Note on the projectable linear connection in the semi-tangent bundle

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Abstract: The present paper is devoted to some results concerning with the projectable linear connection in the semi-tangent (pull-back) bundle \( tM \). In this study, horizontal lift problems of projectable linear connection, which are preliminary to the subject of covarient derivates of almost contact structure and almost paracontact structure on semi-tangent bundle, are discussed.

Keywords: Horizontal lift, Projectable linear connection, Pull-back bundle, Semi-tangent bundle, Vector field.

1 Introduction

Let \( M_n \) be a differentiable manifold of class \( C^\infty \) and finite dimension \( n \), and let \((M_n, \pi_1, B_m)\) be a differentiable bundle over \( B_m \). We use the notation \((x^i) = (x^a, x^\alpha)\), where the indices \( i, j, \ldots \) run from 1 to \( n \), the indices \( a, b, \ldots \) from 1 to \( n - m \) and the indices \( \alpha, \beta, \ldots \) from \( n - m + 1 \) to \( n \). \( x^a \) are coordinates in \( B_m \), \( x^\alpha \) are fibre coordinates of the bundle

\[ \pi_1 : M_n \rightarrow B_m. \]

Let now \((T(B_m), \bar{\pi}, B_m)\) be a tangent bundle [13] over base space \( B_m \), and let \( M_n \) be differentiable bundle determined by a natural projection (submersion) \( \pi_1 : M_n \rightarrow B_m \). The semi-tangent bundle (pull-back) \([2],[3],[9],[10],[14],[15]\) of the tangent bundle \((T(B_m), \bar{\pi}, B_m)\) is the bundle \((t(B_m), \pi_2, M_n)\) over differentiable bundle \( M_n \) with a total space

\[ t(B_m) = \{(x^a, x^\alpha, x^\mu) \in M_n \times T_\alpha(B_m) : \pi_1(x^a, x^\alpha, x^\mu) = \bar{\pi}(x^\alpha) = (x^\alpha)\} \subset M_n \times T_\alpha(B_m) \]

and with the projection map \( \pi_2 : t(B_m) \rightarrow M_n \) defined by \( \pi_2(x^a, x^\alpha, x^\mu) = (x^a, x^\alpha) \), where \( T_\alpha(B_m)(x = \pi_1(\tilde{x}), \tilde{x} = (x^a, x^\alpha) \in M_n) \) is the tangent space at a point \( x \) of \( B_m \), where \( x^\mu = y^\alpha (\alpha, \beta, \ldots = n + 1, \ldots, 2n) \) are fibre coordinates of the tangent bundle \( T(B_m) \).

Where the pull-back (Pontryagin [7]) bundle \( t(B_m) \) of the differentiable bundle \( M_n \) also has the natural bundle structure over \( B_m \), its bundle projection \( \pi : t(B_m) \rightarrow B_m \) being defined by \( \pi : (x^a, x^\alpha, x^\mu) \rightarrow (x^\mu) \), and hence \( \pi = \pi_1 \circ \pi_2 \). Thus \((t(B_m), \pi_1 \circ \pi_2)\) is the composite bundle \([8],[9]\) or step-like bundle [6]. Consequently, we notice the semi-tangent bundle \((t(B_m), \pi_2)\) is a pull-back bundle of the tangent bundle over \( B_m \) by \( \pi_1 \) [9].

If \((x^i) = (x^a, x^\alpha)\) is another local adapted coordinates in differentiable bundle \( M_n \), then we have

\[
\begin{align*}
x^a &= x^\beta (x^\alpha), \\
x^\alpha &= x^\alpha (x^\beta).
\end{align*}
\]

(1)

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The Jacobian of (1) has the components

\[
\left( A^f_j \right) = \left( \frac{\partial x^j}{\partial x^i} \right) = \left( \begin{array}{cc} A^d_b & A^d_p \\ 0 & A^d_p \end{array} \right),
\]

where \( A^d_b = \frac{\partial x^d}{\partial x^b}, A^d_p = \frac{\partial x^d}{\partial x^p}, A^d_p = \frac{\partial x^d}{\partial x^p} \) [9].

