NTMSCI **10**, No. 4, 54-62 (2022) 54

New Trends in Mathematical Sciences

http://dx.doi.org/10.20852/ntmsci.2022.488

Stability Analysis of Fractional PSQ_p Smoking Model and Application in Turkey

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Received: 23 April 2022, Accepted: 15 August 2022
Published online: 5 December 2022.

Abstract: Smoking is one of the most common habits in the world. Due to the harmful substances contained in it cigarette disturb people's health and causes their death. Every year, many people lose their health from smoking or by direct exposure to cigarettes. In this study, we conducted a new application and stability analysis related to a fractional model of cigarette use. This model consists of three compartments: potential smokers (P), smokers (S) and defi-nitely quitters of smoking (Qp). The fractional derivative is used in the sense of Caputo. In this article, we develop a fractional mathematical model to illustrate the consequences of smoking habit in Turkey by the Caputo fractional-order derivative. In addition to stability analysis, numerical solutions are obtained by using Euler method for different values of fractional order.

Keywords: Fractional Smoking Model, Mathematical Modeling, Euler Method, Caputo Derivative, Stability.

1 Introduction

Smoking is one of the most important preventable causes of death in the world. 100 million people have lost their lives due to tobacco use in the last century and 5.4 million people lose their lives due to tobacco use every year. Currently, it is known that smoking, a global problem that threatens the lives of more than 1 billion people, is a risk factor for six of the eight most common life-threatening diseases [1]. Mathematical modeling is important in defining smoking cessation models. These models have been studied using ODE systems, taking into account the fact that we can analyze the spread and control of smoking. It is well known that smoking is a common condition in today's world. It is quite clear that the influence of social behavior of individuals in society on the increase in the prevalence of smoking is great. Smoking extremely affects the social behavior of people in a population [2].

The main purpose of mathematical modeling is to explain the functioning of processes by expressing real-life problems mathematically. However, it is also important that the modeled process can be controlled. Mathematical models have been developed to help explain a system, study the effects of its various components, and make predictions about its behavior. The mathematical modeling method is used not only in the representation of epidemic diseases and smoking cessation models, but also in the modeling of different dynamics [3].

Fractional derivative models give better results in the theory of various physical and biological processes and the control of dynamic systems than integer digit models. In particular, it is an appropriate approach to use fractional operators to explain the memory and hereditary properties of many substances and processes, since such properties are ignored in the integer order derivative [5]. In population models, the future status of a population depends on its past status. This is called the memory effect. The memory effect of the population can be studied by adding a delay term or using a fractional derivative in the model [14, 15, 16, 17, 18, 19, 20].

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In the smoking model, non-smokers, smokers and people who have quit smoking have been considered yet. During the implementation of this model, a numerical study was conducted in Turkey based on the data on smoking use of the Turkish Statistical Institute in 2019 [22].

This paper consists of four parts. In the first chapter, the importance of fractional mathematical modeling and information about smoking are given. In the second part, the formation of the fractional PSQ_p model, the Generalized Euler Method and the stability analysis of the proposed model are presented. In the third part, we obtain the numerical results of the PSQ_p model for real data set in Turkey. In the fourth part, the results are given.

2 Fractional Derivation and Fractional PSQ_p Cigarette Model

The most commonly used definitions of the fractional derivative are Riemann-Liouville, Caputo, Atangana-Baleanu and the Conformable derivative. In this study, because the classical initial conditions are easily applicable and provide ease of calculation, the Caputo derivative operator was preferred. The definition of the Caputo fractional derivative is given below.

