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Sandwich weighted composition operators on weighted hardy spaces

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Abstract: Let C_ϕ, M_ψ and D be the composition, multiplication and differentiation operators defined by $C_\phi f = fo\phi$, $M_\psi f = \psi f$ and $Df = f'$ respectively. In this paper, we study the boundedness and compactness of the sandwich weighted composition operator $DM_\psi C_\phi D$ on the weighted Hardy spaces by using the orthonormal basis of the weighted Hardy spaces.

Keywords: Composition operator, multiplication operator, differentiation operator, weighted Hardy spaces.

1 Introduction

Let ϕ be an analytic self-map of the open unit disc $\mathbb{D}$ in the finite complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the set of all complex valued analytic functions on $\mathbb{D}$. By $\partial\mathbb{D}$ we denote the boundary of $\mathbb{D}$; H^p the classical Hardy space and the space H^∞ consists of all bounded analytic functions f in the disc $\mathbb{D}$. Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be the sequence of positive numbers such that $\beta_0 = 1$ and $\lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 1$. Then $\{\beta_n\}_{n=0}^{\infty}$ is called a weight sequence. For $1 \le p < \infty$, the weighted Hardy space $H^p(\beta)$ is the Banach space of all analytic functions f on the open unit disk $\mathbb{D}$ defined by

$$H^p(\beta) = \left\{ f : z \to \sum_{n=0}^{\infty} a_n z^n \quad s.t \quad \|f\|^p_{H^p(\beta)} = \sum_{n=0}^{\infty} |a_n|^p \beta_n^p < \infty \right\}$$

where $\|.\|_{H^p(\beta)}$ is a norm on $H^p(\beta)$. If $\beta \equiv 1$, then $H^p(\beta)$ becomes the classical Hardy space H^p. For $p = 2$, $H^2(\beta)$ is a Hilbert space w.r.t the inner product

$$< f, g >= \sum_{n=0}^{\infty} a_n.\bar{b_n}\beta_n^2$$

where $f, g \in H^2(\beta)$. For a detailed discussion on $H^p(\beta)$ one can see [13].

Associated with ϕ, the classical linear operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by $f \to fo\phi$ and this operator is called the composition operator induced by self-map ϕ. Let ψ be an analytic function from the open unit disc $\mathbb{D}$ to $\mathbb{C}$, then associated with ψ the multiplication operator $M_\psi f$ is defined by $f \to \psi f$ and the product $M_\psi C_\phi$ of composition and multiplication operators is called weighted composition operator is defined as $f \to \psi.(fo\phi)$. Let D be the differentiation operator defined by $f \to f'$ and the product of composition operator C_ϕ and differentiation operator D is written as $C_\phi D$ and DC_ϕ which are defined as $f \to f'o\phi$ and $f \to (fo\phi)'$ respectively, for function f analytic in the disc $\mathbb{D}$. Similarly, the product of multiplication operator M_ψ and differentiation operator D are defined as $f \to (\psi.f')$ and $f \to (\psi.f)'$.

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Weighted composition operator $M_\psi C_\phi$ followed and preceded by differentiation operator D is denoted by $DM_\psi C_\phi D$ and is defined as $f \to (\psi . f' o \phi)'$. The operator $DM_\psi C_\phi D$ is known as sandwich weighted composition operator and it induces many other operators. For example, if $\psi(z) = 1$, then $DM_\psi C_\phi D = DC_\phi D$, called sandwich composition operator while if $\phi(z) = z$, then we get the sandwich multiplication operator $DM_\psi D$. It has been known that the composition operator C_ϕ is bounded on almost all spaces of analytic functions for example see [2], [3], [9] and D is usually unbounded on spaces of analytic functions. Recently, the above defined operators have received the attention of many researchers see, for example [8], [10],[11] for composition operator and [5], [12], [15], for weighted composition operator.