To a transformation (1) of local coordinates of \( M_n \), there corresponds on \( t(B_m) \) the change of coordinate

\[
\begin{align*}
  x^d &= x^d(\bar{x}^b, \bar{x}^\beta), \\
  x^{\alpha} &= x^{\alpha}(\bar{x}^\beta), \\
  \bar{x}^\beta &= \frac{\partial x^\alpha}{\partial x^\beta} \bar{x}^\beta.
\end{align*}
\]

(2)

The Jacobian of (2) is:

\[
A = \left( A^f_j \right) = \left( \begin{array}{ccc} A^d_b & A^d_p & 0 \\ 0 & A^d_p & 0 \\ 0 & A^d_p & 0 \end{array} \right),
\]

(3)

where \( I = (a, \alpha, \beta), J = (b, \beta, \overline{\beta}), I, J, ..., = 1, ..., 2n; A^d_{b\beta\alpha} = \frac{\partial x^d}{\partial x^b \partial x^\alpha} \) [9].

The purpose of this paper is to study the horizontal lifts of projectable linear connection to semi-tangent (pull-back) bundle \( t(B_m) \) and their properties.

We denote by \( \mathcal{Z}_p^q(M_n) \) the set of all tensor fields of class \( C^\infty \) and of type \( (p, q) \) on \( M_n \), i.e., contravariant degree \( p \) and covariant degree \( q \). We now put \( \mathcal{Z}(M_n) = \sum_{p,q=0}^{n} \mathcal{Z}_p^q(M_n) \), which is the set of all tensor fields on \( M_n \). Similarly, we denote by \( \mathcal{Z}_p^q(B_m) \) and \( \mathcal{Z}(B_m) \) respectively the corresponding sets of tensor fields in the base space \( B_m \).

2 Some lifts of vector and covector fields

If \( f \) is a function on \( B_m \), we write \( ^v f \) for the function on \( t(B_m) \) obtained by forming the composition of \( \pi : t(B_m) \to B_m \) and \( f = f \circ \pi_1 \), so that

\[
^v f = ^v f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi.
\]

Thus, the vertical lift \( ^v f \) of the function \( f \) to \( t(B_m) \) satisfies

\[
^v f(x^a, x^\alpha, x^\beta) = f(x^\alpha).
\]

(4)

We note here that value \( ^v f \) is constant along each fibre of \( \pi : t(B_m) \to B_m \). Let \( X \in \mathcal{Z}_0^1(B_m) \), i.e. \( X = X^\alpha \partial_\alpha \). On putting

\[
^v X = \left( ^v X^\alpha \right) = \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix},
\]

(5)

from (3), we easily see that \( ^v X^\prime = \bar{A} (^v X) \). The vector field \( ^v X \) is called the vertical lift of \( X \) to \( t(B_m) \).
Let $\omega \in \mathfrak{X}^1(B_m)$, i.e. $\omega = \omega_\alpha dx^\alpha$. On putting
\[
\nu^\nu \omega = (\nu^\nu \omega)_\alpha = (0, \omega_\alpha, 0),
\]
from (3), we easily see that $\nu^\nu \omega = \bar{A} \nu^\nu \omega'$. The covector field $\nu^\nu \omega$ is called the vertical lift of $\omega$ to $t(B_m)$.

Let $\tilde{X} \in \mathfrak{X}_0(M_n)$ be a projectable vector field [11] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\tilde{X} = \tilde{X}^\alpha(x^\alpha,x^\alpha)\partial_\alpha + X^\alpha(x^\alpha)\partial_\alpha$.

Now, consider $\tilde{X} \in \mathfrak{X}_0(M_n)$, then $\tilde{c} \tilde{X}$ (complete lift) has the components on the semi-tangent bundle $t(B_m)$ [9]
\[
\tilde{c} \tilde{X} = \left(\tilde{c} \tilde{X}^\alpha \right) = \begin{pmatrix} \tilde{X}^\alpha \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix}
\]
with respect to the coordinates $(x^a, x^\alpha, x^{\varepsilon})$.

For any $F \in \mathfrak{X}_1(B_m)$, if we take account of (3), we can prove that $(\gamma F)' = \bar{A}(\gamma F)$, where $\gamma F$ is a vector field defined by
\[
\gamma F = \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon F^\varepsilon \end{pmatrix}
\]
with respect to the coordinates $(x^a, x^\alpha, x^{\varepsilon})$.

