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A New Hermite-Hadamard inequality for h-convex stochastic processes

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Abstract: Firstly, some new definitions which are the special cases of h-convex stochastic proceses are given. Then, we establish a new refinement of Hermite-Hadamard inequality for h-convex stochastic processes and give some special cases of this result.

Keywords: Hermite-Hadamard inequality, convex stochastic process, h-convex stochastic processes.

1 Introduction

The classical Hermite-Hadamard inequality which was first published in [5] gives us an estimate of the mean value of a convex function $f : I \to \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

An account the history of this inequality can be found in [3]. Surveys on various generalizations and developments can be found in [2] and [9].

In 1980, Nikodem [10] introduced convex stochastic processes and investigated their regularity properties. In 1992, Skwronski [14] obtained some further results on convex functions.

Let $(\Omega, \mathscr{A}, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathscr{A}-$measurable. A function $X : I \times \Omega \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is called a stochastic process if for every $t \in I$ the function $X(t, .)$ is a random variable.

Recall that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called (i) continuous in probability in interval I, if for all $t_0 \in I$ we have

$$P - \lim_{t \to t_0} X(t, .) = X(t_0, .),$$

where $P - \lim$ denotes the limit in probability. (ii) *mean-square continuous* in the interval I, if for all $t_0 \in I$

$$\lim_{t \to t_0} E\left[(X(t) - X(t_0))^2\right] = 0,$$

where $E[X(t)]$ denotes the expectation value of the random variable $X(t, .)$.

Obviously, *mean-square* continuity implies continuity in probability, but the converse implication is not true.

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Definition 1. *Suppose we are given a sequence $\{\Delta^m\}$ of partitions, $\Delta^m = \{a_{m,0},...,a_{m,n_m}\}$. We say that the sequence $\{\Delta^m\}$ is a normal sequence of partitions if the length of the greatest interval in the $n-$th partition tends to zero, i.e.,*

$$\lim_{m\to\infty} \sup_{1\le i\le n_m} |a_{m,i} - a_{m,i-1}| = 0.$$

Now we would like to recall the concept of the mean-square integral. For the definition and basic properties see [15].

Let $X : I\times\Omega \to \mathbb{R}$ be a stochastic process with $E\left[X(t)^2\right] < \infty$ for all $t\in I$. Let $[a,b]\subset I$, $a = t_0 < t_1 < t_2 < ... < t_n = b$ be a partition of $[a,b]$ and $\Theta_k \in [t_{k-1},t_k]$ for all $k = 1,...,n$. A random variable $Y : \Omega \to \mathbb{R}$ is called the mean-square integral of the process X on $[a,b]$, if we have

$$\lim_{n\to\infty} E\left[\left(\sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y\right)^2\right] = 0$$

for all normal sequence of partitions of the interval $[a,b]$ and for all $\Theta_k \in [t_{k-1},t_k]$, $k = 1,...,n$. Then, we write

$$Y(\cdot) = \int_a^b X(s,\cdot)\,ds \text{ (a.e.).}$$

For the existence of the mean-square integral it is enough to assume the mean-square continuity of the stochastic process X.

Throughout the paper we will frequently use the monotonicity of the mean-square integral. If $X(t,\cdot) \le Y(t,\cdot)$ (a.e.) in some interval $[a,b]$, then

$$\int_a^b X(t,\cdot)\,dt \le \int_a^b Y(t,\cdot)\,dt \text{ (a.e.).}$$

Of course, this inequality is the immediate consequence of the definition of the mean-square integral.

Definition 2. *We say that a stochastic processes $X : I\times\Omega \to \mathbb{R}$ is convex, if for all $\lambda \in [0,1]$ and $u,v \in I$ the inequality*

$$X(\lambda u + (1-\lambda)v,\cdot) \le \lambda X(u,\cdot) + (1-\lambda)X(v,\cdot) \text{ (a.e.)} \tag{2}$$

is satisfied. If the above inequality is assumed only for $\lambda = \frac{1}{2}$, then the process X is Jensen-convex or $\frac{1}{2}-$convex. A stochastic process X is concave if $(-X)$ is convex. Some interesting properties of convex and Jensen-convex processes are presented in ([10], [15]).