In [4], Hibschweiles and Portony defined the product $C_\phi D$ and DC_ϕ and studied the boundedness and compactness of these operators between Bergman and Hardy spaces by using the Carleson-type measure, whereas in [8], the author studied the boundedness and compactness of $C_\phi D$ and DC_ϕ between Hardy type spaces.

This paper is organised as follows. In the second section, we discuss the boundedness of the operator $DM_\psi C_\phi D$ on weighted Hardy spaces $H^2(\beta)$. In the third section, we study the compactness of the operator $DM_\psi C_\phi D$ on weighted Hardy spaces $H^2(\beta)$ and in the final section, we give necessary and sufficient condition for the operator $DM_\psi C_\phi D$ to be the Hilbert-Schmidt operator on weighted Hardy spaces.

2 Boundedness of the operator $DM_\psi C_\phi D$

In this section, we characterize the boundedness of the sandwich weighted composition operator $DM_\psi C_\phi D$ on the weighted Hardy Spaces. Recall that a linear operator T on a Hilbert space X is bounded if it takes every bounded set in X into a bounded set in X.

Theorem 1. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ be analytic such that $\{(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1} : n \geq 1\}$ is an orthogonal family. Then the sandwich weighted composition operator $DM_\psi C_\phi D : H^2(\beta) \to H^2(\beta)$ is bounded iff*

$$\|(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1}\|_{H^2(\beta)} \leq \frac{M.\beta_n}{n} \quad for\ all\ n \in \mathbb{N}.$$

Proof. Suppose that the operator $DM_\psi C_\phi D : H^2(\beta) \to H^2(\beta)$ is bounded. Then $\exists$ +ve number M such that

$$\|DM_\psi C_\phi D f\|_{H^2(\beta)} \leq M \|f\|_{H^2(\beta)} \ \ \forall\ f \in\ H^2(\beta). \tag{1}$$

Let $f(z) = z^n$. Then $f \in H^2(\beta)$ and so from (1), we have

$$\|DM_\psi C_\phi D f\|_{H^2(\beta)} = \|DM_\psi C_\phi D(z^n)\|_{H^2(\beta)} = \|n.D(\psi.\phi^{n-1})\|_{H^2(\beta)} = n\|\psi.(n-1)\phi^{n-2}\phi' + \psi'.\phi^{n-1}\|_{H^2(\beta)}$$

$$\therefore\ \ n\|(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1}\|_{H^2(\beta)} \leq M.\|z^n\|_{H^2(\beta)} = M.\beta_n$$

That is

$$\|(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1}\|_{H^2(\beta)} \leq \frac{M.\beta_n}{n} \quad for\ all\ n \in \mathbb{N}.$$

Conversely, assume that

$$\|(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1}\|_{H^2(\beta)} \leq \frac{M.\beta_n}{n} \tag{2}$$

Then, we have to prove that $DM_\psi C_\phi D$ is bounded. Let $f \in H^2(\beta)$ s.t $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since $\{(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1} : n \geq 1\}$ is an orthogonal family, we have

$$\begin{aligned}
\|DM_\psi C_\phi Df\|_{H^2(\beta)}^2 &= \|\sum_{n=1}^{\infty} a_n D(n.\psi.\phi^{n-1})\|_{H^2(\beta)}^2 \\
&= \|\sum_{n=1}^{\infty} n.a_n[(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1}]\|_{H^2(\beta)}^2 \\
&\leq \sum_{n=1}^{\infty} n^2|a_n|^2\|(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1}\|_{H^2(\beta)}^2 \\
&\leq \sum_{n=0}^{\infty} n^2|a_n|^2\frac{M^2.\beta_n^2}{n^2} = M^2\sum_{n=0}^{\infty}|a_n|^2\beta_n^2 \\
&= M^2\|f\|_{H^2(\beta)}^2.
\end{aligned}$$

This implies that $\|DM_\psi C_\phi Df\|_{H^2(\beta)} \leq M\|f\|_{H^2(\beta)}$ and so the operator $DM_\psi C_\phi D$ is bounded.