Let now $\tilde{X} \in \mathfrak{X}_0(M_n)$ be a projectable vector field on $M_n$ with projection $X \in \mathfrak{X}_1(B_m)$ [11]. Then we define the horizontal lift $^{HH} \tilde{X}$ of $\tilde{X}$ by
\[
^{HH} \tilde{X} = \tilde{c} \tilde{X} - \gamma(\nabla \tilde{X})
\]
on $t(M_n)$. Where $\nabla$ is a projectable symmetric linear connection in a differentiable manifold $B_m$. Then, remembering that $\tilde{c} \tilde{X}$ and $\gamma(\nabla \tilde{X})$ have, respectively, local components
\[
\tilde{c} \tilde{X} = \left(\tilde{c} \tilde{X}^\alpha \right) = \begin{pmatrix} \tilde{X}^\alpha \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix}, \gamma(\nabla \tilde{X}) = \begin{pmatrix} \gamma(\nabla \tilde{X})^\varepsilon \\ 0 \\ y^\varepsilon \nabla_\varepsilon X^\alpha \end{pmatrix}
\]
with respect to the coordinates $(x^a, x^\alpha, x^{\varepsilon})$ on $t(B_m)$. $\nabla_a X^\varepsilon$ being the covariant derivative of $X^\varepsilon$, i.e.,
\[
(\nabla_a X^\varepsilon) = \partial_a X^\varepsilon + X^\beta \Gamma^a_{\beta \alpha}.
\]

We find that the horizontal lift $^{HH} \tilde{X}$ of $\tilde{X}$ has the components
\[
^{HH} \tilde{X} = \left({}^{HH} \tilde{X}^\alpha \right) = \begin{pmatrix} \tilde{X}^\alpha \\ X^\alpha \\ -\Gamma^a_{\beta \alpha} X^\beta \end{pmatrix}
\]
with respect to the coordinates $(x^a, x^\alpha, x^{\varepsilon})$ on $t(B_m)$. Where
\[
\Gamma^a_{\beta \alpha} = y^\varepsilon \Gamma^a_{\varepsilon \beta}.
\]
3 Complete lifts of projectable linear connection

Let \( I_a^\beta \) be components of projectable linear connection \([1], [4], [5], [11], [12]\) \( \nabla \) with respect to local coordinates \((x^a)\) in \( B_m \) and \( c^c I_j^l \) components of \( c^c \nabla \) with respect to the induced coordinates \((x^a, x^b, x^c)\) in \( t(B_m) \). We recall from [11] that components \( c^c I_j^l \) of complete lift \( c^c \nabla \) of projectable linear connection \( \nabla \) can be calculated from base manifold \( B_m \) to semi-tangent bundle \( t(B_m) \) also as:

\[
\begin{align*}
\gamma c^c I_a^b & = \gamma c^c I_a^b = \gamma c^c I_a^b = \gamma c^c I_a^b = \gamma c^c I_a^b = \gamma c^c I_a^b = 0, \\
\gamma c^c I_a^\gamma & = \gamma I_a^\gamma, \\
\gamma c^c I_a^\gamma & = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = 0, \\
\gamma c^c I_a^\gamma & = \gamma I_a^\gamma, \\
\gamma c^c I_a^\gamma & = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = 0, \\
\gamma c^c I_a^\gamma & = \gamma I_a^\gamma, \\
\gamma c^c I_a^\gamma & = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = \gamma c^c I_a^\gamma = 0, \\
\gamma c^c I_a^\gamma & = \gamma I_a^\gamma,
\end{align*}
\]

where \( I = (a, \alpha, \overline{a}), J = (b, \beta, \overline{b}), K = (c, \gamma, \overline{\gamma}) \). On the other hand, from (11) we obtain:

**Theorem 1.** Let \( \tilde{X} \) and \( \tilde{Y} \) be projectable vector fields on \( M_n \) with projection \( X \in \mathfrak{F}_0(B_m) \) and \( Y \in \mathfrak{F}_0(B_m) \), respectively. We have:

(i) \( c^c \nabla_{c^c X} (c^c Y) = 0 \),

(ii) \( c^c \nabla_{c^c X} (c^c \nabla_{c^c Y} c^c Z) = 0 \),

(iii) \( c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = 0 \),

(iv) \( c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = 0 \),

(vi) \( c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = c^c \nabla_{c^c X} (c^c Y) = 0 \),

where \( R(X, Y) \) is a tensor field of type of \((1, 1)\) defined by \( F(Z) = R(Z, X)Y \) for any \( Z \in \mathfrak{F}_0(B_m) \) and \( L_X \) is the operator of Lie derivation with respect to \( X \).

