CMMA **5**, No. 1, 1-5 (2020)

Communication in Mathematical Modeling and Applications
http://ntmsci.com/cmma

A new proof of Gronwall inequality with Atangana-Baleanu fractional derivatives

Süleyman Öğrekçi[1], *Yasemin Başcı*[2] *and Adil Mısır*[3]

[1]Department of Mathematics, Amasya University, Amasya, Turkey
[2]Department of Mathematics, Faculty of Arts and Sciences, Bolu Abant Izzet Baysal University, Bolu Turkey
[3]Department of Mathematics, Faculty of Sciences, Gazi University, Ankara Turkey

Received: 10 December 2019, Accepted: 14 June 2020
Published online: 24 June 2020

Abstract: In this study, we establish a new version of Gronwall type integral inequality, which generalizes some previous ones.

Keywords: Gronwall inequality, Atangana-Baleanu fractional derivatives, Atangana-Baleanu fractional integrals, Fractional differential equations

1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. It has been shown that these types of equations have numerous applications in diverse fields and thus have evolved into multidisciplinary subjects. For more details on fractional calculus, we refer the reader to the remarkable books [1,2,3]. Many authors discussed theoretical and application aspects of differential(or difference) equations within fractional integrals and derivatives [4,5,6,7,8,9,10,11,12,13].

One of the most important inequalities in the theory of differential equations is known as the Gronwall inequality. It was published in 1919 in the work by Gronwall [14]. Since then many generalizations and extensions of this inequality has become part of the literature [15].

It is our aim in this study to contribute to the development of this theory, presenting a proof for Gronwall inequality. We have presented some background materials as follows:
Riemann-Liouville fractional integral, $\alpha > 0$,

$$\left({}_aI^{\alpha}x\right)(t) := \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)ds \tag{1}$$

For $0 < \alpha < 1$ Riemann-Liouville fracdional derivative of order α starting from a,

$$\left({}_aD^{\alpha}x\right)(t) := \frac{d}{dt}\left({}_aI^{1-\alpha}x\right)(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_a^t (t-s)^{-\alpha}x(s)ds \tag{2}$$

* Corresponding author e-mail: suleyman.ogrekci@amasya.edu.tr

For $0 < \alpha < 1$ Caputo fracdional derivative of order α starting from a,

$$\left({}_a^C D^\alpha x\right)(t) := \left({}_a I^{1-\alpha} x'\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{-\alpha} x'(s) \underset{\cdot}{s} \tag{3}$$

Definition 1. *[16] Let $x \in H^1(a,b)$, $a < b$ and $\alpha \in [0,1]$. The Atangana-Baleanu fractional derivative of the function x order α in the sense of Caputo is defined by*

$$\left({}_a^{ABC} D^\alpha x\right)(t) := \frac{B(\alpha)}{1-\alpha} \int_a^t x'(s) E_\alpha \left[-\alpha \frac{(t-1)^\alpha}{1-\alpha}\right] \underset{\cdot}{s}, \tag{4}$$

where $E_\alpha(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+1)}$ is the Mittag-Leffler function and $B(\alpha)$ is a normalizing positive function satisfying $B(0) = B(1) = 1$. Similarly, the Atangana-Baleanu fractional derivative of the function x order α in the sense of Riemann-Liouville is defined by

$$\left({}_a^{ABR} D^\alpha x\right)(t) := \frac{B(\alpha)}{1-\alpha} \frac{\cdot}{\underset{\cdot}{t}} \int_a^t x(s) E_\alpha \left[-\alpha \frac{(t-1)^\alpha}{1-\alpha}\right] \underset{\cdot}{s}. \tag{5}$$

The associated Atangana-Baleanu fractional integral of the function x is defined by

$$\left({}_a^{AB} I^\alpha x\right)(t) := \frac{1-\alpha}{B(\alpha)} x(t) + \frac{\alpha}{B(\alpha)} \left({}_a I^\alpha x\right)(t), \tag{6}$$

where $\left({}_a I^\alpha x\right)(t)$ is the Riemann-Liouville fractional integral given with (1).

Lemma 1. *[[17]] Let $\alpha > 0$, $u(t)$ is nonnegative, nondecreasing and locally integrable on the interval $a < t < b < \infty$, $v(t) \leq M$ and $x(t)$ is nonnegative and locally integrable on the interval $[a,b)$ with*

$$x(t) \leq u(t) + v(t) \left({}_a I^\alpha x\right)(t) \tag{7}$$

for all $t \in [a,b)$. Then the inequality

$$x(t) \leq u(t) E_\alpha \left[v(t)(t-a)^\alpha\right] \tag{8}$$

holds for all $t \in [a,b)$.