Corollary 1.*Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$ s.t $\{\phi^n : n \geq 0\}$ is an orthogonal family. Then the sandwich composition operator $DC_\phi\ D : H^2(\beta) \to H^2(\beta)$ is bounded iff*

$$\|\phi^{n-2}.\phi'\|_{H^2(\beta)} \leq \frac{M.\beta_n}{n(n-1)} \quad for\ n \geq 2.$$

Proof. The result following by putting $\psi(z) \equiv 1$.

Corollary 2. *Let $\psi : \mathbb{D} \to \mathbb{C}$ be an analytic such that $\{(n-1)\psi.e_{n-2} + \psi' e_{n-1} : n \geq 1\}$ is an orthogonal family. Then the sandwich multiplication operator $DM_\phi\ D : H^2(\beta) \to H^2(\beta)$ is bounded iff*

$$\|(n-1)\psi.e_{n-2} + \psi' e_{n-1}\| \leq \frac{M.\beta_n}{n} \quad for\ all\ n \geq 2.$$

where $e_n : \mathbb{D} \to \mathbb{D}$ is defined as $e_n(z) = z^n$.

Proof. The result following by putting $\phi(z) = z$. Now we give an example of analytic functions $\phi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ s.t $\{(n-1)\psi.\phi^{n-2}\phi' + \psi'\phi^{n-1} : n \geq 1\}$ is an orthogonal family.

Example 1. Define $\phi : \mathbb{D} \to \mathbb{D}$ as $\phi(z) = z^k$ and $\psi : \mathbb{D} \to \mathbb{C}$ as $\psi(z) = z^m$ for $k, m \in \mathbb{N}$. Then

$$\begin{aligned}
((n-1)\psi.\phi^{n-2}\phi' + \psi'\phi^{n-1})(z) &= (n-1)z^m.z^{k(n-2)}.kz^{k-1} + mz^{m-1}.z^{k(n-1)} \\
&= (n-1)kz^{m+kn-k-1} + mz^{m+kn-k-1} \\
&= (nk-k+m)z^{m+kn-k-1}
\end{aligned}$$

This implies that $\{(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1} : n \geq 1\}$ is an orthogonal family.

Theorem 2. *Let $a \in \mathbb{C}$ and $\beta(n) = (n)^2$. For $\phi(z) = az$ and $\psi(z) = az$, the sandwich weighted composition operator $DM_\psi C_\phi D$ is bounded iff $|a| \leq 1$.*

Proof. We have

$$\begin{aligned}
\|(DM_\psi C_\phi D)\hat{e}_{n+1}\|_{H^2(\beta)} &= \|\frac{(n+1)D(\psi.\phi^n)}{\beta_{n+1}}\|_{H^2(\beta)} \\
&= \frac{(n+1)}{\beta_{n+1}}\|n\psi.\phi^{n-1}\phi' + \psi'\phi^n\|_{H^2(\beta)} \\
&= \frac{(n+1)}{\beta_{n+1}}\|az.na^{n-1}z^{n-1}.a + a.a^n z^n\|_{H^2(\beta)} \\
&= (n+1)^2\|a^{n+1}z^n\|_{H^2(\beta)} \\
&= \frac{(n+1)^2|a^{n+1}|.\beta_n}{\beta_{n+1}} \\
&= |a^{n+1}|
\end{aligned}$$

Hence the Theorem.

3 Compactness of the operator $DM_\psi C_\phi D$

Recall that a linear operator A on a Hilbert space H is called compact if A takes bounded sets into sets with compact closures. This definition is equivalent to the statement that the image of every bounded sequence under A has a convergent subsequence. In this section, we study the compactness of the operator $DM_\psi C_\phi D$ on weighted Hardy space $H^2(\beta)$. For this, we need the following Lemma.