4 Horizontal lifts of projectable linear connection

Let there be given a projectable linear connection \( \nabla \) in \( B_m \). We shall define the horizontal lift \( \nabla \) of a projectable linear connection \( \nabla \) in \( B_m \) to \( t(B_m) \) by the conditions:

(i) \( \nabla_{c^c X} (c^c Y) = 0 \),

(ii) \( \nabla_{c^c X} (c^c Y) = 0 \),

(iii) \( \nabla_{c^c X} (c^c Y) = \nabla_{c^c X} (c^c Y) = \nabla_{c^c X} (c^c Y) = \nabla_{c^c X} (c^c Y) = \nabla_{c^c X} (c^c Y) = \nabla_{c^c X} (c^c Y) = 0 \),

(iv) \( \nabla_{c^c X} (c^c Y) = \nabla_{c^c X} (c^c Y) = \nabla_{c^c X} (c^c Y) = \nabla_{c^c X} (c^c Y) = \nabla_{c^c X} (c^c Y) = \nabla_{c^c X} (c^c Y) = 0 \),

for any \( \tilde{X}, \tilde{Y} \in \mathfrak{F}_0(M_n) \). Thus, if we put

\[
\tilde{S}(\tilde{X}, \tilde{Y}) = \nabla_{c^c X} (c^c Y) = \nabla_{c^c X} (c^c Y)
\]

(13) for any \( \tilde{X}, \tilde{Y} \in \mathfrak{F}_0(M_n) \). Then, from (13) and Theorem 1, the tensor \( \tilde{S} \) of type \((1, 2)\) in \( t(B_m) \) satisfies the conditions

(i) \( \tilde{S}(\tilde{X}, \tilde{Y}) = 0 \),

(ii) \( \tilde{S}(\tilde{X}, \tilde{Y}) = 0 \),

(iii) \( \tilde{S}(\tilde{X}, \tilde{Y}) = 0 \),
\[(iv) S^\text{HH}_\alpha X^\text{HH}_\gamma = -\gamma(R)(X,Y),\] (14)

for any \(\tilde{X}, \tilde{Y} \in \mathfrak{S}_0^1(M_n).\) Therefore \(\tilde{S}\) has the components \(\tilde{S}_{\alpha\gamma}\) such that

\[\tilde{S}_{\alpha\gamma} = -\gamma^e R^\beta_{\epsilon\alpha\gamma}\] (15)

all others being zero, with respect to the induced coordinates \((x^\beta, x^\gamma, x^\pi)\) in \(t(B_m)\).

Since the components \(\gamma^J_I K\) of \(\gamma^\nu\gamma\) are given by (11), it follows from (13) and (15) that the horizontal lift \(\text{HH}\gamma^\nu\) of a projectable linear connection \(\nabla\) has the components \(\gamma^J_I K\) such that

\[
\begin{align*}
\gamma^\nu_{\alpha\beta} &= \gamma^\nu_{\alpha\beta} - c^J_I K \\
-\gamma^e R^\beta_{\epsilon\alpha\gamma} &= \gamma^\nu_{\alpha\beta} - c^J_I K \\
\gamma^\nu_{\alpha\beta} &= \gamma^\nu_{\alpha\beta} - c^J_I K
\end{align*}
\] (16)

with respect to the induced coordinates in \(t(B_m)\). Where \(\gamma^J_I K\) are the components of \(\gamma^\nu\gamma\) in \(t(B_m)\).

**Proof.** For convenience sake we only consider \(\gamma^\nu_{\alpha\beta} \gamma\). According to (11), (13) and (15), in fact:

\[
\tilde{S}_{\alpha\gamma} = -\gamma^e R^\beta_{\epsilon\alpha\gamma} - c^J_I K \gamma
\]

Thus, we have \(\gamma^\nu_{\alpha\beta} = \gamma^\nu_{\alpha\beta} - c^J_I K \gamma\). Similarly, we can easily find other components of \(\gamma^\nu I K\).

