New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2019.378

On the generalized continued fractions

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Received: 20 January 2019, Accepted: 4 March 2019

Published online: 4 October 2019.

Abstract: We introduce a class of continued fractions called Oppenheim continued fractions (OCF). Basic properties of these expansions are discussed and studied in the formal powers series case.

Keywords: Oppenhein continued fraction, Laurent series, finite fields.

1 On generalized continued fractions

Let $\mathbb{F}_q$ be a finite field with q elements of characteristic p, $\mathbb{F}_q[X]$ the set of polynomials of coefficients in $\mathbb{F}_q$ and $\mathbb{F}_q(X)$ its field of fractions. The set $\mathbb{F}_q((X^{-1}))$ is the field of formal power series over $\mathbb{F}_q$

$$\mathbb{F}_q((X^{-1})) = \{\omega = \sum_{j=s}^{+\infty} a_j X^{-j} \ : \ a_j \in \mathbb{F}_q, \ s \in \mathbb{Z}\}.$$

Let $\omega = \sum_{j=s}^{+\infty} a_j X^{-j} \in \mathbb{F}_q((X^{-1}))$, where $a_s \neq 0$. We denote its polynomial part by $[\omega]$ and $\{\omega\}$ its fractional part. We remark that $\omega = [\omega] + \{\omega\}$. We define a non-archimedean absolute value on $\mathbb{F}_q((X^{-1}))$ by $|\ \omega\ | = e^{-s}$. It is clear that, for all $P \in \mathbb{F}_q[X]$, $|\ P\ | = e^{\deg P}$ and, for all $Q \in \mathbb{F}_q[X]$, such that $Q \neq 0$, $|\ \frac{P}{Q}\ | = e^{\deg P - \deg Q}$.

Let $E = (\mathbb{F}_q((X^{-1})))^n$, E is a vectorial space over $\mathbb{F}_q((X^{-1}))$. We define a norm over E as follows, for all $f = (f_1, \ldots, f_n) \in E$,

$$\|\ f\ \| = \max_{1 \leq i \leq n} |\ f_i\ |.$$

Let $A_1, \ldots, A_m \in E$, then we can verify that

$$\|\ A_1 + \cdots + A_m\ \| \leq \max_{1 \leq i \leq m} \|\ A_i\ \|.$$

We begin by giving a few basics facts about the generalized continued fractions over $\mathbb{F}_q((X^{-1}))$.

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1.1 Basics concepts

Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}} \in \mathbb{F}_q((X^{-1}))$, a continued fraction

$$K(\frac{\alpha_n}{\beta_n}) = \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \cfrac{\alpha_3}{\beta_3 + \ddots}}} \tag{1}$$

is said to converge if its sequence of approximants $\{\omega_n\}$ converges. Here

$$\omega_n = K_{i=1}^{n}(\frac{\alpha_i}{\beta_i}) = \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \cfrac{\alpha_3}{\ddots + \frac{\alpha_n}{\beta_n}}}}, \text{ for } n = 1,2,\ldots \tag{2}$$

The value of the continued fraction is then $\omega = K(\frac{\alpha_n}{\beta_n}) = \lim_{n\to+\infty} \omega_n$.

Remark. If $\alpha_i = 1$ and β_i is a non constant polynomial, then we obtain the Regular continued fraction (RCF).
If α_i is a fixed polynomial P and $(\beta_i)_{i\geq 1})$ is a sequence of non constant polynomials, then we obtain the P-continued fraction.

If $K(\frac{\alpha_n}{\beta_n})$ converges, its tails $K_{n=N+1}^{+\infty}(\frac{\alpha_n}{\beta_n})$ for $N = 0,1,2,\ldots$ also converge, and we let $\omega^{(N)} = K_{n=N+1}^{+\infty}(\frac{\alpha_n}{\beta_n})$ denote the values of these tails for $N = 0,1,2,\ldots$. It is easy to see that $\{\omega^{(N)}\}$ is a sequence with $\omega^{(0)} = \omega$, satisfying the recursion relations

$$\omega^{(N)} = \frac{\alpha_{N+1}}{\beta_{N+1} + \omega^{(N+1)}} \text{ for } N = 1,2,\ldots. \tag{3}$$

This sequence is what Waadeland [10] named the sequence of right tails for $K(\frac{\alpha_n}{\beta_n})$.