Now, we present some results proved by Kotrys [6] about Hermite-Hadamard inequality for convex stochastic processes.

Lemma 1. *If $X : I\times\Omega \to \mathbb{R}$ is a stochastic process of the form $X(t,\cdot) = A(\cdot)t + B(\cdot)$, where $A,B : \Omega \to \mathbb{R}$ are random variables, such that $E\left[A^2\right] < \infty, E\left[B^2\right] < \infty$ and $[a,b]\subset I$, then*

$$\int_a^b X(t,\cdot)\,dt = A(\cdot)\frac{b^2-a^2}{2} + B(\cdot)(b-a) \text{ (a.e.).}$$

Proposition 1. *Let $X : I \times \Omega \to \mathbb{R}$ be a convex stochastic process and $t_0 \in intI$. Then there exist a random variable $A : \Omega \to \mathbb{R}$ such that X is supported at t_0 by the process $A(\cdot)(t-t_0)+X(t_0,\cdot)$. That is*

$$X(t,\cdot) \geq A(\cdot)(t-t_0)+X(t_0,\cdot) \text{ (a.e.).}$$

for all $t \in I$.

Theorem 1. *Let $X : I \times \Omega \to \mathbb{R}$ be Jensen-convex, mean-square continuous in the interval I stochastic process. Then for any $u,v \in I$ we have*

$$X\left(\frac{u+v}{2},\cdot\right) \leq \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt \leq \frac{X(u,\cdot)+X(v,\cdot)}{2} \text{ (a.e.)} \tag{3}$$

In [11], Sarıkaya et al. proved the following refinement of the inequality (3):

Theorem 2. *If $X : I \times \Omega \to \mathbb{R}$ be Jensen-convex, mean-square continuous in the interval I stochastic process. Then for any $u,v \in I$ and for all $\lambda \in [0,1]$, we have*

$$X\left(\frac{u+v}{2},\cdot\right) \leq h(\lambda) \leq \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt \leq H(\lambda) \leq \frac{X(u,\cdot)+X(v,\cdot)}{2}, \tag{4}$$

where

$$h(\lambda) := \lambda X\left(\frac{\lambda v+(2-\lambda)u}{2},\cdot\right)+(1-\lambda)X\left(\frac{(1+\lambda)v+(1-\lambda)u}{2},\cdot\right)$$

and

$$H(\lambda) := \frac{1}{2}\left(X(\lambda v+(1-\lambda)u,\cdot)+\lambda X(u,\cdot)+(1-\lambda)X(v,\cdot)\right).$$

In [1], Barraez et al. introduced the concept of $h-$convex stochastic process with following definition.

Definition 3. *Let $h : (0,1) \to \mathbb{R}$ be a non-negative function, $h \neq 0$.we say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is an $h-$convex stochastic process if, for every $t_1,t_2 \in I$, $\lambda \in (0,1)$, the following inequality is satisfied*

$$X(\lambda u+(1-\lambda)v,\cdot) \leq h(\lambda)X(u,\cdot)+h(1-\lambda)X(v,\cdot) \text{ (a.e.)} \tag{5}$$

Obviously, if we take $h(\lambda)=\lambda$ and $h(\lambda)=\lambda^s$ in (5), then the definition of $h-$convex stochastic process reduces to the definition of classical convex stochastic process [10] and $s-$convex stochastic process in the second sense [12] respectively. Moreover, A stochastic process $X : I \times \Omega \to \mathbb{R}$ is:

(1) Godunova-Levin stochastic process if, we take $h(\lambda)=\frac{1}{\lambda}$ in (5),

$$X(\lambda u+(1-\lambda)v,\cdot) \leq \frac{X(u,\cdot)}{\lambda}+\frac{X(v,\cdot)}{1-\lambda} \text{ (a.e.)} \tag{6}$$

(2) $P-$stochastic process if, we take $h(\lambda)=1$ in (5),

$$X(\lambda u+(1-\lambda)v,\cdot) \leq X(u,\cdot)+X(v,\cdot) \text{ (a.e.)} \tag{7}$$

Authors proved the following Hermite-Hadamard inequality for $h-$convex stochastic process in [1].