**Theorem 2.** Let \(X, Y \in \mathfrak{S}_0^1(B_m)\). Then we obtain

\[\gamma^\nu \gamma^\nu_X(Y) = 0.\]

**Proof.** If \(X, Y \in \mathfrak{S}_0^1(B_m)\) and

\[
\begin{pmatrix}
(\gamma^\nu \gamma^\nu_X(Y))_b \\
(\gamma^\nu \gamma^\nu_X(Y))_\beta \\
(\gamma^\nu \gamma^\nu_X(Y))_\pi
\end{pmatrix}
\]

are the components of \((\gamma^\nu \gamma^\nu_X(Y)) J\) with respect to the coordinates \((x^\beta, x^\gamma, x^\pi)\) on \(t(B_m)\), then we have

\[\gamma^\nu \gamma^\nu_X(Y) J = \gamma^\nu X^a \gamma^\nu a(Y) J + \gamma^\nu X^a \gamma^\nu a(Y) J + \gamma^\nu X^\pi \gamma^\nu \pi(Y) J.\]
Firstly, if \( J = b \), we have

\[
\left( HH\nabla_{v_X}(v^Y) \right)^b = v^v X^{aHH} \nabla_a v^v b + v^v X^{aHH} \nabla_a v^v b + v^v X^{\beta HH} \nabla_\beta v^v b
\]

\[= 0 \]

by virtue of (5) and (16). Secondly, if \( J = \beta \), we have

\[
\left( HH\nabla_{v_X}(v^Y) \right)^\beta = v^v X^{aHH} \nabla_a v^v \beta + v^v X^{aHH} \nabla_a v^v \beta + v^v X^{\pi HH} \nabla_\pi v^v \beta
\]

\[= 0 \]

by virtue of (5) and (16). Thirdly, if \( J = \bar{\beta} \), then we have

\[
\left( HH\nabla_{v_X}(v^Y) \right)^{\bar{\beta}} = v^v X^{aHH} \nabla_a v^v \bar{\beta} + v^v X^{aHH} \nabla_a v^v \bar{\beta} + v^v X^{\pi HH} \nabla_\pi v^v \bar{\beta}
\]

\[= X^a \left( \partial_{\pi Y}^\bar{\beta} + HH \Gamma_\pi^\bar{\beta} \right) + v^v X^{\pi HH} \nabla_\pi v^v \bar{\beta}
\]

\[= 0 \]

by virtue of (5) and (16). Thus Theorem 2 is proved.

**Theorem 3.** Let \( \bar{Y} \) be a projectable vector field on \( M \) with projections \( Y \) on \( B_m \). If \( X \in \mathcal{S}_0(B_m) \), then

\[
HH\nabla_{v_X}(HH\bar{Y}) = 0.
\]

**Proof.** If \( \bar{Y} \in \mathcal{S}_0(M) \), \( X \in \mathcal{S}_0(B_m) \) and

\[
\begin{pmatrix}
HH\nabla_{v_X}(HH\bar{Y})^b \\
HH\nabla_{v_X}(HH\bar{Y})^\beta \\
HH\nabla_{v_X}(HH\bar{Y})^{\bar{\beta}}
\end{pmatrix}
\]

are the components of \( \left( HH\nabla_{v_X}(HH\bar{Y}) \right)^f \) with respect to the coordinates \((x^b, x^\beta, x^{\bar{\beta}})\) on \( r(B_m) \), then we have

\[
\left( HH\nabla_{v_X}(HH\bar{Y}) \right)^f = v^v X^{aHH} \nabla_a (HH\bar{Y})^f + v^v X^{aHH} \nabla_a (HH\bar{Y})^f + v^v X^{\beta HH} \nabla_\beta (HH\bar{Y})^f.
\]

Firstly, if \( J = b \), we have

\[
\left( HH\nabla_{v_X}(HH\bar{Y})^b \right) = v^v X^{aHH} \nabla_a (HH\bar{Y})^b + v^v X^{aHH} \nabla_a (HH\bar{Y})^b + v^v X^{\pi HH} \nabla_\pi (HH\bar{Y})^b
\]

\[= X^a \left( \partial_{\pi Y}^b + HH \Gamma_\pi^b \right) + v^v X^{\pi HH} \nabla_\pi (HH\bar{Y})^b
\]

\[= 0 \]
by virtue of (5), (9) and (16). Secondly, if $J = \beta$, we have

$$
\left( HH \nabla_{\alpha \beta}(HH \tilde{Y}) \right)^\beta = \frac{\gamma}{\alpha} X^a HH \nabla_a (HH \tilde{Y})^\beta + \frac{\chi}{\alpha} X^a HH \nabla_a (HH \tilde{Y})^\beta + \frac{\eta}{\alpha} X^a HH \nabla_a (HH \tilde{Y})^\beta
$$

