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Initial smooth topology for a family of mappings

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Abstract: We consider initial smooth topology for a family of mappings. It is shown that the Šostak's initial smooth topology for family of functions is wrong. In this paper by using K.Chattopadhyay and R.Hazra 's result this mistake has been corrected.

Keywords: Smooth topology, initial smooth topology.

1 Introduction

In [4], A. Šostak defined the initial smooth topology for a family of mappings. In this paper we show that it is indeed false, and with the help of K.Chattopadhyay and R.Hazra 's result in [2], it may be corrected.

Firstly, let us recall some necessary notions from the theory of smooth topology.

Definition 1. *Let (X,τ) be a smooth topological space and $r\in(0,1]$. r cut of smooth topology τ is defined as*

$$\tau_r := \left\{\mu\in I^X : \tau(\mu)\geq r\right\}.$$

Definition 2. [5] *A fuzzy topological space is a pair (X,T), where X is a non-empty set and $T\subset I^X$, $(I=[0,1])$ satisfying the following properties:*

(1) $0,1\in T$
(2) $\mu\cap\upsilon\in T, \forall\mu,\upsilon\in T$
(3) $\bigcup\limits_{i\in\triangle}\mu_i\in T, \forall\{\mu_i : i\in\triangle\}\subset T$

Definition 3. [3] *A smooth topological space is a pair (X,τ), where X is a non-empty set and $\tau : I^X \longrightarrow I$ $(I=[0,1])$ is a mapping satisfying the following properties:*

(O1) $\tau(0)=\tau(1)=1_I$
(O2) $\forall\mu,\nu\in I^X, \tau(\mu\cap\nu)\geq\tau(\mu)\wedge(\nu)$
(O3) $\forall\triangle, \tau(\bigcup\limits_{i\in\triangle}\mu_i)\geq\bigwedge\limits_{i\in\triangle}\tau(\mu_i)$.

Note that, sometimes a smooth topology is called gradation of openness. Moreover, it is easy to see that if τ is crisp (i.e. $\tau(I^X)\subset\{0,1\}$), this definition coincides with the known definition of fuzzy topology [1].

Definition 4. [3] *A map $f : X\to Y$ is smooth continuous with respect to the smooth topologies τ_1 and τ_2 respectively, if for every $\mu\in I^Y$ we have*

$$\tau_1\left(f^{-1}(\mu)\right)\geq\tau_2(\mu)$$

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where $f^{-1}(\mu)$ *is defined by* $f^{-1}(\mu)(x) = \mu(f(x))$.

Proposition 1. [3] *Let* $\{T_\alpha : \alpha \in (0,1]\}$ *be a family of fuzzy topologies on* X *such that* $\alpha_1 > \alpha_2$ *implies* $T_{\alpha_1} < T_{\alpha_2}$.

$$\tau(\mu) := \bigvee \{\alpha : \mu \in T_\alpha\}$$

Then τ *is a smooth topology on* X.

Definition 5. [5] *Let* τ_1 *and* τ_2 *be smooth topology on* X*.If* $\tau_1 > \tau_2$ *then* τ_2 *is the weakest than* τ_1.

Definition 6. [4] *Let* X *be a set,* (Y,τ) *be a smooth topological space and* $f : X \to Y$ *be a mapping. The weakest smooth topology* δ *on* X *such that the mapping*

$$f : (X,\delta) \to (Y,\tau)$$

is smooth continuous, is called the initial smooth topology for a mapping.

To construct such a smooth topology consider the set $M := \{f^{-1}(\nu) \ : \ \nu \in I^Y\}$ of smooth subsets of X. For a given $\mu \in M$ let

$$P_\mu := \{\nu \in I^Y \ : \ \mu = f^{-1}(\nu)\}$$

and define $\delta(\mu) := \bigvee\{\tau(\nu) : \nu \in P_\mu\}$.

Definition 6 can be generalized for a family of mappings as follows.

Definition 7. [4] *Let* X *be a set,* $\{(Y_i,\tau_i) \ : \ i \in \triangle\}$ *be a family of smooth topological spaces and for each* $i \in \triangle$

$$f_i : X \longrightarrow (Y_i,\tau_i)$$

be a mapping. The weakest smooth topology τ *on* X *such that the mapping*

$$f_i : (X,\tau) \longrightarrow (Y_i,\tau_i)$$

is smooth continuous for each $i \in \triangle$*, is called initial smooth topology for a family of mappings.*

Let us to remind the Šostak's definition of initial smooth topology for a family of mappings (see [4, (3.2)]). For each $i \in \triangle$, let $\delta_i : I^X \to I$ be the initial smooth topology on X for f_i, i.e. it is the weakest smooth topology on X such that the mapping

$$f_i : (X,\delta_i) \to (Y_i,\tau_i)$$

is smooth continuous. Let the mapping $\tau : I^X \to I$ be defined by the equality

$$\tau(\mu) = \bigwedge\{\delta_i(\mu) : i \in \Delta\}$$

where $\mu \in I^X$. Šostak in [4] asserted that τ is an initial smooth topology for a family (f_i), i.e. τ is the weakest smooth topology on X such that the mapping f_i is smooth continuous for each $i \in \triangle$.

