New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2022.485

An Effective Numerical Approach Based on Cubic Hermite B-spline Collocation Method for Solving the 1D Heat Conduction Equation

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Received: 23 August 2022, Accepted: 2 December 2022
Published online: 5 December 2022.

Abstract: This article is going to deal with the numerical solutions about the most vital problem arising in nature; namely the heat conduction equation given in one-dimension. For this aim, we are going to use cubic Hermite B-spline finite elements based on collocation method. Then, the algorithm of the method has been produced and the stability analysis has also been examined via Fourier stability method. Furthermore, a comparative study between the approximate and exact solutions has been used to demonstrate the accuracy and efficiency of the proposed scheme. The newly obtained results clearly show that the present scheme is a reliable and accurate one and may even be used successfully to find approximate solutions of numerous nonlinear problems encountering widely in many applied sciences.

Keywords: Cubic Hermite Collocation Method (CHCM), Finite Element Method (FEM), Heat Conduction Equation, Stability Analysis.

1 Introduction

First of all, let us take into consideration the following one dimensional heat conduction equation

$$\alpha^2 u_{xx} - u_t = 0, \ \ 0 \le x \le L, \ \ t > 0 \tag{1}$$

with the given initial condition

$$u(x,0) = f(x) \tag{2}$$

and the appropriate boundary conditions

$$u(0,t) = u(L,t) = 0, \quad t > 0 \tag{3}$$

where α is the thermal diffusivity of the rod and $f(x)$ is a predefined function. It is a well known fact that this initial and boundary problem is among the most widely used second order linear partial differential equations (PDEs) [1,2,3,4]. Eq. (1) is frequently seen in several areas of engineering and science and used to describe the change in the temperature (or equivalently the distribution of heat) over a region for a predefined time interval.

In this present model, the flow of the heat is going to be considered only in one-dimension insulated every place with the exception at the two ends of the rod. The newly obtained solutions of the present equation will be given in terms of functions of the state along the rod in $x-$direction and $t-$direction. When studied broadly, it is seen that the heat equation has a fundamental importance in such diverse scientific fields as physics, biology, chemistry etc. In fact, it

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stands for prototypical parabolic PDE in various areas of science. One even can see it in probability theory, because it is directly related to the study of Brownian motion via the Fokker-Planck equation [5]. Because of this widespread usage and fundamental importance, both the approximate and analytical solutions of these kinds of equations come forward and urgently become indispensable for studying several physical phenomena in nature. Eq. (1) can also be utilized for financial mathematical problems in the modeling of various choices [6]. It may also be defined as the heat flowing inside a rod having diffusion $\alpha^2 u_{xx}$ along the rod in which the coefficient α stands for the thermal diffusivity on the given rod and finally L represents the length of the rod [7]. Fluid flow, electrostatics, electrodynamics, elasticity and other practical applications rely heavily on one dimensional parabolic PDEs [8]. When a literature survey is carried out, it is seen that this problem has been investigated both numerically and analytically for a long time by several researchers. In spite of this fact, the problem is still an interesting one due to the fact that many physical phenomena in the nature are formulated into PDEs having appropriate initial and boundary conditions. Since there are so many studies in the literature, especially on numerical solutions of the heat equation, it does not seem possible to refer all of them. Therefore, we would like to express our deep gratitude to all readers for their valuable understanding. Çağlar et al. [6] have examined the boundary value problem about heat equation in one-dimension having a non-local initial condition based on the 3^{rd} degree classical B-splines functions. Dhawan and Kumar [4] have solved heat equation in one-dimension based on Galerkin classical B-spline FEM. Suarez-Carreno and Rosales-Romero [9] have presented several techniques in order to solve differential equations using the finite difference methods: forward time centered space (FTSC) and backward time centered space (BTSC), and the Crank-Nicolson scheme (CN). Goh et al. [10] have obtained approximate solutions of both advection-diffusion and heat equations given in 1D using cubic classical B-spline collocation method. Khabir and Farah [11] have used the collocation method in order to solve heat equation in one-dimension and compared the approximate result with the analytical one. Kaskar [8] has proposed a modified implicit scheme for solving 1D heat equation having appropriate initial and boundary conditions. Mebrate [12] has presented numerical solutions of heat equation in one-dimension having initial condition and Dirichlet type boundary conditions. Lozada-Cruz et al.[13] have studied approximate solutions of heat diffusion equation in one dimension given by Robin type boundary conditions multiplied by a very small parameter epsilon which is greater than zero. Hooshmandasl et al. [14] have developed an effective Chebyschev wavelet method for widely used heat equation in one dimension. Sun and Zhang [15] have applied a high-order compact boundary value method in order to solve heat equations in one dimension. Patel and Pandya [16] have presented heat equation in one dimension given by both Dirichlet and Neumann initial boundary conditions and used a spline collocation method for solving the problem. Tarmizi et al. [17] have attempted for obtaining solutions of 1D heat equations by means of a numerical solution approach. Yosaf et al. [18] have developed a high-order compact finite difference method for solving one-dimensional (1D) heat conduction equation with Dirichlet and Neumann boundary conditions, respectively. Han and Dai [19] have presented two higher-order compact finite difference schemes for solving one-dimensional (1D) heat conduction equations with Dirichlet and Neumann boundary conditions, respectively.