$$
= X^a \frac{\partial \chi}{\partial \alpha} \gamma^\beta + \frac{\partial HH}{\partial \alpha} \chi^\beta \gamma^\gamma + \frac{\partial HH}{\partial \alpha} \chi^\beta \gamma^\gamma
$$

$$
= 0
$$

by virtue of (5), (9) and (16). Thirdly, if $J = \tilde{\beta}$, then we have

$$
\left( HH \nabla_{\alpha \beta}(HH \tilde{Y}) \right)^\tilde{\beta} = \frac{\gamma}{\alpha} X^a HH \nabla_a (HH \tilde{Y})^\tilde{\beta} + \frac{\chi}{\alpha} X^a HH \nabla_a (HH \tilde{Y})^\tilde{\beta} + \frac{\eta}{\alpha} X^a HH \nabla_a (HH \tilde{Y})^\tilde{\beta}
$$

$$
= X^a \frac{\partial \chi}{\partial \alpha} \gamma^\beta + \frac{\partial HH}{\partial \alpha} \chi^\beta \gamma^\gamma + \frac{\partial HH}{\partial \alpha} \chi^\beta \gamma^\gamma
$$

$$
= -X^a \frac{\partial \chi}{\partial \alpha} \gamma^\beta + X^a \frac{\partial \chi}{\partial \alpha} \gamma^\gamma
$$

$$
= 0
$$

by virtue of (5), (9) and (16). The proof is completed.

**Theorem 4.** Let $\tilde{X}$ and $\tilde{Y}$ be projectable vector fields on $M_\alpha$ with projection $X \in \mathcal{X}(M_\alpha)$ and $Y \in \mathcal{X}(M_\alpha)$, respectively. We have:

$$
HH \nabla_{\alpha \beta}(HH \tilde{Y}) = HH \nabla_{\alpha \beta}(HH \tilde{Y}).
$$

**Proof.** (i) If $\tilde{X}, \tilde{Y} \in \mathcal{X}(M_\alpha)$ and

$$
\begin{pmatrix}
HH \nabla_{\alpha \beta}(HH \tilde{Y}) \\
HH \nabla_{\alpha \beta}(HH \tilde{Y}) \\
HH \nabla_{\alpha \beta}(HH \tilde{Y})
\end{pmatrix}
$$

are the components of $(HH \nabla_{\alpha \beta}(HH \tilde{Y}))^J$ with respect to the coordinates $(x^b, \xi^\alpha, \xi^\beta)$ on $\{ B_\alpha \}$, then we have

$$
(HH \nabla_{\alpha \beta}(HH \tilde{Y}))^J = c_c \tilde{X}^a HH \nabla_a (HH \tilde{Y})^J + c_c \tilde{X}^a HH \nabla_a (HH \tilde{Y})^J + c_c \tilde{X}^a HH \nabla_a (HH \tilde{Y})^J.
$$

Firstly, if $J = b$, we have

$$
(HH \nabla_{\alpha \beta}(HH \tilde{Y}))^b = c_c \tilde{X}^a HH \nabla_a (HH \tilde{Y})^b + c_c \tilde{X}^a HH \nabla_a (HH \tilde{Y})^b + c_c \tilde{X}^a HH \nabla_a (HH \tilde{Y})^b
$$

$$
= X^a HH \nabla_a (HH \tilde{Y})^b + X^a HH \nabla_a (HH \tilde{Y})^b + (s \tilde{X}^a HH \nabla_a (HH \tilde{Y})^b
$$

$$
= X^a (\partial_a Y^b + HH \Gamma^a_{b c} Y^c + HH \Gamma^a_{b \gamma} Y^\gamma + HH \Gamma^a_{b Y} Y^\gamma) + X^a (\partial_a Y^b + HH \Gamma^a_{b c} Y^c + HH \Gamma^a_{b \gamma} Y^\gamma + HH \Gamma^a_{b Y} Y^\gamma)
$$

$$
+ (s \tilde{X}^a HH \nabla_a (HH \tilde{Y})^b + s \tilde{X}^a HH \nabla_a (HH \tilde{Y})^b + s \tilde{X}^a HH \nabla_a (HH \tilde{Y})^b
$$