2 General properties

We assert that the assertion of Šostak of before is not true. General speaking, for any $i \in \triangle$ there is no need for f_i to be smooth continuous relative to the smooth topology τ, as shown by the following example.

Example 1. Let Y be a set, τ_1 and τ_2 be the two smooth topology on Y, and for $X = Y$ and some fuzzy set $\mu_o \in I^X$, $\tau_1(\mu_o) = 1$ and $\tau_2(\mu_o) = 0$. Let

$$id_1 : X \to (Y, \tau_1)$$

and $id_2 : X \to (Y, \tau_2)$ be the identity mappings. Assume that δ_1 and δ_2 are the initial smooth topologies corresponding to the mappings id_1 and id_2, respectively; and δ is the initial smooth topology in the sense of Šostak, for a family $\{id_j\}_{j=1}^{2}$. We show that, for example the mapping id_1 isn't smooth continuous relative to δ.

Indeed, it is clear that $\delta_1 = \tau_1$ and $\delta_2 = \tau_2$. According to Šostak for $\mu \in I^X$, $\delta(\mu) = \bigwedge_i \delta_i(\mu)$. Since for $\mu_o \in I^X$

$$\begin{aligned}\delta(id_1^{-1}(\mu_o)) &= \delta(\mu_o)\\ &= \delta_1(\mu_o) \wedge \delta_2(\mu_o)\\ &= \tau_1(\mu_o) \wedge \tau_2(\mu_o)\\ &= 1 \wedge 0 = 0 \not\geq 1 = \tau_1(\mu_o),\end{aligned}$$

then

$$\delta(id_1^{-1}(\mu_o)) \not\geq \tau_1(\mu_o).$$

Hence the mapping id_1 isn't smooth continuous relative to δ.

Essentially the right construction of the initial smooth topology for a family mappings is included in [2, Proposition 5.3]. Let us recall this Proposition.

Theorem 1. [2] *Let X be a set, $\{(X_i, \tau_i') : i \in \triangle\}$ be a family of the smooth topological spaces and for $i \in \triangle$*

$$f_i : X \to X_i$$

be a mapping. Then there exists a smooth topology τ on X such that the following conditions hold:

(i) *for each $i \in \triangle$, $f_i : (X, \tau) \to (X_i, \tau_i')$ is smooth continuous;*
(ii) *if (Z, τ'') is a smooth topological space then $g : (Z, \tau'') \to (X, \tau)$ is smooth continuous iff*

$$f_i \circ g : (Z, \tau'') \to (X_i, \tau_i')$$

is smooth continuous for each $i \in \triangle$.

In the proof of Theorem 1 (i.e. of the Proposition 5.3, [2]) the smooth topology τ on X was defined in [2] as the follows: For each $i \in \triangle$ and $\alpha \in (0, 1]$, define

$$\top_{i,\alpha} := \{f_i^{-1}(\mu) : \mu \in (\tau_i')_\alpha\}.$$

Recall that

$$(\tau_i')_\alpha = \{\mu \in I^X : \tau_i'(\mu) \geq \alpha\}$$

is the α-level fuzzy topology on X_i and $\{\top_{i,\alpha} : \alpha \in (0, 1]\}$ is a descending chain of fuzzy topologies on X. For each $\alpha \in (0, 1]$, define $\mathscr{S}_\alpha := \bigcup_i \top_{i,\alpha}$ and $\top_\alpha$ be the fuzzy topology on X generated by $\mathscr{S}_\alpha$ as a subbase. It can be verified that $\{\top_\alpha : \alpha \in (0, 1]\}$ is a descending chain of fuzzy topologies on X. Moreover we have a gradation of openness, τ, on X associated to $\{\top_\alpha : \alpha \in (0, 1]\}$, where $\tau(\lambda) = \bigvee\{\alpha \in (0, 1] : \lambda \in \top_\alpha\}$.

By Theorem 1 (i) for each $i \in \triangle$, $f_i : (X, \tau) \to (X_i, \tau_i)$ is smooth continuous. By (ii), τ is the weakest smooth topology on

X such that the mapping f_i is smooth continuous for each $i \in \triangle$.