Lemma 1. *Let $\phi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ be analytic mapping. Then the sandwich weighted composition operator $DM_\psi C_\phi D : H^2(\beta) \to H^2(\beta)$ is compact iff for every bounded sequence $\{f_n\}_{n=0}^\infty$ converging to zero uniformly on compact subset of $\mathbb{D}$, we have*

$$||DM_\psi C_\phi D f_n||_{H^2(\beta)} \to 0.$$

Proof. We first suppose that the sandwich weighted composition operator $DM_\psi C_\phi D$ is compact on $H^2(\beta)$. Further, suppose that $\{f_n\}$ is a bounded sequence in $H^2(\beta)$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Then $\{f_n'\}$ and $\{f_n''\}$ converge to zero uniformly on compact subsets of $\mathbb{D}$ and since $DM_\psi C_\phi D$ is compact, $\{(DM_\psi C_\phi D) f_n\}$ has a subsequence which converges in $H^2(\beta)$. That is $\{(\psi.\phi' f_n'' o\phi + \psi'.f_n' o\phi)\}$ has a subsequence which converges in $H^2(\beta)$. Since $\{\phi(z)\}$ is a compact set, $\{f_n'(\phi(z))\}$ and so $\{\psi'(z).f_n'(\phi(z))\}$ converges to zero for each $z \in \mathbb{D}$ and the limit function is necessarily zero. Similarly, $\{\psi(z).\phi'(z).f_n''(\phi(z))\}$ also converges to zero for each $z \in \mathbb{D}$ and the limit function is necessarily zero. Hence $\{(\psi(z).\phi'(z) f_n''(\phi(z)) + \psi'(z) f_n'(\phi(z))\}$ converges to zero for each $z \in \mathbb{D}$ and the limit function is necessarily zero. But this is true for any subsequence of the f_n's, we see that the limit of $\{(\psi.\phi'.f_n'' o\phi + \psi'.f_n' o\phi)\}$ in $H^2(\beta)$ is zero. Hence for every bounded sequence $\{f_n\}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\{(DM_\psi C_\phi D) f_n\}$ converges to zero in $H^2(\beta)$.

Conversely we assume that whenever $\{f_n\}$ is bounded in $H^2(\beta)$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $(DM_\psi C_\phi D) f_n \to 0$ in $H^2(\beta)$. We have to show that $DM_\psi C_\phi D$ is compact on $H^2(\beta)$. Let $\{g_n\}$ be a bounded sequence in $H^2(\beta)$. Since $\{g_n\}$ is a normal family we may extract a subsequence $\{g_{nk}\}$ converging to zero uniformly on compact subsets of $\mathbb{D}$ to some function g. It is easy to check that $g \in H^2(\beta)$ and $\{g_{nk} - g\}$ is a bounded sequence in $H^2(\beta)$ converging almost uniformly to zero. Therefore, by hypothesis $\|(DM_\psi C_\phi D)g_{nk} - (DM_\psi C_\phi D)g\|_{H^2(\beta)} \to 0$. Thus image under $DM_\psi C_\phi D$ of every bounded sequence in $H^2(\beta)$ has a convergent subsequence. Hence $DM_\psi C_\phi D$ is compact on $H^2(\beta)$.

We are now in a position to prove the necessary and sufficient criteria for compactness of sandwich weighted composition operator on $H^2(\beta)$.

Theorem 3. *Let $\phi:\mathbb{D}\to\mathbb{D}$ and $\psi:\mathbb{D}\to\mathbb{C}$ be analytic mappings such that $\{(n-1)\psi.\phi^{n-2}\phi'+\psi'.\phi^{n-1}: n\geq 1\}$ is an orthogonal family. Then the sandwich weighted composition operator $DM_\psi C_\phi D: H^2(\beta)\to H^2(\beta)$ is compact iff*

$$\lim_{n\to\infty}\frac{n}{\beta_n}\|(n-1)\psi.\phi^{n-2}\phi'+\psi'.\phi^{n-1}\|_{H^2(\beta)}=0.$$

Proof. Let us suppose that $DM_\psi C_\phi D: H^2(\beta)\to H^2(\beta)$ is compact. Now the sequence $\{\frac{z^n}{\beta_n}\}_{n=1}^{\infty}$ converges uniformly to zero on compact subsets of $\mathbb{D}$, so by
Lemma 1

$$\|DM_\psi C_\phi D\{\frac{z^n}{\beta_n}\}\|_{H^2(\beta)}\to 0 \text{ as } n\to\infty.$$