$$
= X^a \partial_a Y^b + X^a \Gamma^a_{b Y} Y^\gamma = X^a \left( \partial_a Y^b + \Gamma^a_{b Y} Y^\gamma \right)
$$
by virtue of (7), (9) and (16). Secondly, if $J = \beta$, we have

$$
\left( HH\nabla_{\alpha\beta}(HH\tilde{Y}) \right)^{\beta} = cc\tilde{X}^{aHH}\nabla_{\alpha}(HH\tilde{Y})^{\beta} + cc\tilde{X}^{aHH}\nabla_{\alpha}(HH\tilde{Y})^{\beta} + cc\tilde{X}^{\pi HH}\nabla_{\pi}(HH\tilde{Y})^{\beta}
$$

$$
= X^aHH\nabla_{\alpha}(HH\tilde{Y})^{\beta} + X^aHH\nabla_{\alpha}(HH\tilde{Y})^{\beta} + (\partial_{\alpha}X^aHH\nabla_{\pi}(HH\tilde{Y})^{\beta})
$$

$$
= X^a(\partial_{\alpha}Y^\beta + HH\Gamma_{\beta}^\gamma c + HH\Gamma_{\beta}^\gamma c Y^\gamma + HH\Gamma_{\beta}^\gamma c Y^\gamma Y^\gamma) + X^a(\partial_{\alpha}Y^\beta + HH\Gamma_{\beta}^\gamma c + HH\Gamma_{\beta}^\gamma c Y^\gamma + HH\Gamma_{\beta}^\gamma c Y^\gamma Y^\gamma)
$$

$$
+ (\partial_{\alpha}X^aHH\nabla_{\pi}(HH\tilde{Y})^{\beta})
$$

$$
= X^a\partial_{\alpha}Y^\beta + X^a\Gamma_{\alpha\beta}^\gamma c Y^\gamma = X^a\left( \partial_{\alpha}Y^\beta + \Gamma_{\alpha\beta}^\gamma c Y^\gamma \right)
$$

by virtue of (7), (9) and (16). Thirdly, if $J = \overline{\beta}$, then we have

$$
\left( HH\nabla_{\alpha\overline{\beta}}(HH\tilde{Y}) \right)^{\overline{\beta}} = cc\tilde{X}^{aHH}\nabla_{\alpha}(HH\tilde{Y})^{\overline{\beta}} + cc\tilde{X}^{aHH}\nabla_{\alpha}(HH\tilde{Y})^{\overline{\beta}} + cc\tilde{X}^{\pi HH}\nabla_{\pi}(HH\tilde{Y})^{\overline{\beta}}
$$

$$
= X^aHH\nabla_{\alpha}(HH\tilde{Y})^{\overline{\beta}} - (\partial_{\alpha}X^aHH\nabla_{\pi}(HH\tilde{Y})^{\overline{\beta}})
$$

$$
= -X^a(\partial_{\alpha}Y^\overline{\beta} + HH\Gamma_{\overline{\beta}}^\gamma c + HH\Gamma_{\overline{\beta}}^\gamma c Y^\gamma + HH\Gamma_{\overline{\beta}}^\gamma c Y^\gamma Y^\gamma)
$$

$$
+ (\partial_{\alpha}X^aHH\nabla_{\pi}(HH\tilde{Y})^{\overline{\beta}})
$$

by virtue of (7), (9) and (16). Thus, we have $HH\nabla_{\alpha\beta}(HH\tilde{Y}) = HH(\nabla_X Y)$.

**Theorem 5.** Let $\tilde{X}$ be a projectable vector field on $M_\alpha$ with projections $X$ on $B_\alpha$. If $Y \in \mathfrak{S}_0^1(B_\alpha)$, then

$$
HH\nabla_{\alpha\beta}(\nabla_X Y) = HH(\nabla_X Y).
$$

**Proof.** If $\tilde{X} \in \mathfrak{S}_0^1(M_\alpha), Y \in \mathfrak{S}_0^1(B_\alpha)$ and

$$
\begin{pmatrix}
( HH\nabla_{\alpha\beta}(\nabla_X Y) )^\beta \\
( HH\nabla_{\alpha\beta}(\nabla_X Y) )^\overline{\beta}
\end{pmatrix}
$$

are the components of $(HH\nabla_{\alpha\beta}(\nabla_X Y))^J$ with respect to the coordinates $(x^\alpha, x^\beta, x^\gamma)$ on $t(B_\alpha)$, then we have

$$
( HH\nabla_{\alpha\beta}(\nabla_X Y))^J = HH\tilde{X}^{aHH}\nabla_{\alpha}(\nabla_X Y)^J + HH\tilde{X}^{aHH}\nabla_{\beta}(\nabla_X Y)^J + HH\tilde{X}^{\pi HH}\nabla_{\pi}(\nabla_X Y)^J.
$$