In the present manuscript, we are going to try finding the approximate solution of the heat conduction problem given by Eqs.(1)-(3). For this aim, cubic Hermite basis functions will be used. When looked at its development process, we see that in 1946, Schoenberg firstly put forward the fundamental theory of B-splines [4]. Using the step functions as starting point, one can find linear, quadratic, cubic, quartic etc. B-spline functions in a recursive manner. The recursive relationship to compute the coefficients of those B-spline functions are given by Cox and de Boor [20,21] and thus are known by their names. Both classical and Hermite cubic B-splines collocation methods have been proposed for numerical solutions of the Burgers' equation such as in Refs. [22,23]. In recent years, the theory of spline function is examined in depth and is improved for solving the differential equations approximately by several research articles [6, 24,25]. Moreover, several original problems are approximately investigated by FEMs such as Galerkin, Petrov-Galerkin, subdomain, least square and collocation method based on the second, third, fifth and seventh degree B-spline functions

[26,27,28].

Several schemes, methods and techniques of both the classical and Hermite cubic B-spline collocation methods and their application are developed for obtaining the approximate solutions for the differential equations. Those methods have the pros and cons and also are worth to use in the numerical techniques. Among others, using both classical and Hermite cubic spline collocation procedures have the following the desirable characteristics: (1) the resulting governing system is a diagonal one allowing easy storage in digital computers; (2) it provides high computer speed and low storage cost and at the same time easy problem formulation. The prerequisite of the continuity till the second degree has been under guarantee at the nodes on solution the domain and the 1^{st} and 2^{nd} degree of the derivatives are found directly [29,24,30].

In this manuscript, the Cubic Hermite Collocation Method (CHCM) is going to be utilized in order to solve the heat conduction equation (1) given together with (2) and (3) and the approximate solutions are compared with the exact ones [31]. For building the CHCM, we are going to employ collocation techniques as it has been used in Refs. [29,32,33]. Section two presents the method and also gives information about applying the collocation FEM based on hermite cubic B-splines. The third section has investigated the stability analysis by applying Fourier stability method. In the final section, the numerical results are given in tabular form and also compared with some of the previously published ones.

2 Cubic Hermite Collocation Method for space discretization

In this article, the heat conduction equation is generally given in the following form

$$\alpha^2 u_{xx} - u_t = 0,\ 0 \leq x \leq L$$

together with the following initial

$$u(x,0) = f(x)$$

and boundary conditions

$$u(0,t) = 0,\ u(L,t) = 0.$$

To apply a numerical method, as in general, let us consider the solution interval $[a,b]$ is divided into N finite elements having equal lengths using the nodal points $x_j,\ \ j = 0(1)N$ in such a way that $a = x_0 < x_1 \cdots < x_N = b$ and $h = (x_{j+1} - x_j)$. The cubic hermite B-splines H_j $(j = 1,2,...,N+1)$ are defined over the interval $[x_{j-1}, x_{j+1}]$ as follows [34]

$$H_{2j-1}(x) = \frac{1}{h^3}\begin{cases} (x-x_{j-1})^2\left[3h - 2(x-x_{j-1})\right], & x \in [x_{j-1}, x_j] \\ [h-(x-x_j)]^2\left[h - 2(x-x_j)\right], & x \in [x_j, x_{j+1}] \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

$$H_{2j}(x) = \frac{1}{h^3}\begin{cases} -h(x-x_{j-1})^2\left[h - (x-x_{j-1})\right], & x \in [x_{j-1}, x_j] \\ h[h-(x-x_j)]^2(x-x_j), & x \in [x_j, x_{j+1}] \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

An approximation $u_N(x,t)$ to the exact solution $u(x,t)$ can be written in terms of the cubic hermite B- splines as trial functions:

$$u(x,t) \approx u_N(x,t) = \sum_{j=1}^{N} a_{j+2k-2}(t) H_{ji} \tag{6}$$

where a's are time dependent parameters to be found, k is the number of elements and $i = 1,2$. If the second order Gauss-Legendre quadrature points (η_{ji}) are chosen for each sub interval $\left[x_j, x_{j+1}\right]$, then the Gauss-Legendre quadrature points are taken as follows

$$\eta_{ji} = \frac{x_{j-1}+x_j}{2} + (-1)^i \frac{h_j}{2\sqrt{3}}, \quad 2 \le j \le N+1, \quad 1 \le i \le 2. \tag{7}$$

