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Almost $C(\alpha)$-manifold on $M-$projective curvature tensor

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Abstract: In this article, the behavior of the $C(\alpha)$-manifold satisfying the conditions $R(X,Y)W^* = 0, W^*(X,Y)R = 0, W^*(X,Y)\tilde{Z} = 0$, $W^*(X,Y)S = 0$ and $W^*(X,Y)\tilde{C} = 0$ on the $M-$projective curvature tensor is investigated. The $C(\alpha)-$Manifold is characterized according to these states of the curvature tensor. Here, $W^*, R, S, \tilde{Z}$ and $\tilde{C}$ are $M-$projective, Riemann, Ricci, concircular and quasi-conformal curvature tensors.

Keywords: $M-$Projective Curvature Tensor, Ricci Curvature Tensor, Concircular Curvature Tensor

1 Introduction

A new tensor field

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{1}$$

is defined by Pokhariyal and Mishra in $n-$dimensional Riemannian manifolds [1]. The W^* tensor field is called the $M-$**projective tensor field** where Q is the Ricci operator and S is the Ricci tensor. The definition and properties of the $M-$projective curvature tensor are given by Ojha in Sasakian and Kaehler manifolds [2],[3]. In recent years, many geometers have worked on the $M-$projective curvature tensor [4]-[10]. Again, many authors have worked on curvature tensors in almost $C(\alpha)-$manifold [11]-[13].

Based on the many studies mentioned above, in this article, the curvature conditions of $C(\alpha)-$manifold $R(X,Y)W^* = 0, W^*(X,Y)R = 0, W^*(X,Y)\tilde{Z} = 0, W^*(X,Y)S = 0$ and $W^*(X,Y)\tilde{C} = 0$ are searched.

Let's take an $(2n+1)-$dimensional differentiable M manifold. If it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions;

$$\phi^2 X = -X + \eta(X)\xi \text{ and } \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \text{ and } g(X,\xi) = \eta(X),$$

for all $X,Y \in \chi(M)$ and $\xi \in \chi(M)$, (ϕ,ξ,η,g) is called **almost contact metric structure** and (M,ϕ,ξ,η,g) is called **almost contact metric manifold**. On the $(2n+1)$ dimensional M manifold,

$$g(\phi X, Y) = -g(X, \phi Y),$$

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for all $X,Y\in\chi(M)$, that is, ϕ is an anti-symmetric tensor field according to the g metric. The transformation Φ defined as

$$\Phi(X,Y)=g(X,\phi Y),$$

for all $X,Y\in\chi(M)$, is called the **fundamental 2-form** of the (ϕ,ξ,η,g) almost contact metric structure, where

$$\eta\wedge\Phi^n\neq 0.$$

If the R Riemann curvature tensor of the M almost contact metric manifold satisfies the condition

$$R(X,Y,Z,W)=R(X,Y,\phi Z,\phi W)+\alpha\{-g(X,Z)g(Y,W)+g(X,W)g(Y,Z)+g(X,\phi Z)g(Y,\phi W)-g(X,\phi W)g(Y,\phi Z)\},$$

for all $X,Y,Z,W\in\chi(M),\exists\alpha\in\mathbb{R}$, then M is called the **almost $C(\alpha)$-manifold**. Also, the **Riemann curvature tensor** of a almost $C(\alpha)$-manifold with $c-$constant sectional curvature is given by

$$\begin{aligned}R(X,Y)Z=&\left(\frac{c+3\alpha}{4}\right)\{g(Y,Z)X-g(X,Z)Y\}+\left(\frac{c-\alpha}{4}\right)\{g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X\\&+2g(X,\phi Y)\phi Z+\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}.\end{aligned} \tag{2}$$

For a $(2n+1)-$dimensional M almost $C(\alpha)-$manifold, the following equations are provided.

$$S(X,Y)=\left[\frac{\alpha(3n-1)+c(n+1)}{2}\right]g(X,Y)+\frac{(\alpha-c)(n+1)}{2}\eta(X)\eta(Y), \tag{3}$$

$$S(X,\xi)=2n\alpha\eta(X), \tag{4}$$

$$QX=\left[\frac{\alpha(3n-1)+c(n+1)}{2}\right]X+\frac{(\alpha-c)(n+1)}{2}\eta(X)\xi, \tag{5}$$

$$Q\xi=2n\alpha\xi, \tag{6}$$

$$Q\phi Y=\frac{r-2n\alpha}{2n}QY, \tag{7}$$

for all $X,Y,\in\chi(M)$, where Q and S are the Ricci operator and Ricci tensor of manifold M, respectively.