In this section, we describe a necessary and sufficient conditions for the convergence of (1). For which, we assume the existence of the limits

$$\lim_{n\to+\infty} \alpha_n = \alpha \neq 0 \text{ and } \lim_{n\to+\infty} \beta_n = \beta. \tag{4}$$

The continued fraction expansion (1) can be generated by means of the sequence $\{s_n(\theta)\}$ of linear fractional transformations,

$$s_n(\theta) = \frac{\alpha_n}{\beta_n + \theta}, \text{ for } \theta \in \mathbb{F}_q((X^{-1})) \text{ and } n = 1,2,3,\ldots. \tag{5}$$

Defining $S_n(\theta)$ as their composition,

$$S_0(\theta) = \theta,\ S_n(\theta) = S_{n-1}(s_n(\theta)) \text{ for and } n = 1,2,3,\ldots. \tag{6}$$

gives us $\omega_n = S_n(0)$, from (2). Straightforward computation shows that $S_n(\theta)$ can be written

$$S_n(\theta) = \frac{A_n + A_{n-1}\theta}{B_n + B_{n-1}\theta} \text{ for } n = 0,1,2,\ldots \tag{7}$$

where A_n and B_n, the numerator and denominator of $K_{i=0}^{n}(\frac{\alpha_i}{\beta_i})$, respectively, are given by

$$A_{-1} = 1,\ A_0 = 0,\ A_n = \beta_n A_{n-1} + \alpha_n A_{n-2},\ \text{for } n = 1,2,\ldots \tag{8}$$

$$B_{-1}=1,\ B_0=0,\ B_n=\beta_n B_{n-1}+\alpha_n B_{n-2},\ \text{for } n=1,2,\ldots \tag{9}$$

This notation is in accordance with [9], and it will be used throughout this paper. If we regard the Nth tail $K_{m=N+1}^{+\infty}(\frac{\alpha_m}{\beta_m})$ as a continued fraction, we use the notation $S_n^{(N)}$, $A_n^{(N)}$ and $B_n^{(N)}$ to denote the similar expressions connected with $K_{m=N+1}^{+\infty}(\frac{\alpha_m}{\beta_m})$.

1.2 Convergence results

Theorem 1. *Let* $(\alpha_n)_{n\in\mathbb{N}}$ *and* $(\beta_n)_{n\in\mathbb{N}}\in\mathbb{F}_q((X^{-1}))[z]$*. If in the generalized continued fraction*

$$\omega(z)=K_{n=1}^{+\infty}\frac{\alpha_n(z)}{\beta_n(z)}; \tag{10}$$

$\lim\limits_{i\to+\infty}\alpha_i(z)=\alpha(z)\neq 0$ *and* $\lim\limits_{i\to+\infty}\beta_i(z)=\beta(z)$*, the continued fraction expansion (10) will converge if and only if* $z\in\{z\in\mathbb{F}_q((X^{-1}));a(z)=\frac{|\alpha(z)|}{|\beta(z)|^2}<1\}$ *except possibly at certain isolated points* $p_1,p_1,\ldots$*, which are poles.*

Proof. If a sufficient number of terms of (10) are omitted at the outset in which $(|\alpha_{N+i}(z)|,|\beta_{N+i}(z)|)=(|\alpha(z)|,|\beta(z)|)$ $\forall i\geq 1$, a new continued fraction will be obtained

$$\omega^{(N)}(z)=K_{i=1}^{+\infty}\frac{\alpha_{N+i}(z)}{\beta_{N+i}(z)}. \tag{11}$$

For this continued fraction

$$B_0^{(N)}=1,\ B_1^{(N)}=\beta_{N+1}(z)\text{ and }B_{i+1}^{(N)}=\beta_{N+i+1}(z)B_i^{(N)}+\alpha_{N+i+1}(z)B_{i-1}^{(N)}. \tag{12}$$