Definition 1. *([4]) Let $f(t)$ be a function that can be continuously differentiable n times. The value of the function $f(t)$ for the value of α that satisfies the condition $n-1<\alpha<n$. The Caputo fractional derivative of $\alpha-$th order $f(t)$ is defined by $D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-x)^{(n-\alpha-1)} f^n(x)dx$.*

2.1 The Fractional PSQ_p Cigarette Model

The fractional PSQ_p model basically divides a community into three main groups. The first are those who have not yet smoked, but may smoke in the future, the second are potential smokers, and the third are those who have definitively quit smoking. The expression of the PSQ_p model as a system of fractional differential equations is as follows.
Let us express the number of individuals of groups P, S and Q_P in the form of a system of differential equations.

$$\begin{aligned} \frac{d^\alpha P}{dt^\alpha} &= \mu N - \mu P - \frac{\beta PS}{N} \\ \frac{d^\alpha S}{dt^\alpha} &= \frac{\beta PS}{N} - \mu S - \gamma S - \theta S \\ \frac{d^\alpha Q_p}{dt^\alpha} &= \gamma S - \mu Q_p \end{aligned} \tag{1}$$

where $\frac{d^\alpha}{dt^\alpha}$ is the Caputo fractional derivative with respect to time t and $0<\alpha\leq 1$.
Initial values are given as,

$$P(0)=P_0, S(0)=S_0, Q_p(0)=Q_{p0}$$

Here $P+S+Q_p=N$ and it is easy to see

$$\frac{d^\alpha N}{dt^\alpha}=\frac{d^\alpha P}{dt^\alpha}+\frac{d^\alpha S}{dt^\alpha}+\frac{d^\alpha Q_p}{dt^\alpha}$$

Because fractional-order models have a memory feature in events related to a time variable, they show more realistic and accurate results than integer-order models. Therefore, the established model was created as a fractional order [23]. In the system of (2.1), we reduce the fractional-order differential equation to a full-order differential equation by taking $\alpha=1$. The compartment and parameters of the spell are shown in Table 1 and Table 2.

Table 1: Variables used in the systems and their meanings

Variables used in the systems	Meaning
$P(t)$	the number of individuals who have not yet smoked at the time of t
$S(t)$	the number of individuals infected at the time of t
$Q_p(t)$	the number of individuals who quit smoking definitively at the time of t
$N(t)$	Total population

Table 2: Parameters and their meanings

Parameters	Meaning
β	The annual rate of wannabe-related smoking initiation
μ	Annual birth and death rate
γ	The annual rate of smoking cessation
θ	Annual smoking-related mortality rate

One of the most critical parameters in the model (2.1) is β parameter, which shows the wannabe-related smoking initiation rate. In case of a decrease in this parameter, the numbers of smokers would shift significantly in a positive direction. Natural birth and death rates were considered equal in the model. All births are considered to have entered the non-smoking class. The parameters defined in the model do not change with time. There is no mortality rate associated with some diseases caused by smoking [3, 10, 11, 12, 13, 14].

The N population was dimensionalized and created as follows with the help of new variables. $p = \frac{P}{N}, s = \frac{S}{N}, q_p = \frac{Q_p}{N}$ where $p+s+q_p = 1$. The new form of the fractionalPSQ_p model is written as

$$\begin{aligned} D^{\alpha} p(t) &= \mu - \mu p(t) - \beta p(t)s(t) \\ D^{\alpha} s(t) &= \beta p(t)s(t) - \mu s(t) - \gamma s(t) - \theta s(t) \\ D^{\alpha} q_p(t) &= \gamma s(t) - \mu q_p(t) \end{aligned} \tag{2}$$

2.2 Generalized Euler Method

In this paper, we use the Generalized Euler method to solve the initial value problem with the Caputo fractional derivative. Many of the mathematical models consist of non-linear systems, and finding solutions to these systems can be quite difficult. Analytical solutions cannot be found in most cases and a numerical approach should be considered for this. One of these approaches is the Generalized Euler method [21]. Let $D^{\alpha}y(t) = f(t, y(t)), y(0) = y_0, 0 < \alpha \leq 1, 0 < t < \alpha$ be the initial value problem. Let $[0, b]$ the interval over which we want to find the solution of the problem. For convenience subdivide the $[0, b]$ into n sub-intervals $[t_j, t_{j+1}]$, where $h = \frac{b}{n}$ $j = 0, 1, .., n-1$. Suppose that $y(t), D^{\alpha}y(t)$ and $D^{2\alpha}y(t)$ are continuous in range $[0, b]$ and using the generalized Taylor's formula, the following equality is obtained [21].