2 $C(\alpha)$-manifolds satisfying some important conditions on the $M-$projective curvature tensor

Let M be a $(2n+1)-$dimensional almost $C(\alpha)-$manifold and R be the Riemann curvature tensor of M manifold. So, if we choose $X=\xi$ in (2), we get

$$R(\xi,Y)Z=\alpha[g(Y,Z)\xi-\eta(Z)Y]. \tag{8}$$

Similarly, if we choose $Z=\xi$ in (2), we get

$$R(X,Y)\xi=\alpha[\eta(Y)X-\eta(X)Y]. \tag{9}$$

In addition, if $Y=\xi$ is chosen in (9),

$$R(X,\xi)\xi=\alpha\left[X-\eta(X)\xi\right]$$

is obtained. If the inner product of both sides of (2) is taken by $\xi\in\chi(M)$, we have

$$\eta(R(X,Y)Z)=\alpha\left[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\right].$$

Finally, if we choose $X=\xi$ in the (1), then it reduces the form

$$W^{*}(\xi,Y)Z=\frac{(n+1)(\alpha-c)}{8n}\left[g(Y,Z)\xi-\eta(Z)Y\right], \tag{10}$$

and if we choose $Z=\xi$ in the same equation, we get

$$W^{*}(X,Y)\xi=\frac{(n+1)(\alpha-c)}{8n}\left[\eta(Y)X-\eta(X)Y\right].$$

Theorem 1. *Let M be a $(2n+1)-$dimensional almost $C(\alpha)-$manifold. If M is $M-$projective flat, then M is an Einstein manifold.*

Proof. Let's assume that manifold M is M-projective flat. From (1), we can write

$$W^{*}(X,Y)Z=0,$$

for each $X,Y,Z\in\chi(M)$. Then from (1), we obtain

$$\begin{aligned} R(X,Y)Z=&\frac{1}{4n}[S(Y,Z)X-S(X,Z)Y\\ &+g(Y,Z)QX-g(X,Z)QY], \end{aligned} \tag{11}$$

for each $X,Y,Z\in\chi(M)$. If we choose $Z=\xi$ in (11) and using $(4),(9)$, we obtain

$$\frac{\alpha}{2}\left[\eta(Y)X-\eta(X)Y\right]=\frac{1}{4n}\left[\eta(Y)QX-\eta(X)QY\right].$$

In the last equation, if we first choose $X=\xi$ and we take inner product both sides of the last equation by $Z\in\chi(M)$, then we get

$$S(Y,Z)=2n\alpha g(Y,Z)$$

It is clear from the last equation that M is Einstein manifold.

Theorem 2. *Let M be $(2n+1)-$dimensional a $C(\alpha)-$manifolds. Then $W^{*}(X,Y)R=0$ if and only if either the scalar curvature of M is $r=2n\alpha(2n+1)$ or M reduces real space form with constant sectinal curvature.*

Proof. Suppose that $W^{*}(X,Y)R=0$. Then, we have

$$(W^{*}(X,Y)R)(U,V,Z)=W^{*}(X,Y)R(U,V)Z-R(W^{*}(X,Y)U,V)Z-R(U,W^{*}(X,Y)V)Z-R(U,V)W^{*}(X,Y)Z=0.$$

If we choose $X=\xi$ in here, we get

$$(W^*(\xi,Y)R)(U,V,Z)=W^*(\xi,Y)R(U,V)Z-R(W^*(\xi,Y)U,V)Z-R(U,W^*(\xi,Y)V)Z-R(U,V)W^*(\xi,Y)Z=0, \tag{12}$$

for each $Y,U,V,Z\in\chi(M)$. In (12), using (10), we obtain

$$\begin{aligned}&\frac{(n+1)(\alpha-c)}{8n}[g(Y,R(U,V)Z)\xi-\eta(R(U,V)Z)Y-g(Y,U)R(\xi,V)Z+\eta(U)R(Y,V)Z\\&-g(Y,V)R(U,\xi)Z+\eta(V)R(U,Y)Z-g(Y,Z)R(U,V)\xi+\eta(Z)R(U,V)Y]=0.\end{aligned} \tag{13}$$

Substituting $U=\xi$ in (13) and using $(8),(9)$, we conclude

$$\frac{(n+1)(\alpha-c)}{8n}[R(Y,V)Z-\alpha(g(V,Z)Y-g(Y,Z)V)]=0. \tag{14}$$

From (14), we have

$$c=\alpha. \tag{15}$$

In addition, since the scalar curvature of a $C(\alpha)-$manifold with constant sectional curvature is

$$r=n[\alpha(3n+1)+c(n+1)] \tag{16}$$

if the expression (15) is also put in (16), we get

$$r=2n\alpha(2n+1).$$

On the other hand, from (14) we get

$$R(Y,V)Z=\alpha[g(V,Z)Y-g(Y,Z)V].$$

Thus, M is reduced to the real space form with constant sectional curvature. The converse is obvius and the proof is completed.