When the following shifted Legendre polynomial roots are used in Eq. (7)

$$\xi_1 = \frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right), \quad \xi_2 = \frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right)$$

the following equations are obtained

$$\frac{\eta_{j1}-x_j}{h_j} = -\xi_1, \quad \frac{\eta_{j2}-x_j}{h_j} = -\xi_2$$

But, if Chebyshev polynomial is chosen the following roots

$$\xi_1 = \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right), \quad \xi_2 = \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$$

are used. Throughout the article, Legendre polynomial roots are used for the numerical computations.

If the following local coordinate conversion is used on the k^{th} element

$$\xi = \frac{x-x_k}{h}$$

the interval $[x_k, x_{k+1}]$ is converted into $[0,1]$. Under these conditions,

$$\begin{aligned} H_1(\xi) &= (1+2\xi)(1-\xi)^2, \quad H_2(\xi) = \xi(1-\xi)^2 h \\ H_3(\xi) &= \xi^2(3-2\xi), \qquad H_4(\xi) = \xi^2(\xi-1)h \end{aligned}$$

$$\begin{aligned} A_1(\xi) &= 6\xi^2 - 6\xi, \quad A_2(\xi) = \left(1-4\xi+3\xi^2\right)h \\ A_3(\xi) &= 6\xi - 6\xi^2, \quad A_4(\xi) = \left(3\xi^2-2\xi\right)h \end{aligned}$$

$$\begin{aligned} B_1(\xi) &= 12\xi - 6, \quad B_2(\xi) = (6\xi-4)h \\ B_3(\xi) &= 6-12\xi, \quad B_4(\xi) = (6\xi-2)h \end{aligned}$$

$$\begin{aligned} u(\xi,t) &= \sum_{j=1}^{4} a_{j+2k-2}(t) H_j(\xi) \\ &= a_{2k-1}H_1(\xi) + a_{2k}H_2(\xi) + a_{2k+1}H_3(\xi) + a_{2k+2}H_4(\xi) \end{aligned}$$

$$u_{\xi}(\xi,t)=\frac{1}{h}\sum_{j=1}^{4}a_{j+2k-2}(t)A_j(\xi)$$
$$=\frac{1}{h}\left[a_{2k-1}A_1(\xi)+a_{2k}A_2(\xi)+a_{2k+1}A_3(\xi)+a_{2k+2}A_4(\xi)\right]$$

$$u_{\xi\xi}(\xi,t)=\frac{1}{h^2}\sum_{j=1}^{4}a_{j+2k-2}(t)B_j(\xi)$$
$$=\frac{1}{h^2}\left[a_{2k-1}B_1(\xi)+a_{2k}B_2(\xi)+a_{2k+1}B_3(\xi)+a_{2k+2}B_4(\xi)\right]$$

are obtained. Here A_1,A_2,A_3,A_4 ve B_1,B_2,B_3,B_4 are the first and the second order derivatives of Hermit base functions, respectively. When Eqs. (4) and (5) are used at the nodal points the following approximate solutions are found

$$\left.\begin{aligned} u_i &= a_{2k-1}H_{1i}+a_{2k}H_{2i}+a_{2k+1}H_{3i}+a_{2k+2}H_{4i}\\ hu_i' &= a_{2k-1}A_{1i}+a_{2k}A_{2i}+a_{2k+1}A_{3i}+a_{2k+2}A_{4i}\\ h^2u_i'' &= a_{2k-1}B_{1i}+a_{2k}B_{2i}+a_{2k+1}B_{3i}+a_{2k+2}B_{4i}\end{aligned}\right\},\qquad i=1,2 \tag{8}$$

where $H_{ji}=H_j(\xi_i)$, $A_{ji}=A_j(\xi_i)$, $B_{ji}=B_j(\xi_i)$. During the solution process, firstly, the forward finite difference formula for the time discretization and then the finite element collocation method based on hermite cubic B-spline basis functions for the space discretization are going to be implemented. In fact, the implementation of the presented method based on Hermite B-splines are more efficient because of their many important characteristics such as easy storage and manipulations in computers. Among others, both of the linear and non-linear systems obtained using B-splines are generally well-conditioned and allow the required parameters to be determined easily. Besides, while obtaining the approximations using B-splines, one generally doesn't encounter numerical instability. Moreover, the matrix systems obtained using B-splines are generally sparse band matrixes and easy to be implemented on computers [26].

