Curvature properties of Kenmotsu manifold admitting semi-symmetric metric connection

Phalaksha Murthy B¹, Venkatesha Venkatesha¹ and R. T. Naveen Kumar²

¹Department of Mathematics, Kuvempu University, Shankaraghatta, Shimoga, Karnataka, India
²Department of Mathematics, Siddaganga Institute of Technology, B H Road, Tumakuru, Karnataka, India

Received: 19 November 2018, Accepted: 21 June 2019
Published online: 25 December 2019.

Abstract: In this paper we study Kenmotsu manifold admitting a semi-symmetric metric connection whose concircular curvature tensor satisfies certain curvature conditions.

Keywords: Kenmotsu manifold, concircular curvature tensor, semi-symmetric metric connection, scalar curvature, η-Einstein.

1 Introduction

In 1971, Kenmotsu [14] defined a type of contact metric manifold which is nowadays called Kenmotsu manifold. It may be noticed that a Kenmotsu manifold is not a Sasakian manifold. Also a Kenmotsu manifold is not compact because of $\text{div} \xi = 2n$. In [14], Kenmotsu showed that locally a Kenmotsu manifold is warped product $I \times f \ N$ of an interval $I$ and Kahler manifold $N$ with warping function $f(t) = se^t$, where $s$ is a non-zero constant. Kenmotsu manifolds have been studied by many authors Hui ([8]-[10]), Hui and Lemence [11], Prakash, Hui and Vikas [15], Shaikh and Hui [17] and others.

In 1924, Friedmann and Schouten [6] introduced the idea of semi-symmetric linear connection on a differentiable manifold. A systematic study of semi-symmetric metric connection on a Riemannian manifold with different structures was done by Yano [20], Amur and Pujar [1], Bagewadi et al ([2],[3]), Sharafuddin and Hussain [12] and many others.

The present paper is organized as follows: In section 2, we give some preliminaries that will be needed thereafter. In Section 3, we study Kenmotsu manifold satisfying $\tilde{\mathcal{C}} \cdot \tilde{R} = 0$ and we obtained that the manifold is η-Einstein. In Section 4, we proved that a Kenmotsu manifold is locally concircular $\phi$-symmetric with respect to semi-symmetric metric connection $\tilde{\nabla}$ if and only if it is so with respect to Levi-Civita connection $\nabla$. Finally we obtained that a concircularly $\phi$-recurrent Kenmotsu manifold with respect to semi-symmetric metric connection is a η-Einstein manifold.

2 Preliminaries

Let $(M^n,g)$ be an $n$-dimensional almost contact metric manifold with almost contact metric structure $(\phi, \eta, \xi, g)$, where $\phi$ is a $(1,1)$-tensor field, $\eta$ is a 1-form, $\xi$ is the associated vector field and $g$ is the Riemannian metric. Then the structure
\((\phi, \eta, \xi, g)\) satisfies the following:

\[
\begin{align*}
\phi \xi &= 0, \quad \eta(\phi X) = 0, \quad g(X, \xi) = \eta(X), \quad \eta(\xi) = 1, \\
\phi^2 X &= -X + \eta(X)\xi, \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y),
\end{align*}
\]

for all vector fields \(X, Y \in T_pM\). If moreover

\[
(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \quad \text{and}
\]

\[
\nabla_X \xi = X - \eta(X)\xi,
\]

holds, where \(\nabla\) denotes the Riemannian connection of \(g\), then \(M^n\) is called a Kenmotsu manifold.

In a Kenmotsu manifold \(M^n\), the following relations hold: [14].

\[
\begin{align*}
g(\phi X, Y) &= -g(X, \phi Y), \\
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \\
S(X, \xi) &= (1 - n)\eta(X), \\
(\nabla_X \eta)(Y) &= g(X, Y) - \eta(X)\eta(Y), \\
S(\phi X, \phi Y) &= S(X, Y) + (n - 1)\eta(X)\eta(Y).
\end{align*}
\]

for all vector fields \(X, Y \in T_pM, S(X, Y) = g(QX, Y)\) and \(Q\phi = \phi Q\).