Let M be a $(2n+1)-$dimensional Riemannian manifold. Then the **concircular curvature tensor** $\tilde{Z}$ is defined as

$$\tilde{Z}(X,Y)Z=R(X,Y)Z-\frac{r}{2n(2n+1)}[g(Y,Z)X-g(X,Z)Y], \tag{17}$$

for all $X,Y,Z\in\chi(M)$. If we choose $X=\xi$ in (17), we get

$$\tilde{Z}(\xi,Y)Z=\left(\alpha-\frac{r}{2n(2n+1)}\right)[g(Y,Z)\xi-\eta(Z)Y], \tag{18}$$

and when we choose $Z=\xi$ in (18) we get

$$\tilde{Z}(\xi,Y)\xi=\left(\alpha-\frac{r}{2n(2n+1)}\right)[\eta(Y)\xi-Y].$$

Theorem 3. *Let M be $(2n+1)-$dimensional $C(\alpha)-$manifold. Then $W^*(X,Y)\tilde{Z}=0$ if and only if either the scalar curvature of M is $r=2n\alpha(2n+1)$ or M reduces real space form with constant sectinal curvature-c.*

Proof. Suppose that $W^*(X,Y)\tilde{Z}=0$. Then we have

$$\left(W^*(X,Y)\tilde{Z}\right)(U,V,Z)=W^*(X,Y)\tilde{Z}(U,V)Z-\tilde{Z}(W^*(X,Y)U,V)Z-\tilde{Z}(U,W^*(X,Y)V)Z-\tilde{Z}(U,V)W^*(X,Y)Z=0.$$

If we choose $X=\xi$ in here, we get

$$\left(W^*(\xi,Y)\tilde{Z}\right)(U,V,Z)=W^*(\xi,Y)\tilde{Z}(U,V)Z-\tilde{Z}(W^*(\xi,Y)U,V)Z-\tilde{Z}(U,W^*(\xi,Y)V)Z-\tilde{Z}(U,V)W^*(\xi,Y)Z=0, \tag{19}$$

for each $Y,U,V,Z\in\chi(M)$. In (19), using (10), we obtain

$$\begin{aligned}&\frac{(n+1)(\alpha-c)}{8n}\left[g\left(Y,\tilde{Z}(U,V)Z\right)\xi-\eta\left(\tilde{Z}(U,V)Z\right)Y\right.\\&-g(Y,U)\tilde{Z}(\xi,V)Z+\eta(U)\tilde{Z}(Y,V)Z-g(Y,V)\tilde{Z}(U,\xi)Z\\&\left.+\eta(V)\tilde{Z}(U,Y)Z-g(Y,Z)\tilde{Z}(U,V)\xi+\eta(Z)\tilde{Z}(U,V)Y\right]=0.\end{aligned} \tag{20}$$

Taking $U=\xi$ in (20) and using (18), we obtain

$$\begin{aligned}&\frac{(n+1)(\alpha-c)}{8n}\left[\tilde{Z}(Y,V)Z-\left(\alpha-\tfrac{r}{2n(2n+1)}\right)\right.\\&\left.(g(V,Z)Y-g(Y,Z)V)\right]=0.\end{aligned} \tag{21}$$

In (21), using (17) we conclude

$$\frac{(n+1)(\alpha-c)}{8n}\left[R(Y,Z)V-\alpha(g(V,Z)Y-g(Y,Z)V)\right]=0.$$

This proves our assertion. The converse obvious.

Theorem 4. *Let M be $(2n+1)-$dimensional a $C(\alpha)-$manifold. Then $W^*(X,Y)S=0$ if and only if either the scalar curvature of M is $r=2n\alpha(2n+1)$ or M reduces an Einstein manifold.*

Proof. Suppose that $W^*(X,Y)S=0$. Then we can easily see that

$$S(W^*(X,Y)Z,U)+S(Z,W^*(X,Y)U)=0.$$

If we choose $X=\xi$ in here, we get

$$S(W^*(\xi,Y)Z,U)+S(Z,W^*(\xi,Y)U)=0. \tag{22}$$

In (22), using (10), we obtain

$$\frac{(n+1)(\alpha-c)}{8n}\left[2n\alpha\eta(U)g(Y,Z)-\eta(Z)S(Y,U)+2n\alpha\eta(Z)g(Y,U)-\eta(U)S(Z,Y)\right]=0. \tag{23}$$

Substituting $Z=\xi$ in (23), we find

$$\frac{(n+1)(\alpha-c)}{8n}\left[-S(Y,U)+2n\alpha g(Y,U)\right]=0. \tag{24}$$

From (24), we get

$$c=\alpha.$$

This tell us that the scalar curvature of M is

$$r=2n\alpha(2n+1).$$

On the other hand, from (24) we have

$$S(Y,U)=2n\alpha g(Y,U),$$

which implies M reduces an Einstein manifold. This proves our assertion. The converse is obvious.