Suppose first that $|a(z)|=\frac{|\alpha(z)|}{|\beta(z)|^2}<1$, then, for all $i\geq 1$,

$$\frac{|\beta_{N+i}(z)|^2}{|\alpha_{N+i}(z)|}=\frac{1}{|a(z)|}>1. \tag{13}$$

Let us proof that

$$|B_n^{(N)}|=|\beta(z)|^n. \tag{14}$$

If $|B_s^{(N)}|=|\beta(z)|^s$ for $s\leq n$, then $|\beta_{N+n+1}(z)B_n^{(N)}|=|\beta(z)|^{n+1}$ and $|\alpha_{N+n+1}(z)B_{n-1}^{(N)}|=|\alpha(z)||\beta(z)|^{n-1}$. We have immediately from (12) and (13).

$$|B_{n+1}^{(N)}|=|\beta_{N+n+1}(z)B_n^{(N)}|=|\beta(z)|^{n+1}.$$

We claim that the sequence $(\frac{A_n^{(N)}}{B_n^{(N)}})_n$ converges. The difference between the $(n-1)$th and the nth $(n>0)$ convergent is

$$\frac{A_n^{(N)}}{B_n^{(N)}}-\frac{A_{n-1}^{(N)}}{B_{n-1}^{(N)}}=\frac{(-1)^n\prod_{i=1}^{n}\alpha_{N+i}(z)}{B_{n-1}^{(N)}B_n^{(N)}}, \tag{15}$$

Then from (14), $\left| \frac{A_n^{(N)}}{B_n^{(N)}} - \frac{A_{n-1}^{(N)}}{B_{n-1}^{(N)}} \right| = \frac{|\alpha(z)|^n}{|\beta(z)|^{2n-1}}$.
Consequently for $k \in \mathbb{N}$,

$$\left| \frac{A_{n+k}^{(N)}}{B_{n+k}^{(N)}} - \frac{A_n^{(N)}}{B_n^{(N)}} \right| = \frac{|\alpha(z)|^n}{|\beta(z)|^{2n-1}} = |a(z)|^n |\beta(z)| \longrightarrow 0.$$

Now, suppose that $|a(z)| \geq 1$, one shows, using a simple recurrence on n and (12) that

$$|B_{2n}^{(N)}| \leq |\alpha(z)|^n \text{ and } |B_{2n+1}^{(N)}| \leq |\beta(z)||\alpha(z)|^n . \tag{16}$$

Now, we are able to prove the divergence of $\frac{A_n^{(N)}(z)}{B_n^{(N)}(z)}$ under the assumption $|a(z)| \geq 1$. Indeed, if $\frac{A_n^{(N)}(z)}{B_n^{(N)}(z)}$ converge, then from (15), we deduce that

$$\omega^{(N)}(z) = \sum_{k=1}^{+\infty} \frac{(-1)^k \prod_{i=1}^{k} \alpha_{N+i}(z)}{B_k^{(N)} B_{k-1}^{(N)}}$$

then $A_N^{(N)}(z)$ diverge since from (16)

$$\left| \frac{(-1)^k \prod_{i=1}^{k} \alpha_{N+i}(z)}{B_k^{(N)} B_{k-1}^{(N)}} \right| \geq \left| \frac{\alpha}{\beta} \right| > 0.$$

2 Oppenheim continued fraction expansions (OCF)

Now, we introduce Oppenheim continued fraction expansion. Let $\mathscr{J} = \{\omega \in \mathbb{F}_q((X^{-1})) : |\omega| < 1 \text{ and } \omega \neq 0\}$ and $\{h_j\}_{j\geq 1}$ be a sequence of polynomials valued map defined on $\mathbb{F}_q[X]$. Let $\omega \in \mathscr{J}$, as in the real case [8] we define the Oppenheim algorithm T_0 by

$$T_0(\omega) = \frac{1}{h_1(D_1)+1}\left(\frac{1}{\omega} - D_1\right) \in \mathscr{J} \text{ where } D_1 = \left[\frac{1}{\omega}\right]. \tag{17}$$

Now we define the polynomials $D_j = D_j(\omega)$ and the formal power series ω_j for $j = 1, 2, \ldots$ as follows :

$$\begin{cases} \omega_1 = \omega, & D_j = \left[\frac{1}{\omega_j}\right], \\ \omega_{j+1} = T_0^j(\omega) = T_0(T_0^{j-1}(\omega)) = \frac{1}{h_j(D_j)+1}\left(\frac{1}{\omega_j} - \left[\frac{1}{\omega_j}\right]\right) \end{cases} \tag{18}$$