Theorem 3. *If $X : I \times \Omega \to \mathbb{R}$ Let be $h : (0,1) \to \mathbb{R}$ a non-negative function, $h \neq 0$ and $X : I \times \Omega \to \mathbb{R}$ a non negative, $h-$convex, mean square integrable stochastic process. For every $u,v \in I, (u<v)$, the following inequality is satisfied*

almost everywhere

$$\frac{1}{2h\left(\frac{1}{2}\right)}X\left(\frac{u+v}{2},\cdot\right)\leq\frac{1}{v-u}\int_{u}^{v}X(t,\cdot)\,dt\leq[X(u,\cdot)+X(v,\cdot)]\int_{0}^{1}h(\lambda)d\lambda.$$

For more information and recent developments on Hermite-Hadamard type inequalities for stochastic process, please refer to ([1], [4], [6]-[8], [11]-[13], [16]). The aim of this paper is to establish an improvement of Hermite-Hadamard inequality for $h-$convex stochastic process.

2 Main results

Theorem 4. *If $X: I\times\Omega\to\mathbb{R}$ Let be $h:(0,1)\to\mathbb{R}$ a non-negative function, $h\neq 0$ and $X: I\times\Omega\to\mathbb{R}$ a non negative, $h-$convex, mean square integrable stochastic process. For every $u,v\in I,(u<v)$, we have the following inequality*

$$\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^{2}}X\left(\frac{u+v}{2},\cdot\right)\leq\Delta_1\leq\frac{1}{v-u}\int_{u}^{v}X(t,\cdot)\,dt\leq\Delta_2\leq[X(u,\cdot)+X(v,\cdot)]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_{0}^{1}h(\lambda)d\lambda \tag{8}$$

where

$$\Delta_1:=\frac{1}{4h\left(\frac{1}{2}\right)}\left[X\left(\frac{3u+v}{4},\cdot\right)+\left(\frac{u+3v}{4},\cdot\right)\right]$$

and

$$\Delta_2:=\left[\frac{X(u,\cdot)+X(v,\cdot)}{2}+X\left(\frac{u+v}{2},\cdot\right)\right]\int_{0}^{1}h(\lambda)d\lambda.$$

Proof. Since $X: I\times\Omega\to\mathbb{R}$ is a $h-$convex stochastic process, we have

$$\begin{aligned}X\left(\frac{u+\frac{u+v}{2}}{2},\cdot\right)&=X\left(\frac{\lambda u+(1-\lambda)\frac{u+v}{2}+(1-\lambda)u+\lambda\frac{u+v}{2}}{2},\cdot\right)\\&\leq h\left(\frac{1}{2}\right)\left[X\left(\lambda u+(1-\lambda)\frac{u+v}{2},\cdot\right)+X\left((1-\lambda)u+\lambda\frac{u+v}{2},\cdot\right)\right].\end{aligned} \tag{9}$$

Integrating (9) from 0 to 1 with respect to λ, we get

$$\begin{aligned}X\left(\frac{3u+v}{4},\cdot\right)&\leq h\left(\frac{1}{2}\right)\left[\int_{0}^{1}X\left(\lambda u+(1-\lambda)\frac{u+v}{2},\cdot\right)d\lambda+\int_{0}^{1}X\left((1-\lambda)u+\lambda\frac{u+v}{2},\cdot\right)d\lambda\right]\\&\leq h\left(\frac{1}{2}\right)\left[\frac{2}{v-u}\int_{u}^{\frac{u+v}{2}}X(t,\cdot)\,dt+\frac{2}{v-u}\int_{u}^{\frac{u+v}{2}}X(t,\cdot)\,dt\right]\\&=\frac{4h\left(\frac{1}{2}\right)}{v-u}\int_{u}^{\frac{u+v}{2}}X(t,\cdot)\,dt.\end{aligned} \tag{10}$$