Hence

$$\lim_{n\to\infty}\frac{n}{\beta_n}\|(n-1)\psi.\phi^{n-2}\phi'+\psi'.\phi^{n-1}\|_{H^2(\beta)}=0.$$

Conversely, suppose that

$$\lim_{n\to\infty}\frac{n}{\beta_n}\|(n-1)\psi.\phi^{n-2}\phi'+\psi'.\phi^{n-1}\|_{H^2(\beta)}=0.$$

Then for given any $\varepsilon>0$, there exists +ve integer m, such that

$$\|(n-1)\psi.\phi^{n-2}\phi'+\psi'.\phi^{n-1}\|\frac{n}{\beta_n}<\varepsilon\ \ \forall\ \ n\geq m.$$

Now, let $f\in H^2(\beta)$ s.t $f(z)=\sum_{n=0}^{\infty}a_n z^n$. Define an operator T_k on $H^2(\beta)$ as

$$T_k f=\sum_{n=0}^{k}a_n(DM_\psi C_\phi D)z^n=\sum_{n=0}^{k}n.a_n[(n-1)\psi.\phi^{n-2}\phi'+\psi'.\phi^{n-1}]$$

Then T_k is a finite rank and so a compact operator on $H^2(\beta)$. Since $\{(n-1)\psi.\phi^{n-2}\phi'+\psi'.\phi^{n-1}: n\geq 1\}$ is an orthogonal family, for $k\geq m$

$$\begin{aligned}\|(DM_\psi C_\phi D-T_k)f\|^2_{H^2(\beta)}&=\|\sum_{n=k+1}^{\infty}n.a_n[(n-1)\psi.\phi^{n-2}\phi'+\psi'\phi^{n-1}]\|^2_{H^2(\beta)}\\&\leq\sum_{n=k+1}^{\infty}n^2|a_n|^2\|(n-1)\psi.\phi^{n-2}\phi'+\psi'.\phi^{n-1}\|^2_{H^2(\beta)}\\&\leq\sum_{n=0}^{\infty}|a_n|^2.\varepsilon^2.\beta_n^2=\varepsilon^2\sum_{n=0}^{\infty}|a_n|^2.\beta_n^2\\&=\varepsilon^2\|f\|^2_{H^2(\beta)}.\end{aligned}$$

This implies that $\|DM_\psi C_\phi D-T_k\|<\varepsilon\ \ \forall\ \ k\geq m$ and so the operator $DM_\psi C_\phi D$ is compact.

Remark. It is worthwhile to remark here that the necessary part of the above Theorem is true even if $\{(n-1)\psi.\phi^{n-2}\phi'+\psi'.\phi^{n-1}: n\geq 1\}$ is not an orthogonal family.

Corollary 3. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$ such that $\{\phi^n : n \geq 1\}$ is an orthogonal family. Then the sandwich composition operator $DC_\phi D : H^2(\beta) \to H^2(\beta)$ is compact iff*

$$\lim_{n\to\infty} \frac{n(n-1)}{\beta_n} \|\phi^{n-2}.\phi'\|_{H^2(\beta)} = 0.$$

Proof. The result following by putting $\psi(z) \equiv 1$.

Corollary 4. *Let $\psi : \mathbb{D} \to \mathbb{C}$ be an analytic s.t $\{(n-1)\psi.e_{n-2} + \psi' e_{n-1} : n \geq 1\}$ is an orthogonal family. Then the sandwich multiplication operator $DM_\phi D : H^2(\beta) \to H^2(\beta)$ is compact iff*

$$\lim_{n\to\infty} \frac{n}{\beta_n} \|(n-1)\psi.e_{n-2} + \psi'.e_{n-1}\|_{H^2(\beta)} = 0.$$

where $e_n : \mathbb{D} \to \mathbb{D}$ is defined as $e_n(z) = z^n$.