$$y(t_1) = y(t_0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_0, y(t_0))$$

This process will be repeated to create an array. Let $t_j = t_{j+1} + h$ such that

$$y(t_{j+1}) = y(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_j, y(t_j)$$

$j = 0,1,..,n-1$ the generalized formula in the form is obtained. For every $k = 0,1,..,n-1$ with step size h we get

$$\begin{aligned} p(k+1) &= p(k) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(\mu - \mu p - \beta ps) \\ s(k+1) &= s(k) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(\beta ps - \mu s - \gamma s - \theta s) \\ q_p(k+1) &= q_p(k) + \frac{h^{\alpha}}{\Gamma(\alpha+1)}(\gamma s - \mu q_p) \end{aligned} \tag{3}$$

2.3 Stability Analysis of the Fractional PSQ_p Cigarette Model

In order to find the equilibrium point without cigarettes (2.2) in the system, $D^{\alpha}p = 0$, $D^{\alpha}s = 0$, $D^{\alpha}q_p = 0$ is taken.

$$\begin{aligned} D^{\alpha}p(t) &= \mu - \mu p(t) - \beta p(t)s(t) \\ D^{\alpha}s(t) &= \beta p(t)s(t) - \mu s(t) - \gamma s(t) - \theta s(t) \\ D^{\alpha}q_p(t) &= \gamma s(t) - \mu q_p(t) \end{aligned}$$

In order to determine the equilibrium point without cigarettes in the system (2.2), we take $s(t) = 0$ is taken. From here, we get a non-smoking equilibrium point the following

$$E_0 = (p_0, s_0, q_{p0}) = (1,0,0) \tag{4}$$

the equilibrium point is achieved without smoking. The Jacobian matrix at the non-smoking equilibrium point of the system

$$J(E_0) = \begin{bmatrix} -\mu & -\beta & 0 \\ 0 & \beta - \mu - \gamma - \theta & 0 \\ 0 & \gamma & -\mu \end{bmatrix} \tag{5}$$

and the eigenvalues of Jacobian matrix in (2.5) are

$$\begin{aligned} \lambda_1 &= -\mu \\ \lambda_2 &= -\mu \\ \lambda_3 &= \beta - \mu - \gamma - \theta \end{aligned}$$

where $\beta, \mu, \gamma, \theta$ are the parameters of positively defined real numbers. It is clear that $\lambda_1 < 0$ and $\lambda_2 < 0$. If $\lambda_3 < 0$ the smoke-free equilibrium point is locally asymptotically stable. If $\lambda_3 > 0$, the non-smoking equilibrium point is unstable. If $\beta - \mu - \gamma - \theta < 0$, $\beta < \mu + \gamma + \theta$ is. $R_0 = \dfrac{\beta}{\mu + \gamma + \theta} < 1$ is the basic reproduction rate. If $R_0 < 1$, the contact ratio between potential smokers and non-smokers is less. If $R_0 > 1$, the contact ratio between potential smokers and non-smokers are strong. From above argument we give the following theorem.

Theorem 2.2: For all $t \geq 0$, $P(0) = P_0 \geq 0$, $S(0) = S_0 \geq 0$, $Q_p(0) = Q_{p0} \geq 0$, the solutions of the system in (2.1) with initial conditions $(P(t), S(t), Q_p(t)) \in R^3_+$ are not negative.

Proof: (Generalized Mean Value Theorem) Let $f(x) \in C[a,b]$ and $D^{\alpha}f(x) \in C[a,b]$ for $0 < \alpha \leq 1$. Then we have

$$f(x) = f(\alpha) + \frac{1}{\Gamma(\alpha)} D^{\alpha} f(\varepsilon)(x-a)^{\alpha} \tag{6}$$

with $0 \leq \varepsilon \leq x$, $\forall \in (a,b]$.

The existence and and uniqueness of the solution (2.1) in $(0,\infty)$ can be obtained via Generalized Mean Value Theorem.