3 Implementation of the method for time discretization

At the moment, it is time to discretize the heat equation (1) given as

$$u_t-\alpha^2u_{xx}=0.$$

For this aim, Crank-Nicolson type formula is utilized. Firstly, Eq. (1) is discretized as follows

$$\frac{u^{n+1}-u^n}{\Delta t}-\alpha^2\left[\frac{(u_{xx})^{n+1}+(u_{xx})^n}{2}\right]=0.$$

When the variables for the next time level are put together on the left hand side and the variables for previous time level are put together on the right hand side

$$\frac{u^{n+1}}{\Delta t}-\alpha^2\frac{(u_{xx})^{n+1}}{2}=\frac{u^n}{\Delta t}+\alpha^2\frac{(u_{xx})^n}{2} \tag{9}$$

is obtained.

Substituting (8) into (9), the difference equation system in the following form is obtained for the variables **a** with $2N$ difference equations and $2N+2$ unknowns

$$
\begin{aligned}
&\frac{1}{\Delta t}\left[a_{2k-1}^{n+1}H_{1i}+a_{2k}^{n+1}H_{2i}+a_{2k+1}^{n+1}H_{3i}+a_{2k+2}^{n+1}H_{4i}\right]\\
&-\frac{\alpha^2}{2h^2}\left[a_{2k-1}^{n+1}B_{1i}+a_{2k}^{n+1}B_{2i}+a_{2k+1}^{n+1}B_{3i}+a_{2k+2}^{n+1}B_{4i}\right]\\
&=\frac{1}{\Delta t}\left[a_{2k-1}^{n}H_{1i}+a_{2k}^{n}H_{2i}+a_{2k+1}^{n}H_{3i}+a_{2k+2}^{n}H_{4i}\right]\\
&+\frac{\alpha^2}{2h^2}\left[a_{2k-1}^{n}B_{1i}+a_{2k}^{n}B_{2i}+a_{2k+1}^{n}B_{3i}+a_{2k+2}^{n}B_{4i}\right]
\end{aligned}
\tag{10}
$$

These equations are recursive relationships for the element parameters vector $\mathbf{a}^n=(a_1^n,\ldots,a_{2N+1}^n,a_{2N+2}^n)$ where $t_n=n\Delta t$, $n=1(1)M$ until the final time T. Using the boundary conditions given in Eq.(3) and eliminating the parameters a_1^n,a_{2N+1}^n in Eq. (10) as follows: From the left boundary condition $u(x_0,t)=a_1^nH_{11}+a_2^nH_{21}+a_3^nH_{31}+a_4^nH_{41}=0$, since $H_{21}=H_{31}=H_{41}=0$ and $H_{11}\neq 0$, the condition $a_1^n=0$ is obtained. Similarly from the right boundary condition $u(x_N,t)=a_{2N-1}^nH_{12}+a_{2N}^nH_{22}+a_{2N+1}^nH_{32}+a_{2N+2}^nH_{42}=0$, since $H_{12}=H_{22}=H_{42}=0$ and $H_{32}\neq 0$, the condition $a_{2N+1}^n=0$ is obtained. Now new solvable system is found as follows in the following matrix form

$$
\mathbf{L}\mathbf{a}^{n+1}=\mathbf{R}\mathbf{a}^n. \tag{11}
$$

Here the matrix $\mathbf{L}$ and $\mathbf{R}$ are square $2N\times 2N$ diagonal band matrices, and the matrices $\mathbf{a}^{n+1}$ and $\mathbf{a}^n$ are $2N\times 1$ column matrices.

The values $\mathbf{a}_i$ $(i=1(1)2N)$ obtained by solving the system of equations given by Eq.(11) are found and the approximate solutions of the heat equation at the next time level are computed. This process is repeated successively for $t_n=n\Delta t$ $(n=1(1)M)$ until the final time T. In order to start the iterative process, the initial vector $\mathbf{a}^0$ with entries $\mathbf{a}_{i0}$ $(i=1(1)2N)$ is needed. This vector is calculated by using the initial condition given by the governing equation. With the help of those substitutions, Eq. (10) results in $2N$ unknowns. In order to solve the newly obtained system, an algorithm coded in MatlabR2021a has been implemented.