Proof. The result following by putting $\phi(z) = z$. We now give a sufficient condition for sandwich weighted composition operator $DM_\psi C_\phi D$ to be compact on $H^2(\beta)$.

Theorem 4. *Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be the sequence of positive numbers such that $\beta_0 = 1$, $\lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 1$ and $\beta_n \leq 1 \ \forall \ n$. Further, let $\phi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ be analytic mappings such that $\|\phi\|_\infty < 1$ and ϕ', ψ' and ψ are bounded. Then the sandwich weighted composition operator $DM_\psi C_\phi D$ is compact on $H^2(\beta)$.*

Proof. Suppose $\{f_n\}$ is a bounded sequence in $H^2(\beta)$ converging to zero uniformly on compact subset of $\mathbb{D}$. In view of Lemma 1, to show that $DM_\psi C_\phi D$ is compact, it is sufficient to show that $\|(DM_\psi C_\phi D) f_n\|_{H^2(\beta)} \to 0 \ \ as \ \ n \to \infty$. Since $\|\phi\|_\infty < \infty$, $\phi(\mathbb{D})$ is relatively compact subset of $\mathbb{D}$ and so $f_n \to 0$ uniformly on $\phi(\mathbb{D})$. Therefore, the sequences $\{f_n'\}$ and $\{f_n''\}$ also converge to zero uniformly on $\phi(\mathbb{D})$. Since $\beta_n \leq 1 \ \forall \ n$, $H^2(\mathbb{D}) \subseteq H^2(\beta)$. But $H^\infty \subseteq H^2(\mathbb{D})$. Thus $H^\infty \subseteq H^2(\beta)$, which implies that

$$\begin{aligned}
\|(DM_\psi C_\phi D) f_n\|_{H^2(\beta)} &= \|D(\psi.f_n' o\phi)\|_{H^2(\beta)} \\
&= \|\psi(f_n' o\phi)' + \psi'.f_n' o\phi\|_{H^2(\beta)} \\
&\leq \|\psi.f_n'' o\phi.\phi'\|_{H^2(\beta)} + \|\psi'.f_n' o\phi\|_{H^2(\beta)} \\
&\leq \|\psi\|_\infty \|\phi'\|_\infty \|f_n'' o\phi\|_{H^2(\beta)} + \|\psi'\|_\infty \|f_n' o\phi\|_{H^2(\beta)} \\
&\leq \|\psi\|_\infty \|\phi'\|_\infty \|f_n'' o\phi\|_\infty + \|\psi'\|_\infty \|f_n' o\phi\|_\infty \\
&\leq \|\psi\|_\infty \|\phi'\|_\infty \sup_{z\in\mathbb{D}} |(f_n'' o\phi)(z)| + \|\psi'\|_\infty \sup_{z\in\mathbb{D}} |(f_n' o\phi)(z)| \\
&\leq \|\psi\|_\infty \|\phi'\|_\infty \sup_{w\in\phi(\mathbb{D})} |f_n''(w)| + \|\psi'\|_\infty \sup_{w\in\phi(\mathbb{D})} |f_n'(w)| \\
&\to 0 \ as \ n \to \infty
\end{aligned}$$

Hence $DM_\psi C_\phi D$ is compact on $H^2(\beta)$.

Corollary 5. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be analytic. If $\overline{\phi(\mathbb{D})} \subset \mathbb{D}$, then sandwich weighted composition operator $DM_\psi C_\phi D$ is compact on $H^2(\beta)$.*

Proof. Suppose that $\{f_n\}$ is a bounded sequence and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since $\overline{\phi(\mathbb{D})} \subset \mathbb{D}$, it follows that $f_n \to 0$ uniformly on $\overline{\phi(\mathbb{D})}$ and so $\{f_n'\}$ and $\{f_n''\}$ also converge to zero uniformly on $\overline{\phi(\mathbb{D})}$. Hence, as in the proof of above theorem, $\|(DM_\psi C_\phi D) f_n\|_{H^2(\beta)} \to 0 \ \ as \ \ n \to \infty$. This implies that $DM_\psi C_\phi D$ is compact.