Firstly, if $J = b$, we have

$$
( HH\nabla_{\alpha\beta}(\nabla_X Y))^b = HH\tilde{X}^{aHH}\nabla_{\alpha}(\nabla_X Y)^b + HH\tilde{X}^{aHH}\nabla_{\beta}(\nabla_X Y)^b + HH\tilde{X}^{\pi HH}\nabla_{\pi}(\nabla_X Y)^b
$$

$$
= X^a(\partial_{\alpha}Y^b + HH\Gamma_{\alpha\beta}^\gamma c + HH\Gamma_{\alpha\beta}^\gamma c Y^\gamma + HH\Gamma_{\alpha\beta}^\gamma c Y^\gamma Y^\gamma)
$$

$$
+ X^a(\partial_{\beta}Y^b + HH\Gamma_{\alpha\beta}^\gamma c + HH\Gamma_{\alpha\beta}^\gamma c Y^\gamma + HH\Gamma_{\alpha\beta}^\gamma c Y^\gamma Y^\gamma)
$$

$$
+ HH\tilde{X}^{\pi HH}(\partial_{\pi}Y^b + HH\Gamma_{\alpha\beta}^\gamma c + HH\Gamma_{\alpha\beta}^\gamma c Y^\gamma + HH\Gamma_{\alpha\beta}^\gamma c Y^\gamma Y^\gamma)
$$

$$
= 0
$$

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by virtue of (5), (9) and (16). Secondly, if \( J = \beta \), we have

\[
(HH\nabla_{HH}\nabla^{\gamma Y})^\beta = HH\nabla_a HH\nabla^{\gamma Y}_a \beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta
\]

\[
= X^a (\partial_a \gamma Y^\beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta)
\]

\[
+ X^a (\partial_a \gamma Y^\beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta)
\]

\[
+ HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta
\]

\[
= 0
\]

by virtue of (5), (9) and (16). Thirdly, if \( J = \bar{\beta} \), then we have

\[
(HH\nabla_{HH}\nabla^{\gamma Y})^\bar{\beta} = HH\nabla_a HH\nabla^{\gamma Y}_a \beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta
\]

\[
= X^a (\partial_a \gamma Y^\beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta)
\]

\[
+ X^a (\partial_a \gamma Y^\beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta)
\]

\[
+ HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta + HH\nabla^\gamma HH\nabla^{\gamma Y}_a \beta
\]

\[
= X^a \partial_a Y^\beta + X^a \Gamma^\beta_a \gamma Y^\gamma = X^a (\partial_a Y^\beta + \Gamma^\beta_a \gamma Y^\gamma)
\]

\[
= (\nabla_X Y)^\beta
\]

by virtue of (5), (9) and (16). On the other hand, we know that \( \gamma Y (\nabla_X Y) \) have the components

\[
\gamma Y (\nabla_X Y) = \begin{pmatrix} 0 \\ 0 \\ (\nabla_X Y)^\beta \end{pmatrix}
\]

with respect to the coordinates \((x^h, x^\beta, x^\bar{\beta})\) on \(t(B_m)\). Thus, we have

\[ HH\nabla_{HH}\nabla^{\gamma Y} = \gamma Y (\nabla_X Y) \]

in \(t(B_m)\).

5 Conclusion

In this paper, we consider horizontal lifting problem of projectable linear connection on \(M\) to the semi-tangent bundle \(tM\). In this context, the following equations have been obtained:

\[
(i) \quad HH\nabla_{\nabla_X} (\gamma Y) = 0,
\]

\[
(ii) \quad HH\nabla_{\nabla_X} (\gamma Y) = 0,
\]

\[
(iii) \quad HH\nabla_{\nabla_X} (\gamma Y) = HH (\nabla_X Y),
\]

\[
(iv) \quad HH\nabla_{HH\nabla^{\gamma Y}} (\gamma Y) = \gamma Y (\nabla_X Y).
\]
Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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