2 Main results

Theorem 1. *Suppose $\alpha > 0$, $\varepsilon_1, \varepsilon_2 > 0$ and $x(t)$ is a nonnegative locally integrable function satisfying*

$$x(t) \leq \varepsilon_1 + \varepsilon_2 \left({}_a^{AB} I^\alpha x\right)(t) \tag{9}$$

on $[a,b)$. Then the following inequality holds for all $n \in \mathbb{N}$

$$x(t) \leq \varepsilon_1 \sum_{k=0}^n \left[A(\alpha)\varepsilon_2\right]^k + \left[A(\alpha)\varepsilon_2\right]^{n+1} x(t) + \frac{\alpha}{B(\alpha)} \sum_{k=0}^n \left[A^k(\alpha) \varepsilon_2^{k+1}\right] \left({}_a I^\alpha x\right)(t), \tag{10}$$

where $A(\alpha) := (1-\alpha)/B(\alpha)$.

Proof. We will prove by induction. By using (6) in the inequality (9), we obtain

$$\begin{aligned} x(t) &\leq \varepsilon_1 + \varepsilon_2 \left({}^{AB}_{a}I^{\alpha}x\right)(t) \\ &= \varepsilon_1 + \varepsilon_2 \left[A(\alpha)x(t) + \frac{\alpha}{B(\alpha)}\left({}_{a}I^{\alpha}x\right)(t)\right] \\ &= \varepsilon_1 + A(\alpha)\varepsilon_2 x(t) + \frac{\alpha}{B(\alpha)}\varepsilon_2\left({}_{a}I^{\alpha}x\right)(t), \end{aligned}$$

that is, the inequality (10) holds for $n=1$. Now for a $N \in \mathbb{N}$, assume that the inequality (10) holds for $n=N$, i.e.

$$x(t) \leq \varepsilon_1 \sum_{k=0}^{N}[A(\alpha)\varepsilon_2]^k + [A(\alpha)\varepsilon_2]^{N+1}x(t) + \frac{\alpha}{B(\alpha)}\sum_{k=0}^{N}\left[A^k(\alpha)\varepsilon_2^{k+1}\right]\left({}_{a}I^{\alpha}x\right)(t).$$

By using the relations (9) and (6) in this inequality, we obtain

$$\begin{aligned} x(t) &\leq \varepsilon_1 \sum_{k=0}^{N}[A(\alpha)\varepsilon_2]^k + [A(\alpha)\varepsilon_2]^{N+1}\cdot\left[\varepsilon_1 + \varepsilon_2\left({}^{AB}_{a}I^{\alpha}x\right)(t)\right] + \frac{\alpha}{B(\alpha)}\sum_{k=0}^{N}\left[A^k(\alpha)\varepsilon_2^{k+1}\right]\left({}_{a}I^{\alpha}x\right)(t) \\ &= \varepsilon_1 \sum_{k=0}^{N}[A(\alpha)\varepsilon_2]^k + [A(\alpha)\varepsilon_2]^{N+1}\cdot\left[\varepsilon_1 + \varepsilon_2\left(A(\alpha)x(t) + \frac{\alpha}{B(\alpha)}\right)\left({}_{a}I^{\alpha}x\right)(t)\right] + \frac{\alpha}{B(\alpha)}\sum_{k=0}^{N}\left[A^k(\alpha)\varepsilon_2^{k+1}\right]\left({}_{a}I^{\alpha}x\right)(t) \\ &= \varepsilon_1 \sum_{k=0}^{N}[A(\alpha)\varepsilon_2]^k + \varepsilon_1[A(\alpha)\varepsilon_2]^{N+1} + [A(\alpha)\varepsilon_2]^{N+2}x(t) + \frac{\alpha}{B(\alpha)}\sum_{k=0}^{N}\left[A^k(\alpha)\varepsilon_2^{k+1}\right]\left({}_{a}I^{\alpha}x\right)(t) + \frac{\alpha}{B(\alpha)}[A(\alpha)\varepsilon_2]^{N+1}\left({}_{a}I^{\alpha}x\right)(t) \\ &= \varepsilon_1 \sum_{k=0}^{N+1}[A(\alpha)\varepsilon_2]^k + [A(\alpha)\varepsilon_2]^{N+2}x(t) + \frac{\alpha}{B(\alpha)}\sum_{k=0}^{N+1}\left[A^k(\alpha)\varepsilon_2^{k+1}\right]\left({}_{a}I^{\alpha}x\right)(t), \end{aligned}$$

that is, the inequality (10) holds for $k=N$. Thus, the proof is complete.