The concept of the **quasi-conformal curvature tensor** was defined by Yano and Sowaki as

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY] \\ &-\frac{r}{2n+1}\left[\frac{a}{2n}+2b\right][g(Y,Z)X-g(X,Z)Y],\end{aligned} \tag{25}$$

where a and b are constants, Q is the Ricci operator, S is the Ricci tensor and r is the scalar curvature of the manifold. If $\tilde{C}=0$, then this manifold is called a **quasi-conformal flat**. If $X=\xi$ is chosen in (25),

$$\tilde{C}(\xi,Y)Z=\left[\frac{bc(n+1)+\alpha(2a+7bn-b)}{2}-\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\otimes[g(Y,Z)\xi-\eta(Z)Y], \tag{26}$$

and if $Z=\xi$ is chosen in (26), we reach at

$$\tilde{C}(\xi,Y)\xi=\left[a\alpha+2nb\alpha-\frac{r}{2n+1}\left(\tfrac{a}{2n}+2b\right)\right][\eta(Y)\xi-Y]+b[2n\alpha\eta(Y)\xi-QY]. \tag{27}$$

Theorem 5. *Let M be $(2n+1)-$dimensional a $C(\alpha)-$manifolds. Then $W^{*}(X,Y)\tilde{C}=0$ if and only if either the scalar curvature of M is $r=2n\alpha(2n+1)$ or M reduces real space form with constant sectinal curvature.*

Proof. Suppose that $W^{*}(X,Y)\tilde{C}=0$. Then, we have

$$\begin{aligned}\left(W^{*}(X,Y)\tilde{C}\right)(U,V,Z) &= W^{*}(X,Y)\tilde{C}(U,V)Z-\tilde{C}(W^{*}(X,Y)U,V)Z \\ &-\tilde{C}(U,W^{*}(X,Y)V)Z-\tilde{C}(U,V)W^{*}(X,Y)Z=0.\end{aligned}$$

If we choose $X=\xi$ in here

$$\begin{aligned}\left(W^{*}(\xi,Y)\tilde{C}\right)(U,V,Z) &= W^{*}(\xi,Y)\tilde{C}(U,V)Z-\tilde{C}(W^{*}(\xi,Y)U,V)Z \\ &-\tilde{C}(U,W^{*}(\xi,Y)V)Z-\tilde{C}(U,V)W^{*}(\xi,Y)Z=0,\end{aligned} \tag{28}$$

for each $Y,U,V,Z \in \chi(M)$. Using (10) in (28), we get

$$\begin{aligned}&\frac{(n+1)(\alpha-c)}{8n}\left[g\left(Y,\tilde{C}(U,V)Z\right)\xi-\eta\left(\tilde{C}(U,V)Z\right)Y\right.\\&-g(Y,U)\tilde{C}(\xi,V)Z+\eta(U)\tilde{C}(Y,V)Z-g(Y,V)\tilde{C}(U,\xi)Z\\&\left.+\eta(V)(U)\tilde{C}(U,Y)Z-g(Y,Z)\tilde{C}(U,V)\xi+\eta(Z)\tilde{C}(U,V)Y\right]=0.\end{aligned}\tag{29}$$

Taking $U=\xi$ in (29) and using $(26),(27)$, we obtain

$$\begin{aligned}&\left[\frac{(n+1)(\alpha-c)}{8n}\right]\otimes\left\{\tilde{C}(Y,Z)V-\left[\frac{bc(n+1)+\alpha(2a+7bn-b)}{2}-\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\right.\\&\left.[g(V,Z)Y-g(Y,Z)V]\right\}=0.\end{aligned}$$

In the last equation, if (25) is written in its place and necessary adjustments are made, we get

$$\begin{aligned}aR(Y,V)Z=&\left[\frac{\alpha(2a+bn+b)-bc(n+1)}{2}\right][g(V,Z)Y-g(Y,Z)V]\\&-\frac{b(\alpha-c)(n+1)}{2}[\eta(V)\eta(X)Y-\eta(Y)\eta(Z)V+g(V,Z)\eta(Y)\xi-g(Y,Z)\eta(V)\xi].\end{aligned}\tag{30}$$

Substituting $Y\to\phi Y$ and $V\to\phi V$ in (30), we conclude

$$R(\phi Y,\phi V)Z=\left[\frac{\alpha(2a+bn+b)-bc(n+1)}{2}\right][g(V,Z)Y-g(Y,Z)V].$$

This proves our assertion. The converse is obvious.

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