3.1 The initial state

The initial vector $\mathbf{a}^0$ is determined from the initial and boundary conditions. So, the approximation in Eq. (6) must be rewritten for the initial condition as

$$
u(x,t)\approx u_N(x,t)=\sum_{j=1}^{N}a_{j+2k-2}^0(t)H_{ji}
$$

where the a_m^0's are unknown parameters to be computed. We require the initial numerical approximation $u_N(x,0)$ such that it satisfies the following conditions

$$
u_N(x_i,0)=u(x_i,0),\qquad i=0,\ldots,N
$$

Thus, these conditions lead to the matrix equation of the form

$$
\mathbf{W}\mathbf{a}^0=\mathbf{b} \tag{12}
$$

where

$$\mathbf{W}=\begin{bmatrix} H_{21} & H_{31} & H_{41} & & & & & & \\ H_{22} & H_{32} & H_{42} & & & & & & \\ & H_{11} & H_{21} & H_{31} & H_{41} & & & & \\ & H_{12} & H_{22} & H_{32} & H_{42} & & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & H_{11} & H_{21} & H_{31} & H_{41} & & \\ & & & H_{12} & H_{22} & H_{32} & H_{42} & & \\ & & & & & H_{11} & H_{21} & H_{41} & \\ & & & & & H_{12} & H_{22} & H_{42} & \end{bmatrix},$$

$$\mathbf{a}^0=(a_2,a_3,a_4,\ldots,a_{2N-1},a_{2N},a_{2N+2})^T$$

and

$$\mathbf{b}=(u(x_{11},0),u(x_{12},0),u(x_{21},0),\ldots,u(x_{N1},0),u(x_{N2},0))^T.$$

Thus the initial values required to start the numerical scheme in Eq. (11) are computed from Eq. (12) and then the iteration process is repeatedly continued until the desired final time T.

4 The Analysis of Stability

To investigate the stability of the linear numerical scheme (10), we have used von-Neumann (Fourier) method. For this aim, substituting the Fourier mode

$$a_j^n=\xi^n e^{ij\varphi}$$

into Eq. (10), one gets

$$\begin{aligned}&\xi^{n+1}e^{i(2j-1)\varphi}(\alpha_1)+\xi^{n+1}e^{i(2j)\varphi}(\alpha_2)+\xi^{n+1}e^{i(2j+1)\varphi}(\alpha_3)+\xi^{n+1}e^{i(2j+2)\varphi}(\alpha_4)=\\&\xi^{n}e^{i(2j-1)\varphi}(\beta_1)+\xi^{n}e^{i(2j)\varphi}(\beta_2)+\xi^{n}e^{i(2j+1)\varphi}(\beta_3)+\xi^{n}e^{i(2j+2)\varphi}(\beta_4)\end{aligned}\tag{13}$$

where $\varphi=\beta h$, β is the mode number, h is the spatial step size, $i=\sqrt{-1}$ and

$$\alpha_1=H_{1i}-\frac{\alpha^2}{2h^2}\Delta tB_{1i}\qquad \beta_1=H_{1i}+\frac{\alpha^2}{2h^2}\Delta tB_{1i}$$

$$\alpha_2=H_{2i}-\frac{\alpha^2}{2h^2}\Delta tB_{2i}\qquad \beta_2=H_{2i}+\frac{\alpha^2}{2h^2}\Delta tB_{2i}$$

$$\alpha_3=H_{3i}-\frac{\alpha^2}{2h^2}\Delta tB_{3i}\qquad \beta_3=H_{3i}+\frac{\alpha^2}{2h^2}\Delta tB_{3i}$$

$$\alpha_4=H_{4i}-\frac{\alpha^2}{2h^2}\Delta tB_{4i}\qquad \beta_4=H_{4i}+\frac{\alpha^2}{2h^2}\Delta tB_{4i}.$$

When some simple calculations are carried out, one obtains the following iterative scheme

$$\begin{aligned}&\xi^{n+1}e^{i(2j-1)\varphi}(\alpha_1)+\xi^{n+1}e^{i(2j)\varphi}(\alpha_2)+\xi^{n+1}e^{i(2j+1)\varphi}(\alpha_3)+\xi^{n+1}e^{i(2j+2)\varphi}(\alpha_4)=\\&\xi^{n}e^{i(2j-1)\varphi}(\beta_1)+\xi^{n}e^{i(2j)\varphi}(\beta_2)+\xi^{n}e^{i(2j+1)\varphi}(\beta_3)+\xi^{n}e^{i(2j+2)\varphi}(\beta_4).\end{aligned} \tag{14}$$

Again, if one makes the required simplification and mathematical operations, one results in

$$\xi=\frac{P-iQ}{R+iS} \tag{15}$$

where

$$\begin{aligned}P&=\beta_4\cos 2\varphi+(\beta_1+\beta_3)\cos\varphi+\beta_2\\Q&=-i(-\beta_4\sin 2\varphi+(\beta_1-\beta_3)\sin\varphi)\\R&=\alpha_4\cos 2\varphi+(\alpha_3+\alpha_1)\cos\varphi+\alpha_2\\S&=i(\alpha_4\sin 2\varphi+(\alpha_3-\alpha_1)\sin\varphi).\end{aligned} \tag{16}$$

When the modulus of Eq. (15) is taken, it is seen that the condition $|\xi|\leq 1$ is satisfied. Thus, it is concluded that the numerical scheme is unconditionally stable.