Let \(\nabla\) be the Levi-Civita connection on \(M^n\). A linear connection \(\tilde{\nabla}\) on \((M^n, g)\) is said to be semi-symmetric [6] if the torsion tensor \(T\) of the connection \(\tilde{\nabla}\) satisfies

\[
T(X, Y) = U(Y)X - U(X)Y,
\]

where \(U\) is a 1-form on \(M^n\) with \(\rho\) as associated vector field, i.e., \(U(X) = g(X, \rho)\) for any differentiable vector field \(X\) on \(M^n\). A semi-symmetric connection \(\tilde{\nabla}\) is called semi-symmetric metric connection if it further satisfies \(\tilde{\nabla}g = 0\).

In an almost contact manifold, semi-symmetric metric connection is defined by identifying the 1-form \(\phi\) of (11) with the contact-form \(\eta\),

\[
T(X, Y) = \eta(Y)X - \eta(X)Y,
\]

with \(\xi\) as associated vector field i.e., \(g(X, \xi) = \eta(X)\). The relation between the semi-symmetric metric connection \(\tilde{\nabla}\) and the Levi-Civita connection \(\nabla\) of \((M^n, g)\) has been obtained by K.Yano [20], which is given by

\[
\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi.
\]

If \(R\) and \(\tilde{R}\) are the curvature tensors of the Levi-Civita connection \(\nabla\) and the semi-symmetric metric connection \(\tilde{\nabla}\), respectively, then we have

\[
\tilde{R}(X, Y)Z = R(X, Y)Z + 3\{g(X, Z)Y - g(Y, Z)X\} + 2\eta(Z)\{\eta(Y)X - \eta(X)Y\} + 2\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi.
\]
Let us consider a Kenmotsu manifold admitting a semi-symmetric metric connection satisfying

\[ \tilde{S}(Y, Z) = S(Y, Z) - (3n - 5)g(Y, Z) + 2(n-2)\eta(Y)\eta(Z), \]  

where \( \tilde{S} \) and \( S \) are the Ricci tensor of the connections \( \tilde{\nabla} \) and \( \nabla \) respectively. Contracting above equation, we get

\[ \tilde{r} = r - (3n^2 - 7n + 4), \]  

where \( \tilde{r} \) and \( r \) are the scalar curvatures of the connections \( \tilde{\nabla} \) and \( \nabla \) respectively.

In a Riemannian manifold \((M^n, g)\), the concircular curvature tensor is defined by [19]

\[ C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)} \{ g(Y, Z)X - g(X, Z)Y \}, \]

for all \( X, Y, Z \in T_pM \), where \( r \) is scalar curvature.

Let \( \tilde{C} \) be the concircular curvature tensor with respect to semi-symmetric metric connection and is given by

\[ \tilde{C}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{\tilde{r}}{n(n-1)} \{ g(Y, Z)X - g(X, Z)Y \}. \]

**Definition 1.** A Kenmotsu manifold \((M^n, g)\) is said to be \( \eta \)-Einstein manifold if its Ricci tensor \( S \) satisfies the condition

\[ S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \quad \text{for all } X, Y \in T_pM, \]

where \( a \) and \( b \) are functions on \( M^n \). In particular, if \( b = 0 \), then \( M^n \) is an Einstein manifold.

### 3 Kenmotsu manifold admitting semi-symmetric metric connection with \( \tilde{R}(\xi, X) \cdot \tilde{C} = 0 \)

Let us consider a Kenmotsu manifold admitting a semi-symmetric metric connection satisfying

\[ (\tilde{R}(\xi, X) \cdot \tilde{C})Y, Z)W = 0. \]

The above relation \((20)\), gives

\[ \tilde{R}(\xi, X)\tilde{C}(Y, Z)W - \tilde{C}(\tilde{R}(\xi, X)Y, Z)W - \tilde{C}(Y, \tilde{R}(\xi, X)Z)W - \tilde{C}(Y, Z)\tilde{R}(\xi, X)W = 0. \]

Putting \( W = \xi \) in \((21)\) and simplifying, we get

\[ \tilde{C}(Y, Z)X = \left[ 2 + \frac{r - (3n^2 - 7n + 4)}{n(n-1)} \right] g(Y, Z)X - g(X, Z)Y. \]

