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Curvature properties of Kenmotsu manifold admitting semi-symmetric metric connection

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Abstract: In this paper we study Kenmotsu manifold admitting a semi-symmetric metric connection whose concircular curvature tensor satisfies certain curvature conditions.

Keywords: Kenmotsu manifold, concircular curvature tensor, semi-symmetric metric connection, scalar curvature, η-Einstein.

1 Introduction

In 1971, Kenmotsu [14] defined a type of contact metric manifold which is nowadays called Kenmotsu manifold. It may be noticed that a Kenmotsu manifold is not a Sasakian manifold. Also a Kenmotsu manifold is not compact because of $div\xi = 2n$. In [14], Kenmotsu showed that locally a Kenmotsu manifold is warped product $I \times_f N$ of an interval I and Kahler manifold N with warping function $f(t) = se^t$, where s is a non-zero constant. Kenmotsu manifolds have been studied by many authours Hui ([8]-[10]), Hui and Lemence [11], Prakasha, Hui and Vikas [15], Shaikh and Hui [17] and others.

In 1924, Friedmann and Schouten [6] introduced the idea of semi-symmetric linear connection on a differentiable manifold. A systematic study of semi-symmetric metric connection on a Riemannian manifold with different structures was done by Yano [20], Amur and Pujar [1], Bagewadi et, al ([2],[3]), Sharafuddin and Hussain [12] and many others.

The present paper is organized as follows: In section 2, we give some preliminaries that will be needed thereafter. In Section 3, we study Kenmotsu manifold satisfying $\tilde{C} \cdot \tilde{R} = 0$ and we obtained that the manifold is η-Einstein. In Section 4, we proved that a Kenmotsu manifold is locally concircular ϕ-symmetric with respect to semi-symmetric metric connection $\tilde{\nabla}$ if and only if it is so with respect to Levi-Civita connection ∇. Finally we obtained that a concircularly ϕ-recurrent Kenmotsu manifold with respect to semi-symmetric metric connection is a η-Einstein manifold.

2 Preliminaries

Let (M^n, g) be an n-dimensional almost contact metric manifold with almost contact metric structure (ϕ, η, ξ, g), where ϕ is a $(1,1)$-tensor field, η is a 1-form, ξ is the associated vector field and g is the Riemannian metric. Then the structure

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(ϕ,η,ξ,g) satisfies the following:

$$\phi\xi=0,\quad \eta(\phi X)=0,\quad g(X,\xi)=\eta(X),\quad \eta(\xi)=1, \tag{1}$$
$$\phi^2X=-X+\eta(X)\xi, \tag{2}$$
$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{3}$$

for all vector fields $X,Y\in T_pM$. If moreover

$$(\nabla_X\phi)Y=-g(X,\phi Y)\xi-\eta(Y)\phi X,\ and \tag{4}$$
$$\nabla_X\xi=X-\eta(X)\xi, \tag{5}$$

holds, where ∇ denotes the Riemannian connection of g, then M^n is called a Kenmotsu manifold.

In a Kenmotsu manifold M^n, the following relations hold: [14].

$$g(\phi X,Y)=-g(X,\phi Y), \tag{6}$$
$$R(X,Y)\xi=\eta(X)Y-\eta(Y)X, \tag{7}$$
$$S(X,\xi)=(1-n)\eta(X), \tag{8}$$
$$(\nabla_X\eta)(Y)=g(X,Y)-\eta(X)\eta(Y), \tag{9}$$
$$S(\phi X,\phi Y)=S(X,Y)+(n-1)\eta(X)\eta(Y). \tag{10}$$

for all vector fields $X,Y\in T_pM$, $S(X,Y)=g(QX,Y)$ and $Q\phi=\phi Q$.

Let ∇ be the Levi-Civita connection on M^n. A linear connection $\tilde{\nabla}$ on (M^n,g) is said to be semi-symmetric [6] if the torsion tensor T of the connection $\tilde{\nabla}$ satisfies

$$T(X,Y)=U(Y)X-U(X)Y, \tag{11}$$

where U is a 1-form on M^n with ρ as associated vector field, i.e., $U(X)=g(X,\rho)$ for any differentiable vector field X on M^n . A semi-symmetric connection $\tilde{\nabla}$ is called semi-symmetric metric connection if it further satisfies $\tilde{\nabla}g=0$.