Indeed, if τ_o is an arbitrary smooth topology on X such that $f_i : (X, \tau_o) \to (X_i, \tau_i)$ is smooth continuous for each $i \in \triangle$, then by Theorem 1 (ii), $id : (X, \tau_0) \to (X, \tau)$ must be smooth continuous, since for each

$$f_i = f_i \circ id : (X, \tau_0) \to (X_i, \tau_i)$$

is smooth continuous, where id is the identity mapping. Hence for each $\mu \in I^X$, $\tau_o(id^{-1}(\mu)) \geq \tau(\mu)$, i.e. $\tau_o(\mu) \geq \tau(\mu)$. Therefore $\tau_o \geq \tau$. The above proof can be given in more detail as below. Let $\{(X_{i,} \tau_i') : i \in \triangle\}$ be a family of smooth topological spaces and $\{f_i : X \to X_i : i \in \triangle\}$ be a family of functions. Let

$$\left(\tau_i'\right)_r = \left\{\lambda \in I^{X_i} : \tau_i'(\lambda) \geq r\right\}, \ \ \forall i \in \triangle \ \text{ and } \ \forall\, r \in (0,1]$$

and

$$T_{i,r} := \left\{f_i^{-1}(\lambda) : \lambda \in \left(\tau_i'\right)_r\right\}, \ \ \forall i \in \triangle \ \text{ and } \ \forall r \in (0,1].$$

In addition, let's define as

$$\mathscr{S}_r := \bigcup_{i \in \triangle} T_{i,r} \ \ ,\forall r \in (0,1].$$

Let T_r be the fuzzy topology which accepts $\mathscr{S}_r$ a subbase on X.

$$\tau(\mu) := \bigvee \{r \in (0,1] : \mu \in T_r\} \text{ for } \mu \in I^X$$

is a smooth topology on X [3]. Moreover, τ is the weakest topology that makes smooth continuous to each function

$$f_i : (X, \tau) \longrightarrow (X_{i,} \tau_i)$$

on X. Now, let's prove it.

Let τ' be any smooth topology that makes smooth continuous to functions

$$f_i : (X, \tau') \longrightarrow (X_{i,} \tau_i)$$

on X and be $\mu \in I^X$. Let

$$s := \tau(\mu) = \bigvee \{r \in (0,1] : \mu \in T_r\}.$$

There exists at least $r \in (0,1]$ for each $\varepsilon > 0$ such that $r > s - \varepsilon$ and $\mu \in T_r$. Since $\mu \in T_r$ ve $\mathscr{S}_r$ is subbase,

$$\exists \mu_{\gamma,k} \in \mathscr{S}_r, \ (\gamma \in \Lambda \ \text{ and } k \in \{1,2,...,n\}) \text{ such that } \ \mu = \bigcup_{\gamma \in \Lambda} \bigcap_{k=1}^{n} \mu_{\gamma,k}. \tag{1}$$

For every $\mu_{\gamma,k} \in \mathscr{S}_r$, there exists at least $i_{\gamma,k} \in \triangle$ such that $\mu_{\gamma,k} \in T_{i_{\gamma,k},r}$. From the definiton of $T_{i_{\gamma,k},r}$,

$$\exists \lambda_{\gamma,k} \in \left(\tau_{i_{\gamma,k}}'\right)_r \text{ such that } \mu_{\gamma,k} := f_{i_{\gamma,k}}^{-1}\left(\lambda_{\gamma,k}\right), \ \forall \gamma \in \Lambda, \forall k \in \{1,2,...,n\}. \tag{2}$$

Therefore,

$$\tau_{i_{\gamma,k}}'\left(\lambda_{\gamma,k}\right) \geq r, \ \ \forall \gamma \in \Lambda \ \text{ and } \forall k \in \{1,2,...,n\}. \tag{3}$$

From the hypothesis, since $f_i : (X,\tau') \longrightarrow (X_i,\tau_i)$ is smooth continuous for every $i \in \triangle, f_{i_{\gamma,k}} : (X,\tau') \longrightarrow \left(X_{i_{\gamma,k}}, \tau_{i_{\gamma,k}}\right)$ is smooth continuous for every $\gamma \in \Lambda$ and every $k \in \{1,2,...,n\}$ and

$$\tau'\left(f^{-1}_{i_{\gamma,k}}(\lambda_{\gamma,k})\right) \geq \tau'_{i_{\gamma,k}}(\lambda_{\gamma,k}) \geq r.$$

From here,

$$\tau'\left(f^{-1}_{j_{\gamma,k}}(\lambda_{\gamma,k})\right) \geq r \text{ for } \forall\gamma \in \Lambda \text{ and } \forall k \in \{1,2,...,n\} \quad (4)$$

From (2) and (4), $\tau'(\mu_{\gamma,k}) \geq r$ for every $\gamma \in \Lambda$ and every $k \in \{1,2,...,n\}$. Since τ' is the smooth topology,

$$\tau'\left(\bigcup_{\gamma\in\Lambda}\bigcap_{k=1}^{n}\mu_{\gamma,k}\right) \geq r.$$

From (1), $\tau'(\mu) \geq r > s-\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\tau'(\mu) \geq s = \tau(\mu)$. That is, $\tau'(\mu) \geq \tau(\mu)$. Since $\mu \in I^X$ is arbitrary, $\tau' \geq \tau$. Therefore, since τ' is also arbitrary, τ is the weakest topology that makes smooth continuous to every function $f_i : (X,\tau) \longrightarrow (X_i,\tau_i)$ on X.

3 Conclusion

In this paper, we considered initial smooth topology for a family of mappings. We showed that the Šostak's initial smooth topology for family of functions is wrong. After that, we gave the correct concept.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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