We need to show that the domain R^3_+ is positively invariant. Since

$$D^\alpha P = \mu N - \mu P - \frac{\beta PS}{N} \geq 0$$
$$D^\alpha S = \frac{\beta PS}{N} - \mu S - \gamma S - \theta S \geq 0$$
$$D^\alpha Q_p = \gamma S - \mu Q_p \geq 0$$

on each hyperplane bounding the nonnegative orthant, the vector field points into R^3_+.

3 Numerical Simulation of Fractional PSQ_p Cigarette Model for Turkey

This section will show the numeric simulation and graphics of the fractional PSQ_p cigarette model. Let's now use the Generalized Euler method to obtain a numerical simulation of the fractionalPSQ_p cigarette model. [22] take into account Turkey's population in 2019 and consider the following parameters based on smoking data. [22] According to the smoking usage data in Turkey in 2019, we can show the non-smoking rate, the smoking rate and the smoking cessation rate in Fig-1 as follows.

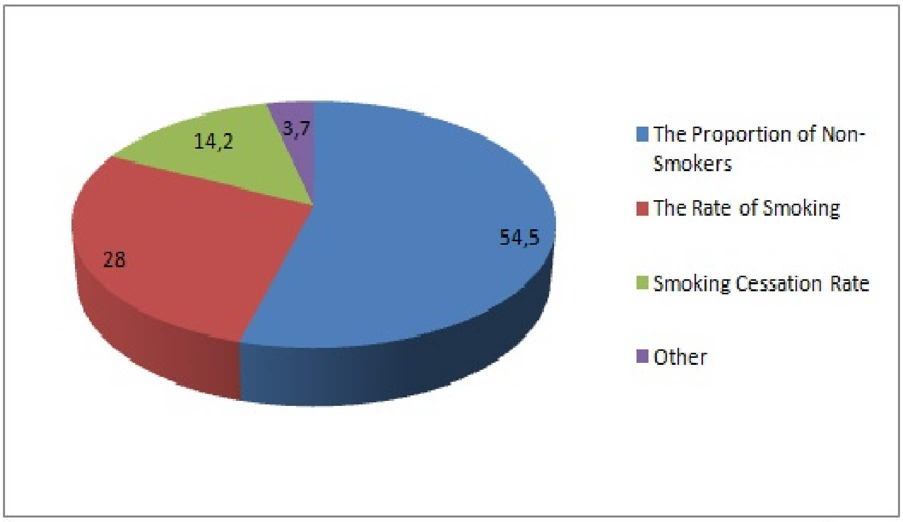

Fig. 1: Smoking Data of Turkey in 2019

$N = 83000043, P = 45046935, S = 23003604, Q_p = 11084706, \beta = 0.1798, \gamma = 0.272, \mu = 0.0053, \theta = 0.001$ and let's take size of step h=1. Hence we get the following results and tables. Using the Euler method, we obtain the following tables for different values of α.

If we calculate the basic reproduction rate R_0, $R_0 = \dfrac{\beta}{\mu+\gamma+\theta}$ then according to the parameters we have, $R_0 = 0.64606539705$ is obtained. $R_0 < 1$, the contact ratio between potential smokers and non-smokers is less.

In Table 3, Table 4 and Table 5, the changes of P, S and Q_p are observed for different states of α .

By the above figures, we observe the following highlights:

*It is observed that the number of individuals who do not smoke yet, but may smoke in the future, decreases over time and increases at some point after taking the minimum value (see Fig-2).

*It is observed that individuals who smoke decrease over time (see Fig-3).

*It is observed that individuals who quit smoking increase over time (see Fig-4).