5 Numerical results

In the present section, we are going to apply the collocation FEM based on cubic hermite B-spline basis functions to a test problem. In this test problem the initial condition function will be taken as $f(x)=\sin(\pi x)$. The solution domain is the closed interval $[0,1]$. The analytical solution for the problem has been given as [18,19]

$$u(x,t)=\sin(\pi x)e^{-\alpha^2\pi^2 t}$$

Since the analytical solution of the test problem does exist, the validity and accuracy of the current method is going to be checked utilizing the error norms L_2 and L_∞ given as follows, respectively:

$$L_2=\left(h\sum_{i=1}^{N}|u_i-(u_N)_i|^2\right)^{1/2},\qquad L_\infty=\max_{1\leq i\leq N}|u_i-(u_N)_i|.$$

Throughout the paper, all of the numerical computations have been carried out by using both Cubic Hermite Collocation Method with Legendre roots (CHCM-L) and Cubic Hermite Collocation Method with Chebyshev roots (CHCM-C). These computations are made using MATLAB R2021a on Intel (R) Core(TM) i7 8565U CPU @1.80Ghz computer having 8 GB of RAM.

In order to test the validity and accuracy of the current method, the newly obtained scheme has been tested using various values of space step size, time step size and final value of time. In Table 1, a comparison of the calculated error norms L_2 of the present scheme with those in Ref.[18] for values of $N=1000$ and $k=\Delta t=0.01,0.005,0.0025$ ($\alpha=1$, $0\leq x\leq 1$, $t_{final}=1$) is presented. From the table, it is obviously seen that as the values of Δt decrease, also the values of the error norm L_2 decrease. Moreover, one can see that both of the results obtained using CHCM-C and CHCM-L are much more better than those in Ref. [1] From Table 2, one can see a comparison of the calculated error norms L_2 of the present

Table 1: A comparison of the calculated error norms L_2 of the present scheme with those in Ref.[18] for $N = 1000$ and $k = \Delta t = 0.01, 0.005, 0.0025$ ($\alpha = 1$, $0 \le x \le 1$, $t_{final} = 1$).

	L_2		
Δt	[18]	CHCM-L	CHCM-C
0.01	4.2273×10^{-4}	4.1333×10^{-7}	4.1343×10^{-7}
0.005	1.0560×10^{-4}	1.0353×10^{-7}	1.0363×10^{-7}
0.0025	2.6395×10^{-5}	2.5895×10^{-8}	2.6000×10^{-8}

Table 2: A comparison of the calculated error norms L_2 of the present scheme with those in Ref.[19] for various values of $N = 5, 10, 20$ and $k = \Delta t = 10^{-6}$ ($\alpha = 1$, $0 \le x \le 1$, $t_{final} = 1$).

	L_2			
N	[19](CN-I)	[19](CN-II)	CHCM-L	CHCM-C
5	4.7696×10^{-6}	8.5859×10^{-3}	3.5716×10^{-8}	4.0123×10^{-6}
10	7.7143×10^{-9}	2.1412×10^{-3}	2.2848×10^{-9}	1.0378×10^{-6}
20	1.8820×10^{-11}	5.3498×10^{-4}	1.4361×10^{-10}	2.6167×10^{-7}

Table 3: A comparison of the calculated error norms L_2 of the present scheme with those in Ref.[18] for $N = 10, 20, 40$ and $k = \Delta t = 10^{-6}$ ($\alpha = 1$, $0 \le x \le 1$, $t_{final} = 0.1$).