This algorithm generates the Oppenheim continued fraction expansion of ω as follows

$$\omega = \cfrac{1}{D_1 + \cfrac{h_1(D_1)+1}{D_2 + \cfrac{h_2(D_2)+1}{D_3 + \ddots + \cfrac{h_{j-1}(D_{j-1})+1}{D_j + \ddots}}}}, \tag{19}$$

where $D_j \in F_q[X] \setminus F_q$

Proposition 1. *we have*

$$|D_{j+1}| > |h_j(D_j)+1| \text{ for all } j \geq 1. \tag{20}$$

In fact,

$$D_{j+1} = \left[\frac{1}{T_0^j(\omega)}\right] = \left[\frac{h_j(D_j)+1}{\left\{\frac{1}{T_0^{j-1}(\omega)}\right\}}\right]$$

then $\mid D_{j+1} \mid > \mid h_j(D_j)+1 \mid$.

Proposition 2. *Let A_n and B_n, the numerator and denominator of $K_{i=0}^{n}(\frac{h_i(D_i)+1}{D_i})$, then from (8) and (9), (A_n), (B_n) are recursively defined by*

$$A_0 = 0, \quad A_1 = 1, \quad A_n = D_nA_{n-1} + (h_{n-1}(D_{n-1})+1)A_{n-2}, \quad for \quad n \geq 2 \tag{21}$$

$$B_0 = 1, \quad B_1 = B_1, \quad B_n = D_nB_{n-1} + (h_{n-1}(D_{n-1})+1)B_{n-2}, \quad for \quad n \geq 2 \tag{22}$$

Then, for $n \geq 2$

$$A_nB_{n-1} - A_{n-1}B_n = (-1)^n \prod_{j=1}^{n-1} (h_{n-1}(D_{n-1})+1) \tag{23}$$

and

$$\cfrac{1}{D_1 + \cfrac{(h_1(D_1)+1)}{D_2 + \cfrac{(h_2(D_2)+1)}{D_3 + \ddots + \cfrac{(h_{n-1}(D_{n-1})+1)}{D_n}}}} = \frac{A_n}{B_n} \tag{24}$$

$$\mid B_n \mid > \mid A_n \mid \tag{25}$$

$$\mid B_{n+1} \mid \geq \left| \prod_{i=0}^{n} h_i(D_i+1) \right| \tag{26}$$

Remark. (i) It is clear that the Oppenheim continued fraction is a particular case of the generalized continued fraction (1).

(ii) If $h_j(D_j) = 0$, then we obtain the Regular continued fraction (RCF).

(iii) If $h_j(D_j) = D_j - 1$, then we obtain the Engel continued fraction (ECF).

Proposition 3. *A formal power series $\omega \in \mathscr{J}$ has a finite Oppenheim continued fraction expansion if and only if $\omega \in \mathbb{F}_q(X)$.*

Proof. Using the expression (19) of ω, we state that if ω has a finite expansion then $\omega \in \mathbb{F}_q(X)$. Suppose now ω is rational fraction. By the algorithm, we know that for $j \geq 1$, ω_j is a rational fraction in $\mathscr{J}$, then $\omega_j := \frac{R_j}{S_j} = \frac{R_j}{D_jR_j + R_{j+1}}$ where $\mid R_{j+1} \mid < \mid R_j \mid$ and $D_j = \left[\frac{S_j}{R_j}\right]$. Thus, by the algorithm, we have

$$\omega_{j+1} = \frac{1}{h_j(D_j)+1}\left(\frac{1}{\omega_j} - D_j\right) = \frac{1}{h_j(D_j)+1}\frac{R_{j+1}}{R_j} := \frac{R_{j+1}}{S_{j+1}}. \tag{27}$$

Since $\mid R_{j+1} \mid < \mid R_j \mid$, then this procedure will stop at finite steps, it follows that $\omega_j = 0$ for some j.