Table 3: The values of P, S and Q_p at the moment t $\alpha = 1$.

t	$P(t)$	$S(t)$	$Q_p(t)$
0	45046935,00	23003604,00	11084706,00
1	43003313,11	18846474,36	17282937,34
2	41459623,69	15357172,63	22317578,80
3	40300522,03	12462537,37	26376446,59
4	39438830,45	10082212,26	29626461,59
5	38808331,91	8137705,55	32211803,08
6	38358418,48	6557111,59	34254536,43
7	38050159,45	5277127,07	35856521,75
8	37853417,73	4243478,72	37101860,75
9	37744727,49	3410485,94	38059447,10
10	37705722,14	2740206,23	38785384,20
11	37721960,49	2201428,37	39325157,76
12	37782043,02	1768662,17	39715522,95
13	37876940,71	1421201,19	39986106,78
14	37999481,54	1142292,52	40160747,14

Table 4: The values of P, S and Q_p at the moment t $\alpha = 0.9$.

t	$P(t)$	$S(t)$	$Q_p(t)$
0	45046935,00	23003604,00	11084706,00
1	42922079,18	18681228,54	17529316,91
2	41336899,50	15081619,35	22715994,05
3	40162300,61	12121756,42	26856076,45
4	39301825,61	9710715,52	30136259,45
5	38683016,76	7760414,90	32716495,15
6	38251079,57	6190996,39	34730943,60
7	37964286,97	4932944,28	36290440,70
8	37790649,64	3927350,33	37485551,69
9	37705493,21	3125213,43	38389682,79
10	37689682,28	2486308,58	39061977,57
11	37728307,18	1977930,08	39549877,60
12	37809704,03	1573672,41	39891313,44
13	37924716,91	1252327,14	40116538,81
14	38066137,83	996925,27	40249642,75

4 Conclusions and Comments

In this study, a new application of the fractional PSQ_p cigarette model of smoking for the first time has been made taking into account the data in [22] and graphs are drawn with the help of the numerical results obtained. Stability analysis has been performed by obtaining the non-smoking equilibrium point of the fractional PSQ_p model and the number R_0 the basic reproduction rate has been found. In the graphs obtained, it is observed that the number of individuals who are not yet smokers but may smoke in the future decreases over time and increases after taking the minimum value at some point, smoking individuals decrease over time and quitting individuals increase over time.

Competing interests

The authors declare that they have no competing interests.

Table 5: The values of P, S and Q_p at the moment t $\alpha = 0.8$.

t	$P(t)$	$S(t)$	$Q_p(t)$
0	45046935,00	23003604,00	11084706,00
1	42852759,42	18540218,50	17739561,23
2	41233326,74	14848246,12	23053068,05
3	40047012,16	11835541,04	27258141,48
4	39189031,15	9401456,46	30559461,03
5	38581412,44	7449202,13	33131148,78
6	38165722,72	5891814,66	35118069,54
7	37897846,52	4654330,19	36638867,11
8	37744243,32	3673860,34	37789617,87
9	37679250,05	2898621,36	38647485,63
10	37683121,11	2286531,12	39274072,30
11	37740590,93	1803714,11	39718339,66
12	37839809,32	1423088,38	40019077,90
13	37971545,67	1123111,95	40206947,53
14	38128589,13	886712,44	40306143,50

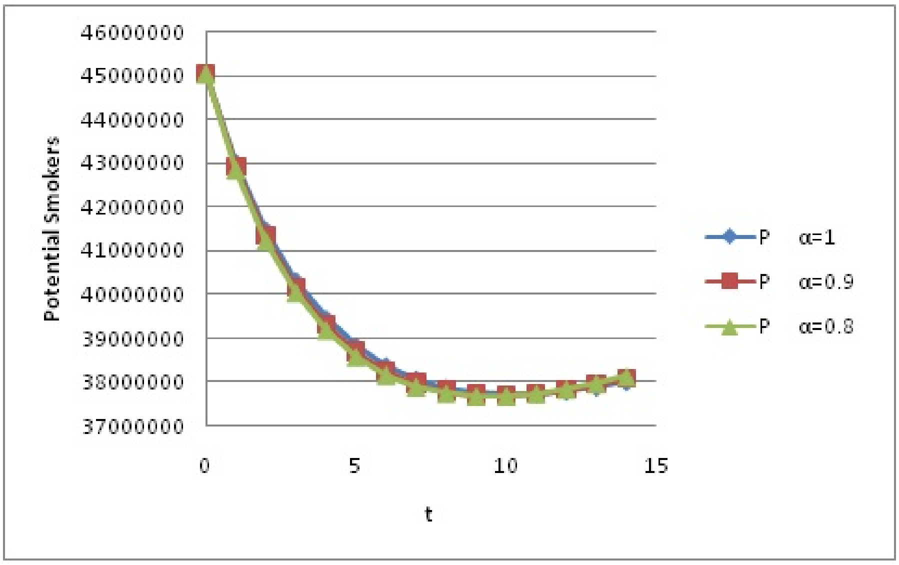

Fig. 2: The graph of change of the P compartment model.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

