

and

$$\begin{aligned} \gamma 1 = \Gamma(1,1) &= \Gamma(a+1,1) \\ &= \Gamma(a,1) + \gamma 1. \end{aligned}$$

If $\gamma 1 \neq 1$, then we get $\Gamma\left(a', 1\right) = \gamma 1$. Replacing a by 1, we get $\gamma 1 = 0$. For all $a \in A$,

$$0 = \gamma 1 = \Gamma(a,1) + \gamma 1 = \Gamma(a,1)$$

and

$$\Gamma(a,1) = \Gamma(a,a+1) = \gamma a + \Gamma(a,1) = \gamma a.$$

Thus $\gamma a = 0$ for all $a \in A$. That is, $\gamma = 0$.

Theorem 5. *Let A be a linearly ordered MV-algebra, Γ_1 and Γ_2 be bi-additive generalized symmetric bi-derivations on A related to $D_1$and D_2 respectively and d_1, d_2 be traces of D_1, D_2, γ_1, γ_2 be traces of Γ_1 and Γ_2 respectively. If $\gamma_1\gamma_2 = 0$ where $(\gamma_1\gamma_2)(a) = \gamma_1(\gamma_2 a)$ for all $a \in A$, then $\gamma_1 = 0$ or $\gamma_2 = 0$.*

Proof. Let $\gamma_1\gamma_2 = 0$ and $\gamma_2 \neq 0$. Then $\gamma_2 1 = 1$. For all $a \in A$,

$$0 = (\gamma_1\gamma_2)(a) = \gamma_1(\gamma_2 a) = \gamma_1(\gamma_2 a + (a \cdot D_2(a,1))).$$

Replacing a by 1, we get

$$0 = \gamma_1(\gamma_2 1 + (1 \cdot d_2 1)) = \gamma_1(1 + d_2 1) = \gamma_1 1$$

From here, for all $a \in A$ we get

$$0 = \gamma_1 1 = \Gamma_1(1,1) = \Gamma_1(a+1,1) = \Gamma_1(a,1) + \gamma_1 1 = \Gamma_1(a,1),$$

and

$$0 = \Gamma_1(a,1) = \Gamma_1(a,a+1) = \Gamma_1(a,a) + \Gamma_1(a,1) = \gamma_1 a.$$

That is, $\gamma_1 = 0$. Similarly it can be shown in the proof of the other case.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] N. O. Alshehri, *Derivations of MV-algebras*, Int. J. Math. Math. Sci., Volume 2010, Article ID 312027, 7 pages, doi:10.1155/2010/312027.

[2] G. Cattaneo, R. Giuntini and R. Pilla, *BZMVdM algebras and Stonian MV-algebras (applications to fuzzy sets and rough approximations)*, Fuzzy Sets and Systems 108 (1999), 201-222, 1999.

[3] C. C. Chang, *Algebraic analysis of many valued logics*, Trans. Am. Math. Soc., vol. 88 (1958), pp. 467–490.

[4] C. C. Chang, *A new proof of the completeness of the łukasiewicz axioms*, Trans. Am. Math. Soc., vol. 93 (1959), pp. 74–80.

[5] R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici, *Algebraic foundations of many-valued reasoning*, ser. Trends in Logic-Studia Logica Library. Dordrecht: Kluwer Academic Publishers, vol. 7 (2000).

[6] E. C. Posner, *Derivations in prime rings*, Proceedings of the American Mathematical Society, vol. 8 (1957), pp. 1093-1100.

[7] S. Rasouli and B. Davvaz, *Roughness in MV-algebras*, Inf. Sci., vol. 180 (2010), no. 5, pp. 737–747.

[8] G. Sz´asz, *Derivations of lattices*, Acta Scientiarum Mathematicarum, vol. 37 (1975), pp. 149–154.

[9] X. L. Xin, T. Y. Li and J. H. Lu, *On derivations of lattices*, Inf. Sci. 178 (2008), pp. 307-316.

[10] H. Yazarli, *A note on derivations in MV-algebras*, Miskolc Mathematical Notes, Vol. 14 (2013), No 1, pp. 345-354.