Proposition 4. *For all $\omega \in \mathscr{I}$, we have*

$$\lim_{n\to+\infty} \frac{A_n(\omega)}{B_n(\omega)} = \omega. \tag{28}$$

Proof. If ω is rational we conclude (28) by (20). Now let ω be irrational, (24) implies that

$$\omega = \frac{A_n(\omega) + (h_n(D_n(\omega))+1)\omega_{n+1}A_{n-1}(\omega)}{B_n(\omega) + (h_n(D_n(\omega))+1)\omega_{n+1}B_{n-1}(\omega)} \tag{29}$$

$$\mid \omega - \frac{A_n}{B_n} \mid = \frac{\mid (h_n(D_n(\omega))+1)\omega_{n+1}\prod_{j=1}^{n-1}(h_j(D_j)+1) \mid}{\mid B_n \mid\mid (B_n + (h_n(D_n(\omega))+1)\omega_{n+1}B_{n-1}) \mid}.$$

Since $\mid (h_n(D_n(\omega))+1)\omega_{n+1} \mid < 1$, $\mid B_{n-1} \mid < \mid B_n \mid$ and $\mid \prod_{j=1}^{n-1}(h_j(D_j)+1) \mid < \mid B_n \mid$, then

$$\mid \omega - \frac{A_n}{B_n} \mid < \frac{1}{\mid B_n \mid} \longrightarrow 0. \tag{30}$$

Proposition 5. *Let $(D_1,\dots,D_n,\dots)$ and $(h_1(D_1),\dots,h_n(D_n),\dots)$ be two sequences of polynomials such that $\mid D_{i+1} \mid > \mid h_i(D_i)+1 \mid$. Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be given by (21) and (22), Then $\frac{A_n}{B_n}$ converge to some $\omega \in \mathscr{I}$, $D_n(\omega) = D_n$ and $h_j(D_j) = h_j(D_j(\omega))$ for all $n \geq 1$.*

Proof. Let $k \in \mathbb{N}$, we have

$$\begin{aligned}
\mid \frac{A_{n+k}}{B_{n+k}} - \frac{A_n}{B_n} \mid &= \mid \sum_{i=n}^{n+k-1} \left(\frac{A_i}{B_i} - \frac{A_{i-1}}{B_{i-1}}\right) \mid \\
&\leq \max_{n\leq i\leq n+k-1} \mid \frac{A_i}{B_i} - \frac{A_{i-1}}{B_{i-1}} \mid \\
&< \max_{n\leq i\leq n+k-1} \frac{1}{\mid B_i \mid} = \frac{1}{\mid B_n \mid} \longrightarrow 0
\end{aligned}$$

then $\frac{A_n}{B_n}$ is a cauchy sequence which implies that it converge. Let $\omega \in F_q((X^{-1}))$ be its limit.

Let us prove that $\omega \in \mathscr{I}$, $D_n(\omega) = D_n$ and $h_n(D_n(\omega)) = h_n(D_n)$. Since $\exists n_0 \in N$ such that $\forall n \geq n_0$ we have $\mid \omega - \frac{A_n}{B_n} \mid < 1$ then we obtain that

$$\mid \omega \mid \leq \max(\mid \omega - \frac{A_n}{B_n} \mid, \mid \frac{A_n}{B_n} \mid) < 1.$$

For the third part, let

$$C_n = \left[0; \begin{pmatrix} B_1 \\ A_1 \end{pmatrix}, \dots, \begin{pmatrix} B_n \\ A_n \end{pmatrix}\right] = \frac{1}{B_1 + A_1\left[0; \begin{pmatrix} B_2 \\ A_2 \end{pmatrix}, \dots, \begin{pmatrix} B_n \\ A_n \end{pmatrix}\right]}$$

$= \frac{1}{B_1 + A_1\widetilde{C}_n}$. It follows from the first part of the proof that there exists $\widetilde{\omega} \in \mathscr{I}$ such that $\lim_{n\to+\infty} \widetilde{C}_n = \widetilde{\omega}$. We find that $\omega = \frac{1}{B_1 + A_1\widetilde{\omega}}$ which implies that $\widetilde{\omega} = \frac{\frac{1}{\omega} - B_1}{A_1} \in \mathscr{I}$. Since D_1 and $h_1(D_1)$ are unique for which $\frac{\frac{1}{\omega} - D_1}{h_1(D_1)} \in \mathscr{I}$, then $D_1 = B_1$ et $A_1 = h_1(D_1)$. By induction we find the result.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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