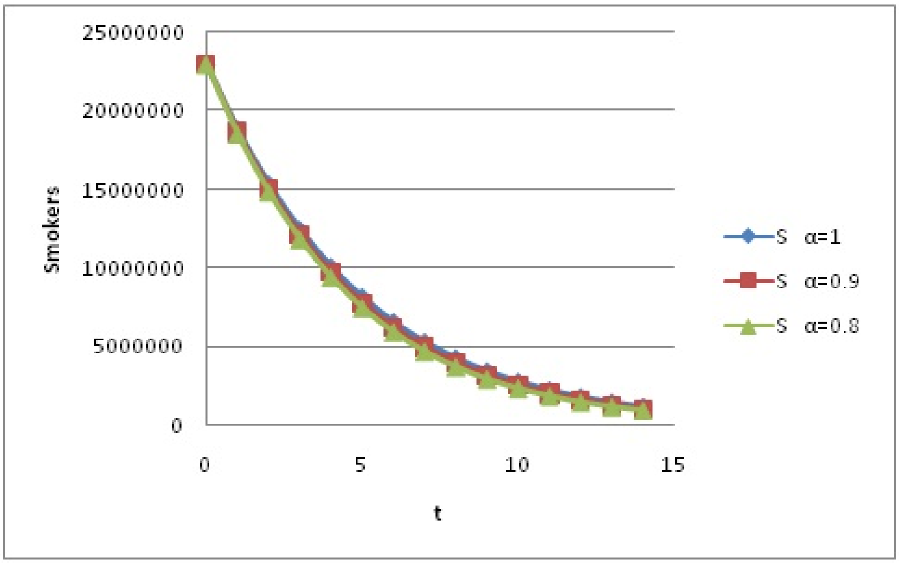

Fig. 3: The graph of change of the S compartment model.

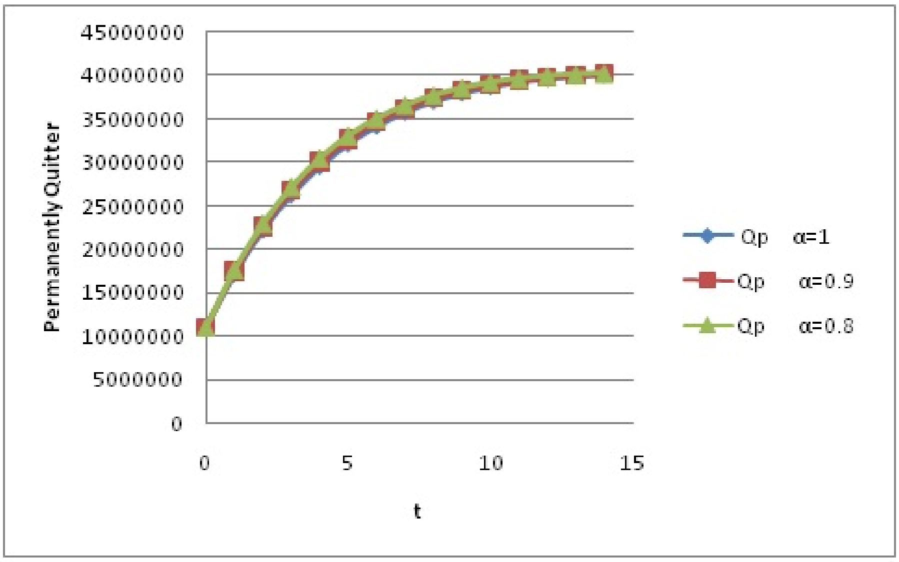

Fig. 4: The graph of change of the Q_p compartment model.

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