Example 2. Let $\beta_0 = 1$ and $\beta_n = \frac{n}{n+1}, n \geq 2$. Further, let $\phi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ be defined as $\phi(z) = \frac{z}{2}$ and $\psi(z) = P(z)$, a polynomial of degree n. Then clearly $\beta_n \leq 1$, $\|\phi\|_\infty < 1$, $\psi'(z) = \frac{1}{2}$ and ψ, ψ' are bounded. Hence by Theorem 4, the sandwich composition operator $DM_\psi C_\phi D$ is compact

4 Necessary and Sufficient Condition for the operator $DM_{\psi}C_{\phi}D$ to be Hilbert Schmidt operator on $H^2(\beta)$

In this section, we give a necessary and sufficient condition for the operator $DM_{\psi}C_{\phi}D$ to be Hilbert Schmidt operator on $H^2(\beta)$. Recall that a linear operator T on Hilbert space H is said to be Hilbert-Schmidt operator if $\sum_{n=0}^{\infty}\|Te_n\|^2 < \infty$ for some orthonormal basis $\{e_n\}$ of H.

Theorem 5. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic such that $\{(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1} : n \geq 1\}$ is an orthogonal family and $\psi : \mathbb{D} \to \mathbb{C}$ be analytic map. Then the sandwich weighted composition operator $DM_{\psi}C_{\phi}D$ is Hilbert-Schmidt operator on $H^2(\beta)$ iff*

$$\sum_{n=0}^{\infty}\frac{n^2}{\beta_n^2}\|(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1}\|^2_{H^2(\beta)} < \infty$$

Proof. Since $\{\frac{z^n}{\beta_n} : n \geq 0\}$ is an orthonormal basis for $H^2(\beta)$. The operator $DM_{\psi}C_{\phi}D$ is Hilbert Schmidt operator

$$iff \quad \sum_{n=0}^{\infty}\|DM_{\psi}C_{\phi}D(\frac{z^n}{\beta_n})\|^2_{H^2(\beta)} < \infty.$$

$$iff \quad \sum_{n=0}^{\infty}\|\frac{n}{\beta_n}[(n-1)\psi.\phi^{n-2}\phi' + \psi'\phi^{n-1}]\|^2_{H^2(\beta)} < \infty$$

$$iff \quad \sum_{n=0}^{\infty}\frac{n^2}{\beta_n^2}\|(n-1)\psi.\phi^{n-2}\phi' + \psi'.\phi^{n-1}\|^2_{H^2(\beta)} < \infty.$$

This completes the proof.

Corollary 6. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$ such that $\{\phi^n : n \geq 1\}$ is an orthogonal family . Then the sandwich composition operator $DC_{\phi}D : H^2(\beta) \to H^2(\beta)$ is Hilbert Schmidt operator iff*

$$\sum_{n=0}^{\infty}\frac{n^2}{\beta_n^2}\|(n-1)\phi^{n-2}\phi'\|^2_{H^2(\beta)} < \infty$$

Proof. The result following by putting $\psi(z) \equiv 1$.

Corollary 7. *Let $\psi : \mathbb{D} \to \mathbb{C}$ be an analytic such that $\{(n-1)\psi.e_{n-2} + \psi' e_{n-1} : n \geq 1\}$ is an orthogonal family. Then the sandwich multiplication operator $DM_{\phi}\ D : H^2(\beta) \to H^2(\beta)$ is Hilbert Schmidt operator iff*

$$\sum_{n=0}^{\infty}\frac{n^2}{\beta_n^2}\|(n-1)\psi.e_{n-2} + \psi'.e_{n-1}\|^2_{H^2(\beta)} < \infty$$

where $e_n : \mathbb{D} \to \mathbb{D}$ is defined as $e_n(z) = z^n$.

Proof. The result following by putting $\phi(z) = z$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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