By rearranging the inequality (10), we obtain the following useful result. Note that $A(\alpha)\varepsilon_2 > 0$ for $\alpha \in [0,1]$.

Corollary 1. *Suppose $\alpha \in [0,1]$, $\varepsilon_1, \varepsilon_2 > 0$ and $x(t)$ is a nonnegative locally integrable function satisfying (9) on $[a,b)$. If there exists a $n_1 \in \mathbb{N}$ such that $A^{n_1+1}(\alpha)\varepsilon_2^{n_1+1} < 1$, then the function $x(t)$ satisfies*

$$x(t) \leq K + L\left({}_{a}I^{\alpha}x\right)(t) \tag{11}$$

for all $t \in [a,b)$, where

$$K := \varepsilon_1 \frac{\sum_{k=0}^{n_1}[A(\alpha)\varepsilon_2]^k}{1-[A(\alpha)\varepsilon_2]^{n_1+1}} \qquad \text{and} \qquad L := \frac{\alpha}{B(\alpha)}\frac{\sum_{k=0}^{n_1}\left[A^k(\alpha)\varepsilon_2^{k+1}\right]}{1-[A(\alpha)\varepsilon_2]^{n_1+1}}.$$

Now, by using Lemma 1 in Corollary 1, we obtain following Gronwall type inequality for Atangana-Baleanu fractional integrals.

Corollary 2. *Suppose $\alpha \in [0,1]$, $\varepsilon_1, \varepsilon_2 > 0$ and $x(t)$ is a nonnegative locally integrable function satisfying (9) on $[a,b)$. If there exists a $n_1 \in \mathbb{N}$ such that $A^{n_1+1}(\alpha)\varepsilon_2^{n_1+1} < 1$, then the inequality*

$$x(t) \leq KE_{\alpha}\left[L(t-a)^{\alpha}\right] \tag{12}$$

holds for all $t \in [a,b)$.

By letting $n \to \infty$ in (10) and applying Lemma 1, one can obtain following special case of Gronwall inequality stated in Corollary 2. Note that if $\varepsilon_2 < A^{-1}(\alpha)$, then we have

$$\varepsilon_1 \sum_{k=0}^{\infty} [A(\alpha)\varepsilon_2]^k = \frac{\varepsilon_1 B(\alpha)}{B(\alpha)-(1-\alpha)\varepsilon_2} \quad \text{and} \quad \lim_{n\to\infty} [A(\alpha)\varepsilon_2]^{n+1} x(t) = 0.$$

Corollary 3. *Suppose $\alpha \in [0,1]$, $\varepsilon_1, \varepsilon_2 > 0$ and $x(t)$ is a nonnegative locally integrable function satisfying (9) on $[a,b)$. If $\varepsilon_2 < A^{-1}(\alpha)$, then the function $x(t)$ satisfies*

$$x(t) \le K^* E_\alpha \left[L^*(t-a)^\alpha\right] \tag{13}$$

for all $t \in [a,b)$, where

$$K^* := \frac{\varepsilon_1 B(\alpha)}{B(\alpha)-(1-\alpha)\varepsilon_2} \quad \textit{and} \quad L^* := \frac{\alpha\varepsilon_2}{B(\alpha)-(1-\alpha)\varepsilon_2}.$$

It is easy to see that the result of Theorem 1 is valid for variable $\varepsilon_1(t)$ and $\varepsilon_2(t)$ provided that $\varepsilon_1(t), \varepsilon_2(t) > 0$ for all $t \in [a,b)$. Therefore, we have the following results analogue to Corollary 2 and Corollary 3 succesively.

Corollary 4. *Suppose $\alpha \in [0,1]$, $\varepsilon_1(t), \varepsilon_2(t) > 0$ and $x(t)$ is a nonnegative locally integrable function satisfying*

$$x(t) \le \varepsilon_1(t) + \varepsilon_2(t) \left({}^{AB}_{a}I^{\alpha} x\right)(t) \tag{14}$$

on $[a,b)$. If there exists a $n_1 \in \mathbb{N}$ such that $A^{n_1+1}(\alpha)\varepsilon_2^{n_1+1}(t) < 1$, then the inequality

$$x(t) \le K(t) E_\alpha \left[L(t)(t-a)^\alpha\right] \tag{15}$$

holds for all $t \in [a,b)$, where

$$K(t) := \varepsilon_1(t) \frac{\sum_{k=0}^{n_1} [A(\alpha)\varepsilon_2(t)]^k}{1-[A(\alpha)\varepsilon_2(t)]^{n_1+1}} \qquad \textit{and} \qquad L(t) := \frac{\alpha}{B(\alpha)} \frac{\sum_{k=0}^{n_1} \left[A^k(\alpha)\varepsilon_2(t)^{k+1}\right]}{1-[A(\alpha)\varepsilon_2(t)]^{n_1+1}}.$$