In an almost contact manifold, semi-symmetric metric connection is defined by identifying the 1-form ϕ of (11) with the contact-form η,

$$T(X,Y)=\eta(Y)X-\eta(X)Y, \tag{12}$$

with ξ as associated vector field i.e., $g(X,\xi)=\eta(X)$. The relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of (M^n,g) has been obtained by K.Yano [20], which is given by

$$\tilde{\nabla}_XY=\nabla_XY+\eta(Y)X-g(X,Y)\xi. \tag{13}$$

If R and $\tilde{R}$ are the curvature tensors of the Levi-Civita connection ∇ and the semi-symmetric metric connection $\tilde{\nabla}$, respectively, then we have

$$\tilde{R}(X,Y)Z=R(X,Y)Z+3\{g(X,Z)Y-g(Y,Z)X\}+2\eta(Z)\{\eta(Y)X-\eta(X)Y\}+2\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi. \tag{14}$$

Putting $X = W = e_i$ in above equation and summing over i, $1 \leq i \leq n$, we get

$$\tilde{S}(Y,Z) = S(Y,Z) - (3n-5)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z), \tag{15}$$

where $\tilde{S}$ and S are the Ricci tensor of the connections $\tilde{\nabla}$ and ∇ respectively. Contracting above equation, we get

$$\tilde{r} = r - (3n^2 - 7n + 4), \tag{16}$$

where $\tilde{r}$ and r are the scalar curvatures of the connections $\tilde{\nabla}$ and ∇ respectively.

In a Riemannian manifold (M^n, g), the concircular curvature tensor is defined by [19]

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{17}$$

for all $X,Y,Z \in T_pM$, where r is scalar curvature.

Let $\tilde{C}$ be the concircular curvature tensor with respect to semi-symmetric metric connection and is given by

$$\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{18}$$

Definition 1. *A Kenmotsu manifold (M^n, g) is said to be η-Einstein manifold if its Ricci tensor S satisfies the condition*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \ \ for\ all\ \ X,Y \in T_pM, \tag{19}$$

where a and b are functions on M^n. In particular, if $b = 0$, then M^n is an Einstein manifold.

3 Kenmotsu manifold admitting semi-symmetric metric connection with $\tilde{R}(\xi,X)\cdot\tilde{C} = 0$

Let us consider a Kenmotsu manifold admitting a semi-symmetric metric connection satisfying

$$(\tilde{R}(\xi,X)\cdot\tilde{C})(Y,Z)W = 0. \tag{20}$$

The above relation (20), gives

$$\tilde{R}(\xi,X)\tilde{C}(Y,Z)W - \tilde{C}(\tilde{R}(\xi,X)Y,Z)W - \tilde{C}(Y,\tilde{R}(\xi,X)Z)W - \tilde{C}(Y,Z)\tilde{R}(\xi,X)W = 0. \tag{21}$$

Putting $W = \xi$ in (21) and simplifying, we get

$$\tilde{C}(Y,Z)X = \left[2 + \frac{r-(3n^2-7n+4)}{n(n-1)}\right][g(X,Y)Z - g(X,Z)Y]. \tag{22}$$

Simplifying (22), gives

$$\begin{aligned} &R(Y,Z)X + 3[g(Y,X)Z - g(Z,X)Y] + 2\eta(X)[\eta(Z)Y - \eta(Y)Z] + 2[g(Z,X)\eta(Y) - g(Y,X)\eta(Z)]\xi \\ &- \left[\frac{r-(3n^2-7n+4)}{n(n-1)}\right][g(Z,X)Y - g(Y,X)Z] - \left[2 + \frac{r-(3n^2-7n+4)}{n(n-1)}\right][g(X,Y)Z - g(X,Z)Y] = 0. \end{aligned} \tag{23}$$

Taking innerproduct of (23) with U, we get

$$g(R(Y,Z)X,U)+3[g(Y,X)g(Z,U)-g(Z,X)g(Y,U)]+2\eta(X)[\eta(Z)g(Y,U)-\eta(Y)g(Z,U)] \tag{24}$$
$$+2[g(Z,X)\eta(Y)\eta(U)-g(Y,X)\eta(Z)\eta(U)]-\left[\frac{r-(3n^2-7n+4)}{n(n-1)}\right][g(Z,X)g(Y,U)-g(Y,X)g(Z,U)]$$
$$-\left[2+\frac{r-(3n^2-7n+4)}{n(n-1)}\right][g(X,Y)g(Z,U)-g(X,Z)g(Y,U)]=0.$$