N	[18]		CHCM-L	CHCM-C
	$\theta = 0.1$	$\theta = 0.2$		
10	1.6534×10^{-4}	2.4968×10^{-4}	1.6957×10^{-6}	7.5463×10^{-4}
20	4.3906×10^{-7}	7.8152×10^{-7}	1.0426×10^{-7}	1.8898×10^{-4}
40	8.6154×10^{-10}	8.0111×10^{-10}	6.4916×10^{-9}	4.7266×10^{-5}
	$\theta = 0.3$	$\theta = 0.4$		
10	3.2357×10^{-4}	3.6276×10^{-4}	1.6957×10^{-6}	7.5463×10^{-4}
20	1.0791×10^{-6}	1.2556×10^{-6}	1.0426×10^{-7}	1.8898×10^{-4}
40	9.2171×10^{-10}	1.1166×10^{-9}	6.4916×10^{-9}	4.7266×10^{-5}
	$\theta = 0.5$	$\theta = 0.6$		
10	3.5225×10^{-4}	2.8316×10^{-4}	1.6957×10^{-6}	7.5463×10^{-4}
20	1.2494×10^{-6}	1.0109×10^{-6}	1.0426×10^{-7}	1.8898×10^{-4}
40	1.2167×10^{-9}	1.1107×10^{-9}	6.4916×10^{-9}	4.7266×10^{-5}
	$\theta = 0.7$	$\theta = 0.8$		
10	1.5954×10^{-4}	1.9640×10^{-4}	1.6957×10^{-6}	7.5463×10^{-4}
20	5.4419×10^{-7}	9.1503×10^{-7}	1.0426×10^{-7}	1.8898×10^{-4}
40	7.4804×10^{-10}	8.6751×10^{-10}	6.4916×10^{-9}	4.7266×10^{-5}
	$\theta = 0.9$	$\theta = 0.95$		
10	4.5197×10^{-4}	6.1645×10^{-4}	1.6957×10^{-6}	7.5463×10^{-4}
20	2.0820×10^{-6}	2.8818×10^{-6}	1.0426×10^{-7}	1.8898×10^{-4}
40	2.4668×10^{-9}	3.6858×10^{-9}	6.4916×10^{-9}	4.7266×10^{-5}

scheme with those in Ref.[19] for various values of $N = 5, 10, 20$ and $k = \Delta t = 10^{-6}$ ($\alpha = 1$, $0 \le x \le 1$, $t_{final} = 1$). One can see from the table that as the number of elements increase, the values of the error norm L_2 decrease. At the same time, from the table it is obvious that the values of the error norm L_2 found by CHCM-L are much more better than those found by CHCM-C. Again, from the same table it is obvious that the results obtained using the current scheme are generally better than those in Ref. [19].

Table 3 clearly shows a comparison of the calculated error norms L_2 of the present scheme with those in Ref.[18] for $N = 10, 20, 40$ and $k = \Delta t = 10^{-6}$ ($\alpha = 1$, $0 \le x \le 1$, 0, $t_{final} = 0.1$). While, from the table it is obvious that the values of the error norm L_2 found by CHCM-L are much more better than those found by CHCM-C, it is again obvious that the results obtained using the current scheme are generally in good agreement with those in Ref. [18]. Table 4 illustrates a comparative table of the error norm L_∞ of the present scheme with those in Refs. [10] and [15] for various values of $h = \Delta x = k = \Delta t$ ($\alpha = 1$, $0 \le x \le 1$, $t_{final} = 1$). From the table it is easy to see that the present results are much better than both of the compared ones. Table 5 shows a comparison of the error norms L_2 and L_∞ for $h = 1/16$, $k = 0.01$, $\alpha = 1$ with those of Ref.[14] at various t_{final} values. It is seen from the table that the calculated results are much better than

Table 4: A comparison of the calculated error norms L_∞ of the present scheme with those in Refs. [10] and [15] for various values of $h=\Delta x=k=\Delta t(\alpha=1,\ 0\leq x\leq 1,\ t_{final}=1)$.

$h=k$	CHCM-L	CHCM-C	[10]	[15](CN)	[15](CBVM)
0.2	5.8498×10^{-5}	5.7997×10^{-5}	1.4145×10^{-1}	1.1×10^{-1}	2.8×10^{-2}
0.1	3.1584×10^{-5}	3.2175×10^{-5}	3.7195×10^{-2}	3.0×10^{-2}	3.8×10^{-3}
0.05	9.7065×10^{-6}	9.9357×10^{-6}	8.4588×10^{-3}	6.9×10^{-3}	2.7×10^{-4}
0.025	2.5485×10^{-6}	2.6120×10^{-6}	2.0698×10^{-3}	1.7×10^{-3}	1.3×10^{-5}
0.0125	6.4488×10^{-7}	6.6116×10^{-7}	5.1473×10^{-4}	4.2×10^{-4}	5.1×10^{-7}
0.00625	1.6171×10^{-7}	1.6580×10^{-7}	-	1.1×10^{-4}	3.6×10^{-8}
0.01	4.1332×10^{-7}	4.2376×10^{-7}			
0.005	1.0353×10^{-7}	1.0615×10^{-7}			
0.0025	2.5895×10^{-8}	2.6551×10^{-8}			
0.002	1.6574×10^{-8}	1.6993×10^{-8}			
0.001	4.1437×10^{-9}	4.2487×10^{-9}			