That is,

$$\frac{1}{4h\left(\frac{1}{2}\right)}X\left(\frac{3u+v}{4},\cdot\right)\leq\frac{1}{v-u}\int_{u}^{\frac{u+v}{2}}X(t,\cdot)\,dt. \tag{11}$$

Since X is a $h-$convex stochastic process, we also have

$$X\left(\frac{\frac{u+v}{2}+v}{2},\cdot\right)=X\left(\frac{\lambda\frac{u+v}{2}+(1-\lambda)v+(1-\lambda)\frac{u+v}{2}+\lambda v}{2},\cdot\right) \tag{12}$$
$$\leq h\left(\frac{1}{2}\right)\left[X\left(\lambda\frac{u+v}{2}+(1-\lambda)v,\cdot\right)+X\left((1-\lambda)\frac{u+v}{2}+\lambda v,\cdot\right)\right].$$

Integrating (12) from 0 to 1 with respect to λ, we get

$$X\left(\frac{u+3v}{4},\cdot\right)\leq h\left(\frac{1}{2}\right)\left[\int_0^1 X\left(\lambda\frac{u+v}{2}+(1-\lambda)v,\cdot\right)d\lambda+\int_0^1 X\left((1-\lambda)\frac{u+v}{2}+\lambda v,\cdot\right)d\lambda\right]$$
$$\leq h\left(\frac{1}{2}\right)\left[\frac{2}{v-u}\int_{\frac{u+v}{2}}^{v} X(t,\cdot)\,dt+\frac{2}{v-u}\int_{\frac{u+v}{2}}^{v} X(t,\cdot)\,dt\right]$$
$$=\frac{4h\left(\frac{1}{2}\right)}{v-u}\int_{\frac{u+v}{2}}^{v} X(t,\cdot)\,dt,$$

i.e.

$$\frac{1}{4h\left(\frac{1}{2}\right)}X\left(\frac{u+3v}{4},\cdot\right)\leq\frac{1}{v-u}\int_{\frac{u+v}{2}}^{v} X(t,\cdot)\,dt. \tag{13}$$

Summing inequalities (11) and (13), we obtain

$$\Delta_1=\frac{1}{4h\left(\frac{1}{2}\right)}\left[X\left(\frac{3u+v}{4},\cdot\right)+\left(\frac{u+3v}{4},\cdot\right)\right]\leq\frac{1}{v-u}\int_u^v X(t,\cdot)\,dt$$

which finishes the proof of second inequality in (8).

Applying the Hermite-Hadamard inequality for $h-$convex stochastic process (Theorem 3), we have

$$\frac{1}{v-u}\int_u^v X(t,\cdot)\,dt=\frac{1}{2}\left[\frac{2}{v-u}\int_u^{\frac{u+v}{2}} X(t,\cdot)\,dt+\frac{2}{v-u}\int_{\frac{u+v}{2}}^{v} X(t,\cdot)\,dt\right]$$
$$\leq\frac{1}{2}\left[\left[X(u,\cdot)+X\left(\frac{u+v}{2},\cdot\right)\right]\int_0^1 h(\lambda)d\lambda\right]+\frac{1}{2}\left[\left[X\left(\frac{u+v}{2},\cdot\right)+X(v,\cdot)\right]\int_0^1 h(\lambda)d\lambda\right]$$
$$=\left[\frac{X(u,\cdot)+X(v,\cdot)}{2}+X\left(\frac{u+v}{2},\cdot\right)\right]\int_0^1 h(\lambda)d\lambda$$
$$=\Delta_2.$$

This completes the proof of third inequality in (8).