Corollary 5. *Suppose $\alpha \in [0,1]$, $\varepsilon_1(t), \varepsilon_2(t) > 0$ and $x(t)$ is a nonnegative locally integrable function satisfying (9) on $[a,b)$. If $\varepsilon_2(t) < A^{-1}(\alpha)$ for all $t \in [a,b)$, then the function $x(t)$ satisfies*

$$x(t) \le K^*(t) E_\alpha \left[L^*(t)(t-a)^\alpha\right] \tag{16}$$

for all $t \in [a,b)$, where

$$K^*(t) := \frac{\varepsilon_1(t) B(\alpha)}{B(\alpha)-(1-\alpha)\varepsilon_2(t)} \quad \textit{and} \quad L^*(t) := \frac{\alpha\varepsilon_2(t)}{B(\alpha)-(1-\alpha)\varepsilon_2(t)}.$$

Remark. The results given in Corollary 3 and Corollary 5 are same with the result given in Theorem 2.1 of the paper [18] (see also Remark 2.1 of [18]), but the results given in Corollary 2 and Corollary 4 are more general than the results of [18].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Diethelm, K., *The Analysis of Fractional Differential Equations*, Springer, (2010), New York.

[2] Kilbas, A. A. and Srivastava, H. M. and Trujillo, J. J., *Theory and Applications of Fractional Differential Equations*, Elsevier Science, (2006), Amsterdam.

[3] Miller, K. S. and Ross, B., *An Introduction to Fractional Calculus and Fractional Differential Equations*, Wiley Interscience, (1993), New York.

[4] Alzabut, J., Muthulakshmi, V., Özbekler, A., & Adiguzel, H., *On the Oscillation of Non-Linear Fractional Difference Equations with Damping*, Mathematics, 7(8), 687, (2019).

[5] Grace, S. R., & Zafer, A., *On the asymptotic behavior of nonoscillatory solutions of certain fractional differential equations*, The European Physical Journal Special Topics, 226(16-18):3657–3665, (2017).

[6] Adiguzel, H., *Oscillation properties of solutions of fractional neutral differential equations*. Thermal Science, 23(Suppl. 1):175–183, (2019).

[7] Abdeljawad, T., *On conformable fractional calculus*. Journal of computational and Applied Mathematics, 279:57-66, (2015).

[8] Ferreira, R. A.,*A discrete fractional Gronwall inequality*. Proceedings of the American Mathematical Society, 1605–1612, (2012).

[9] Adiguzel, H., *Oscillatory behavior of solutions of certain fractional difference equations*. Advances in Difference Equations, 2018(1):1–13, (2018).

[10] Grace, S. R., *On the asymptotic behavior of positive solutions of certain fractional differential equations*, Mathematical Problems in Engineering, 2015, (2015).

[11] Abdeljawad, T., Alzabut, J., & Baleanu, D., *A generalized q-fractional Gronwall inequality and its applications to nonlinear delay q-fractional difference systems*, Journal of Inequalities and Applications, 2016(1):1–13, (2016).

[12] Adjabi, Y., Jarad, F., & Abdeljawad, T., *On generalized fractional operators and a Gronwall type inequality with applications*, Filomat, 31(17):5457-5473, (2017).

[13] Adiguzel, H., *Oscillation theorems for nonlinear fractional difference equations*, Boundary Value Problems, 2018(1), 178, (2018).

[14] Gronwall, T. H., *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Annals of Mathematics (2) 20(4):292-296, (1919).

[15] Pachpatte, B. G., *Integral and finite difference inequalities and applications*, Elsevier, (2006).

[16] Atangana, A. and Baleanu, D., *New Fractional derivatives with non-local and non-singular kernel: theory and applications to heat transfer model*, Thermal Science, 20(2): 763–769, (2016).

[17] Ye, H. and Gao, J. and Ding, Y., *A generalized Gronwall inequality and its application to a fractional differential equation*, Journal of Mathematical Analysis and Applications, 328(2): 1075–1081, (2007).

[18] Jarad, F. and Abdeljawad, T. and Hammouch, Z., *On a class of ordinary differential equations in the frame of Atangana-?Baleanu fractional derivative*, Chaos, Solitons & Fractals, 117:16–20, (2018).