Now let $\{e_i\}$ be an orthonoral basis of the tangent space at each point of the manifold M^n for $i=1,2,...,n$. Putting $Y=U=e_i$ in (24) and then taking summation over i, we get

$$S(Z,X)=(n-3)g(Z,X)-2(n-2)\eta(Z)\eta(X). \tag{25}$$

In view of (25), we conclude the following:

Theorem 1. *If a kenmotsu manifold admitting semi-symmetric metric connection satisfies $\tilde{R}(\xi,X)\cdot\tilde{C}=0$, then M^n is an η-Einstein manifold.*

4 Kenmotsu manifold admitting semi-symmetric metric connection with $\tilde{C}(\xi,X)\cdot\tilde{R}=0$

Let us consider Kenmotsu manifold admitting a semi-symmetric metric connection satisfying $(\tilde{C}(\xi,X)\cdot\tilde{R})(Y,Z)W=0$. Then we have

$$\tilde{C}(\xi,X)\tilde{R}(Y,Z)W-\tilde{R}(\tilde{C}(\xi,X)Y,Z)W-\tilde{R}(Y,\tilde{C}(\xi,X)Z)W-\tilde{R}(Y,Z)\tilde{C}(\xi,X)W=0. \tag{26}$$

On plugging $W=\xi$ in (26) and then by virtue of (18), we get either $r=(n-1)(n-4)$ or

$$\tilde{R}(Y,Z)X=2[g(X,Y)Z-g(X,Z)Y]. \tag{27}$$

Using (14) in (27), we get

$$R(Y,Z)X+3\{g(Y,X)Z-g(Z,X)Y\}+2\eta(X)\{\eta(Z)Y-\eta(Y)Z\} \tag{28}$$
$$+2\{g(Z,X)\eta(Y)-g(Y,X)\eta(Z)\}\xi=2[g(X,Y)Z-g(X,Z)Y].$$

On contracting above equation with respect to Y, we get

$$S(X,Z)=(n-3)g(X,Z)-2(n-2)\eta(X)\eta(Z). \tag{29}$$

Thus, we can state the following:

Theorem 2. *A kenmotsu manifold M^n satisfying $\tilde{C}(\xi,X)\cdot\tilde{R}=0$ with respect to semi-symmetric metric connection is either η-Einstein or the manifold is of constant scalar curvature with respect to Levi-civita connection.*

5 Locally Concircular ϕ-symmetric Kenmotsu manifold with respect to semi-symmetric metric connection

Definition 2. *A Kenmotsu manifold M^n is said to be locally concirular ϕ-symmetric with respect to semi-symmetric metric connection if*

$$\phi^2((\tilde{\nabla}_W\tilde{C})(X,Y)Z)=0, \tag{30}$$

for all vector fields X,Y,Z,W orthogonal to vector field ξ.

From equation (13), we have

$$(\tilde{\nabla}_W\tilde{C})(X,Y)Z = (\nabla_W\tilde{C})(X,Y)Z + \eta(\tilde{C}(X,Y)Z)W - g(W,\tilde{C}(X,Y)Z)\xi. \quad (31)$$

Now, differentiating (18) covariantly with respect to W, we get

$$\begin{aligned}(\nabla_W\tilde{C})(X,Y)Z =&(\nabla_W C)(X,Y)Z + 2(\nabla_W\eta)(Z)[\eta(Y)X - \eta(X)Y] + 2\eta(Z)[(\nabla_W\eta)(Y)X - (\nabla_W\eta)(X)Y] \\ &+ 2[g(Y,Z)(\nabla_W\eta)(X) - g(X,Z)(\nabla_W\eta)(Y)]\xi + 2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\nabla_W\xi.\end{aligned} \quad (32)$$