Simplifying \((22)\), gives

\[ R(Y, Z)X + 3[g(Y, X)Z - g(Z, X)Y] + 2\eta(Y)[\eta(Z)Y - \eta(Y)Z] + 2[g(Z, X)\eta(Y) - g(Y, X)\eta(Z)]\xi \]

\[ - \left[ \frac{r - (3n^2 - 7n + 4)}{n(n-1)} \right] [g(Z, X)Y - g(Y, X)Z] - \left[ 2 + \frac{r - (3n^2 - 7n + 4)}{n(n-1)} \right] g(Y, X)Z - g(X, Z)Y = 0. \]
Taking inner product of (23) with $U$, we get
\[
g(R(Y,Z)X, U) + 3[g(Y,X)g(Z, U) - g(Z, X)g(Y, U)] + 2\eta(X)\eta(Z)g(Y, U) - \eta(Y)g(Z, U)] \tag{24}
\]
\[
+ 2[g(Z, X)\eta(Y)\eta(U) - g(Y, X)\eta(Z)\eta(U)] - \left[\frac{r - (3n^2 - 7n + 4)}{n(n - 1)}\right] [g(Z, X)g(Y, U) - g(Y, X)g(Z, U)]
\]
\[
- \left[\frac{r - (3n^2 - 7n + 4)}{n(n - 1)}\right] [g(X, Y)g(Z, U) - g(X, Z)g(Y, U)] = 0.
\]

Now let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold $M^n$ for $i = 1, 2, \ldots, n$. Putting $Y = U = e_i$ in (24) and then taking summation over $i$, we get

\[
S(Z, X) = (n - 3)g(Z, X) - 2(n - 2)\eta(Z)\eta(X). \tag{25}
\]

In view of (25), we conclude the following:

**Theorem 1.** If a Kenmotsu manifold admitting semi-symmetric metric connection satisfies $\hat{R}(\xi, X)\cdot \hat{C} = 0$, then $M^n$ is an $\eta$-Einstein manifold.

### 4 Kenmotsu manifold admitting semi-symmetric metric connection with $\hat{C}(\xi, X)\cdot \hat{R} = 0$

Let us consider Kenmotsu manifold admitting a semi-symmetric metric connection satisfying $(\hat{C}(\xi, X)\cdot \hat{R})(Y, Z)W = 0$. Then we have

\[
\hat{C}(\xi, X)\hat{R}(Y, Z)W - \hat{R}(\hat{C}(\xi, X)Y, Z)W - \hat{R}(Y, \hat{C}(\xi, X)Z)W - \hat{R}(Y, Z)\hat{C}(\xi, X)W = 0. \tag{26}
\]

On plugging $W = \xi$ in (26) and then by virtue of (18), we get either $r = (n - 1)(n - 4)$ or

\[
\hat{R}(Y, Z)X = 2[g(X, Y)Z - g(X, Z)Y]. \tag{27}
\]

Using (14) in (27), we get

\[
R(Y, Z)X + 3\{g(Y, X)Z - g(Z, X)Y\} + 2\eta(X)\eta(Z)Y - \eta(Y)Z] \tag{28}
\]
\[
+ 2\{g(Z, X)\eta(Y) - g(Y, X)\eta(Z)\} \hat{\xi} = 2[g(X, Y)Z - g(X, Z)Y].
\]

On contracting above equation with respect to $Y$, we get

\[
S(X, Z) = (n - 3)g(X, Z) - 2(n - 2)\eta(X)\eta(Z). \tag{29}
\]

Thus, we can state the following:

**Theorem 2.** A Kenmotsu manifold $M^n$ satisfying $\hat{C}(\xi, X)\cdot \hat{R} = 0$ with respect to semi-symmetric metric connection is either $\eta$-Einstein or the manifold is of constant scalar curvature with respect to Levi-civita connection.

### 5 Locally Concircular $\phi$-symmetric Kenmotsu manifold with respect to semi-symmetric metric connection

**Definition 2.** A Kenmotsu manifold $M^n$ is said to be locally concircular $\phi$-symmetric with respect to semi-symmetric metric connection if

\[
\phi^2((\tilde{\nabla}_W \tilde{C})(X, Y)Z) = 0. \tag{30}
\]
for all vector fields $X, Y, Z, W$ orthogonal to vector field $\xi$.