Table 5: A comparison of the calculated error norms L_2 and L_∞ of the present scheme with those in Ref. [14] for $h=\Delta x=1/16$, $k=\Delta t=0.01$ at various t_{final} values ($\alpha=1,\ 0\leq x\leq 1$).

t_{final}	L_2			L_∞		
	CHCM-L	CHCM-C	[14]	CHCM-L	CHCM-C	[14]
0.1	2.9917×10^{-4}	5.9459×10^{-4}	4.86×10^{-3}	2.9891×10^{-4}	5.9435×10^{-4}	6.79×10^{-3}
0.3	1.2457×10^{-4}	2.4739×10^{-4}	8.87×10^{-5}	1.2447×10^{-4}	2.4729×10^{-4}	3.76×10^{-4}
0.5	2.8818×10^{-5}	5.7185×10^{-5}	1.73×10^{-3}	2.8793×10^{-5}	5.7162×10^{-5}	2.44×10^{-4}
0.7	5.5999×10^{-6}	1.1103×10^{-5}	2.04×10^{-4}	5.5953×10^{-6}	1.1099×10^{-5}	3.17×10^{-4}
0.9	9.9934×10^{-7}	1.9800×10^{-6}	2.14×10^{-3}	9.9856×10^{-7}	1.9794×10^{-6}	3.14×10^{-3}
1.0	4.1368×10^{-7}	8.1936×10^{-7}	2.15×10^{-3}	4.1338×10^{-7}	8.1913×10^{-7}	3.32×10^{-3}

those in Ref.[14] both in terms of the error norms L_2 and L_∞.

To see visually how good the calculated results approach to their exact ones, in Figure 1 numerical simulation of the problem for values of (a) $\alpha=1, h=0.1$, $\Delta t=0.01$ and (b) $\alpha=1, h=0.01$, $\Delta t=0.01$ is given. From the figure it is clearly seen that as the number of partition in spatial direction increases, so the numerical solution approaches to the exact one. One can clearly see that the graphs in Figure 1 (a) are very similar to those given in Ref.[5].

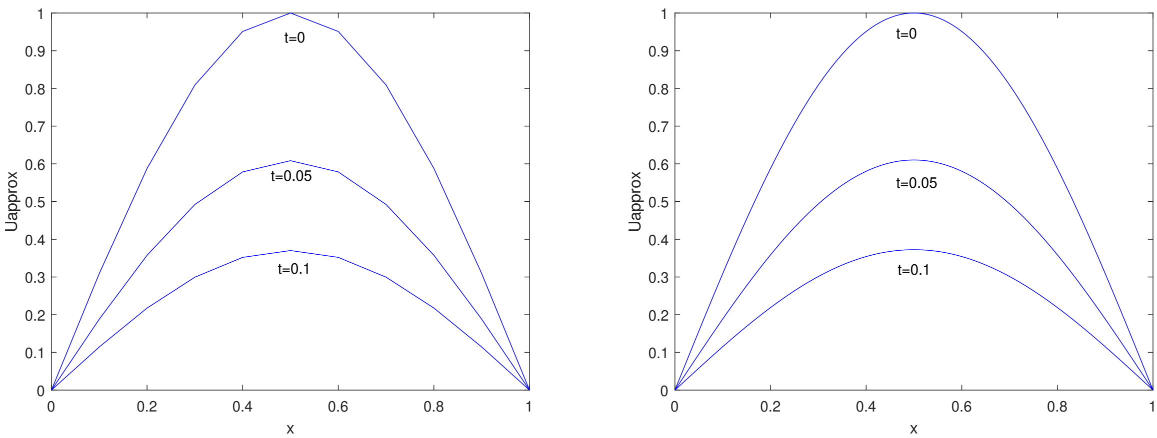

Fig. 1: Numerical simulation of the problem for values of (a) $\alpha=1, h=0.1$, $\Delta t=0.01$ and (b) $\alpha=1, h=0.01$, $\Delta t=0.01$.

6 Conclusion

In this study, the proposed scheme resulting an implicit linear algebraic system is successfully applied for obtaining the approximate solutions of the heat conduction equation in one dimension. The numerical experiments showed that the approximate solutions are in good harmony with the exact ones. In conclusion, the present numerical scheme, which can be easily implemented, produces accurate and reliable results. As a prospective study, the method may be successfully used to find approximate solutions of such PDEs that play a crucial role in describing phenomena which is nonlinear in nature encountered in physics and applied mathematics.

7 Declaration of Ethical Standards

The authors declare that the methods and techniques utilized in their manuscript do not require any ethical committee and/or legal special permission.

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