For the first inequality, using the $h-$convexity of X, we have

$$\begin{aligned}
\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}X\left(\frac{u+v}{2},\cdot\right) &= \frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}X\left(\frac{1}{2}\frac{3u+v}{2}+\frac{1}{2}\frac{u+3v}{4},\cdot\right)\\
&\leq \frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2}\left[h\left(\frac{1}{2}\right)X\left(\frac{3u+v}{2},\cdot\right)+h\left(\frac{1}{2}\right)X\left(\frac{u+3v}{4},\cdot\right)\right]\\
&= \Delta_1.
\end{aligned}$$

Finally,

$$\begin{aligned}
\Delta_2 &= \left[\frac{X(u,\cdot)+X(v,\cdot)}{2}+X\left(\frac{u+v}{2},\cdot\right)\right]\int_0^1 h(\lambda)d\lambda\\
&\leq \left[\frac{X(u,\cdot)+X(v,\cdot)}{2}+h\left(\frac{1}{2}\right)[X(u,\cdot)+X(v,\cdot)]\right]\int_0^1 h(\lambda)d\lambda\\
&= [X(u,\cdot)+X(v,\cdot)]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\int_0^1 h(\lambda)d\lambda.
\end{aligned}$$

This completes completely the proof of the Theorem.

Remark. Under assumption of Theorem 4 with $h(t)=t$, we have

$$X\left(\frac{u+v}{2},\cdot\right)\leq\Delta_1\leq\frac{1}{v-u}\int_u^v X(t,\cdot)dt\leq\Delta_2\leq\frac{X(u,\cdot)+X(v,\cdot)}{2}$$

where

$$\Delta_1 := \frac{1}{2}\left[X\left(\frac{3u+v}{4},\cdot\right)+\left(\frac{u+3v}{4},\cdot\right)\right]$$

and

$$\Delta_2 := \frac{1}{2}\left[\frac{X(u,\cdot)+X(v,\cdot)}{2}+X\left(\frac{u+v}{2},\cdot\right)\right].$$

This inequality is a special case of the Theorem 2 with $\lambda=\frac{1}{2}$.

Corollary 1. *Under assumption of Theorem 4 with $h(t)=t^s$, we have the refinement Hermite-Hadamard inequality for $s-$convex stochastic processes in the second sense*

$$2^{2s-2}X\left(\frac{u+v}{2},\cdot\right)\leq\Delta_1\leq\frac{1}{v-u}\int_u^v X(t,\cdot)dt\leq\Delta_2\leq[X(u,\cdot)+X(v,\cdot)]\left[\frac{1}{2}+\frac{1}{2^s}\right]\frac{1}{s+1}$$

where

$$\Delta_1 = 2^{s-2}\left[X\left(\frac{3u+v}{4},\cdot\right)+\left(\frac{u+3v}{4},\cdot\right)\right]$$

and

$$\Delta_2 = \left[\frac{X(u,\cdot)+X(v,\cdot)}{2}+X\left(\frac{u+v}{2},\cdot\right)\right]\frac{1}{s+1}.$$

Corollary 2. *Under assumption of Theorem 4 with $h(t)=1$, we have the following Hermite-Hadamard type inequality for $P-$stochastic processes*

$$\frac{1}{4}X\left(\frac{u+v}{2},\cdot\right)\leq\Delta_1\leq\frac{1}{v-u}\int_u^v X(t,\cdot)\,dt\leq\Delta_2\leq\frac{3}{2}\left[X(u,\cdot)+X(v,\cdot)\right]$$

where

$$\Delta_1=\frac{1}{4}\left[X\left(\frac{3u+v}{4},\cdot\right)+\left(\frac{u+3v}{4},\cdot\right)\right]$$

and

$$\Delta_2=\left[\frac{X(u,\cdot)+X(v,\cdot)}{2}+X\left(\frac{u+v}{2},\cdot\right)\right].$$

Corollary 3. *Under assumption of Theorem 4 with $h(t)=\frac{1}{t}$, we have the following Hermite-Hadamard type inequality for Godunova-Levin stochastic processes*

$$\frac{1}{16}X\left(\frac{u+v}{2},\cdot\right)\leq\Delta\leq\frac{1}{v-u}\int_u^v X(t,\cdot)\,dt$$

where

$$\Delta=\frac{1}{8}\left[X\left(\frac{3u+v}{4},\cdot\right)+\left(\frac{u+3v}{4},\cdot\right)\right].$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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