From equation (13), we have

$$\left(\tilde{\nabla}_W \tilde{C}\right)(X, Y)Z = \left(\nabla_W \tilde{C}\right)(X, Y)Z + \eta(\tilde{C}(X, Y)Z)W - g(W, \tilde{C}(X, Y)Z)\xi. \tag{31}$$

Now, differentiating (18) covariantly with respect to $W$, we get

$$\left(\tilde{\nabla}_W \tilde{C}\right)(X, Y)Z = \left(\nabla_W C\right)(X, Y)Z + 2(\nabla_W \eta)(Y)X - \eta(Y)\xi + 2\eta(Y)\left(\nabla_W \eta\right)(Y)X - \left(\nabla W \eta\right)(Y)X \tag{32}$$

Using equations (18) and (32) in (31), we get

$$\left(\tilde{\nabla}_W \tilde{C}\right)(X, Y)Z = \left(\nabla_W C\right)(X, Y)Z + 2\eta(Y)\nabla_W \eta(Y)X - \eta(Y)\xi + \left[2\eta(Y)\nabla W \eta(Y)X - \eta(Y)\xi\right] + \left(\tilde{\nabla}_W \tilde{C}\right)(X, Y)Z - g(W, \tilde{C}(X, Y)Z)\xi. \tag{33}$$

Operating $\phi^2$ on both sides of equation (33) and using equation (2), we get

$$\phi^2(\left(\tilde{\nabla}_W \tilde{C}\right)(X, Y)Z) = \phi^2\left(\left(\nabla_W C\right)(X, Y)Z\right) + 2\eta(Y)\phi^2(\nabla_W \eta(Y)X - \eta(Y)\xi) + \left[\phi^2\left(\eta(Y)\nabla W \eta(Y)X - \eta(Y)\xi\right) - \phi^2(\eta(Y)\nabla W \eta(Y)X\right] \tag{34}$$

If we consider $X, Y, Z$ and $W$ are orthogonal to $\xi$ then equation (34) yields

$$\phi^2(\left(\tilde{\nabla}_W \tilde{C}\right)(X, Y)Z) = \phi^2\left(\left(\nabla_W C\right)(X, Y)Z\right). \tag{35}$$

In view of (35), we conclude the following:

**Theorem 3.** A Kenmotsu manifold is locally concircular $\phi$-symmetric with respect to semi-symmetric metric connection $\tilde{\nabla}$ if and only if it is so with respect to Levi-Civita connection $\nabla$.

6 Concircular $\phi$-Recurrent Kenmotsu manifold with respect to semi-symmetric metric connection

**Definition 3.** A Kenmotsu manifold $M^n$ is said to be concircularly $\phi$-recurrent with respect to semi-symmetric metric connection if

$$\phi^2(\left(\tilde{\nabla}_W \tilde{C}\right)(X, Y)Z) = A(W)\tilde{C}(X, Y)Z, \tag{36}$$

for arbitrary vector fields $X, Y, Z$ and $W$.

Let us consider an concircular $\phi$-recurrent Kenmotsu manifold with respect to semi-symmetric metric connection. Then by virtue of (36) and (2), we get

$$-(\left(\tilde{\nabla}_W \tilde{C}\right)(X, Y)Z) + \eta(\left(\tilde{\nabla}_W \tilde{C}\right)(X, Y)Z)\xi = A(W)\tilde{C}(X, Y)Z, \tag{37}$$
from which it follows that

$$-g((\nabla W C)(X, Y)Z, U) + \eta((\nabla W C)(X, Y)Z)\eta(U) = A(W)g(C(X, Y)Z, U), \quad (38)$$

which on simplification, we get

$$\langle \nabla W S(Y, \xi) \rangle + 2(n - 2)g(W, Y) - 2(n - 2)\eta(W)\eta(Y) - \left[\frac{dr(W) + r - (3n^2 - 7n + 4)}{n} + (n - 1)(A(W) + 1)\right]\eta(Y) = 0. \quad (39)$$

Also we have

$$\langle \nabla W S(Y, \xi) \rangle = -S(Y, W) - (n - 1)g(Y, W). \quad (40)$$

In view of (39) and (40), we get

$$S(Y, W) - (n - 3)g(Y, W) + 2(n - 2)\eta(W)\eta(Y) + \left[\frac{dr(W) + r - (3n^2 - 7n + 4)}{n} + (n - 1)(A(W) + 1)\right]\eta(Y) = 0. \quad (41)$$

Replacing $Y$ and $W$ by $\phi Y$ and $\phi W$ in above equation and using (3) and (10), we get

$$S(Y, W) = (n - 3)g(Y, W) - 2(n - 2)\eta(W)\eta(Y). \quad (42)$$

Hence, we can state the following theorem.

**Theorem 4.** A concircular $\phi$-recurrent Kenmotsu manifold $(M^n, g)$ with respect to semi-symmetric metric connection is an $\eta$-Einstein manifold.

### Competing interests

The authors declare that they have no competing interests.

### Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

### References