Using equations (18) and (32) in (31), we get

$$\begin{aligned}(\tilde{\nabla}_W\tilde{C})(X,Y)Z =&(\nabla_W C)(X,Y)Z + 2[g(W,Z) - \eta(W)\eta(Z)][\eta(Y)X - \eta(X)Y] + 2\eta(Z)[g(W,Y)X - \eta(W)\eta(Y)X \\ &- g(W,X)Y + \eta(W)\eta(X)Y] + 2[g(Y,Z)g(W,X) - g(Y,Z)\eta(W)\eta(X) \\ &- g(X,Z)g(W,Y) + g(X,Z)\eta(W)\eta(Y)]\xi + 2[g(Y,Z)\eta(X)W - g(X,Z)\eta(Y)W \\ &- g(Y,Z)\eta(X)\eta(W)\xi + g(X,Z)\eta(Y)\eta(W)\xi] + \eta(\tilde{C}(X,Y)Z)W - g(W,\tilde{C}(X,Y)Z)\xi.\end{aligned} \quad (33)$$

Operating ϕ^2 on both sides of equation (33) and using equation (2), we get

$$\begin{aligned}\phi^2((\tilde{\nabla}_W\tilde{C})(X,Y)Z) =&\phi^2((\nabla_W C)(X,Y)Z) + 2[g(W,Z) - \eta(W)\eta(Z)][\eta(X)Y - \eta(Y)X] \\ &+ 2\eta(Z)[\eta(W)\eta(Y)X - g(W,Y)X + g(W,X)Y - \eta(W)\eta(X)Y] \\ &+ 2[g(X,Z)\eta(Y)W - g(Y,Z)\eta(X)W] - \left[6 + \frac{r - (3n^2 - 7n + 4)}{n(n-1)}\right] \\ &[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]W.\end{aligned} \quad (34)$$

If we consider X,Y,Z and W are orthogonal to ξ then equation (34) yields

$$\phi^2((\tilde{\nabla}_W\tilde{C})(X,Y)Z) = \phi^2((\nabla_W C)(X,Y)Z). \quad (35)$$

In view of (35), we conclude the following:

Theorem 3. *A Kenmotsu manifold is locally concircular ϕ-symmetric with respect to semi-symmetric metric connection $\tilde{\nabla}$ if and only if it is so with respect to Levi-Civita connection ∇.*

6 Concircular ϕ-Recurrent Kenmotsu manifold with respect to semi-symmetric metric connection

Definition 3. *A Kenmotsu manifold M^n is said to be concircularly ϕ-recurrent with respect to semi-symmetric metric connection if*

$$\phi^2((\tilde{\nabla}_W\tilde{C})(X,Y)Z) = A(W)\tilde{C}(X,Y)Z, \quad (36)$$

for arbitrary vector fields X,Y,Z and W.

Let us consider an concircular ϕ-recurrent Kenmotsu manifold with respect to semi-symmetric metric connection. Then by virtue of (36) and (2), we get

$$-((\tilde{\nabla}_W\tilde{C})(X,Y)Z) + \eta((\tilde{\nabla}_W\tilde{C})(X,Y)Z)\xi = A(W)\tilde{C}(X,Y)Z, \quad (37)$$

from which it follows that

$$-g((\tilde{\nabla}_W \tilde{C})(X,Y)Z,U)+\eta((\tilde{\nabla}_W \tilde{C})(X,Y)Z)\eta(U)=A(W)g(\tilde{C}(X,Y)Z,U), \tag{38}$$

which on simplification, we get

$$(\nabla_W S)(Y,\xi)+2(n-2)g(W,Y)-2(n-2)\eta(W)\eta(Y)-\left[\frac{dr(W)+r-(3n^2-7n+4)}{n}+(n-1)(A(W)+1)\right]\eta(Y)=0. \tag{39}$$

Also we have

$$(\nabla_W S)(Y,\xi)=-S(Y,W)-(n-1)g(Y,W). \tag{40}$$

In view of (39) and (40), we get

$$S(Y,W)-(n-3)g(Y,W)+2(n-2)\eta(W)\eta(Y)+\left[\frac{dr(W)+r-(3n^2-7n+4)}{n}+(n-1)(A(W)+1)\right]\eta(Y)=0. \tag{41}$$

Replacing Y and W by ϕY and ϕW in above equation and using (3) and (10), we get

$$S(Y,W)=(n-3)g(Y,W)-2(n-2)\eta(W)\eta(Y). \tag{42}$$

Hence, we can state the following theorem.

Theorem 4. *A concircular ϕ-recurrent Kenmotsu manifold (M^n,g) with respect to semi-symmetric metric connection is an η-Einstein manifold .*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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