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Two finite iterative algorithms for finding the reflexive and Hermitian reflexive solutions of coupled complex of conjugate and transpose matrix equations

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Abstract: In this paper, two finite iterative algorithms for finding the reflexive and Hermitian reflexive solutions of coupled complex of conjugate and transpose matrix equations are constructed. With these two iterative algorithms, for any initial reflexive and Hermitian reflexive matrices, the solutions can be obtained within finite iterative steps in the absence of round off errors. Some needed lemmas and theorems are stated and proved to investigate the convergence of the proposed algorithms. Finally, we report two numerical examples to verify the theoretical results.

Keywords: Coupled complex matrix equations, reflexive matrix, Hermitian reflexive matrix, iterative algorithm, inner product, Frobenius norm.

1 Introduction

Matrix equations are often encountered in many areas of computational mathematics, control and system theory. Iterative approaches for solving matrix equations have given much attention from many researches. In [1], an iterative method for solving the linear matrix equation $AXB = C$ over a skew-symmetric matrix X was proposed. In [2], a finite iterative algorithm was presented to solve the pair of linear matrix equations $(AXB, CXD) = (E, F)$. In [3], by using a real inner product in complex matrix spaces, an iterative algorithm is constructed for solving coupled Sylveter-conjugate matrix equations. In [4], two iterative algorithms to obtain the reflexive and Hermitian reflexive solutions to the generalized Sylvester matrix equation $A_1V + A_2\overline{V} + B_1W + B_2\overline{W} = E_1VF_1 + E_2\overline{V}F_2 + C$ are presented. In [5], two new iterative algorithms based on a two-dimensional projection technique for solving $A_1V + A_2\overline{V} + B_1W + B_2\overline{W} = E_1VF_1 + E_2\overline{V}F_2 + C$ over reflexive and Hermitian reflexive matrices are developed. In [6], an iterative algorithm to solve the generalized coupled Sylvester equations $(AY - ZB, CY - ZD) = (E, F)$ over unknown reflexive matrices$Y, Z$are presented. In [7], two gradient based iterative (GI) methods extending the Jacobi and Gauss-Seidel iterations for solving the generalized Sylvester-conjugate matrix equation $A_1XB_1 + A_2\overline{X}B_2 + C_1YD_1 + C_2\overline{Y}D_2 = E$ over reflexive and Hermitian reflexive matrices are presented. In [8], the necessary and sufficient conditions for the solvability of the matrix equation $A^HXB = C$ over reflexive and anti-reflexive matrices are given, and the general expression of the reflexive and anti-reflexive solutions for a solvable case is obtained. In [9], an iterative algorithm is presented for solving a class of complex matrix equations, in which there exist the conjugate and the transpose of the unknown matrices. In [10], the reflexive and anti-reflexive solutions of a linear matrix equation and systems of matrix equations are presented. In [11], two iterative algorithms for finding the Hermitian reflexive and skew-Hermitian solutions of the Sylvester matrix equation $AX + XB = C$ are presented.

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This paper is organized as follows: First, in Section 2, we introduce some notations, definition and a theorem that will be needed to develop this work. In Section 3, we propose two finite iterative algorithms for finding the reflexive and Hermitian reflexive solutions of coupled complex of conjugate and transpose matrix equations and their convergence analysis is also given. In Section 4, two numerical examples are given to explore the simplicity and the neatness of the presented methods.

2 Preliminaries

The following notations, definition and theorem will be used to develop the proposed work. We use $A^T, \overline{A}, A^H$ and $tr(A)$ to denote the transpose, conjugate, conjugate transpose and the trace of a matrix$A$, respectively. We denote the set of all $m\times n$ complex matrices by $\mathbb{C}^{m\times n}$; $Re(a)$ denote the real part of the number a. A matrix $P\in\mathbb{C}^{n\times n}$ is called generalized reflection matrix if $P=P^H$ and $PP^H=I$. The Frobenius norm of the matrix A is denoted by $\|A\|=\sqrt{\langle A,A\rangle}=\sqrt{Re(tr(A^HA))}$. An $n\times n$ matrix A is said to be reflexive with respect to P if $A=PAP$. $\mathbb{C}_r^{n\times n}(P)$ denotes the set of all $n\times n$ reflexive matrices with respect to P. An $n\times n$ matrix A is said to be Hermitian reflexive with respect to P if $A=A^H=PAP$. H $\mathbb{C}_r^{n\times n}(P)$ denotes the set of all $n\times n$ Hermitian reflexive matrices with respect to P.

Definition 1. *[12] A real inner product space is a vector space V over the real field $\mathbb{R}$ together with an inner product that is with a map $\langle .,.\rangle : V\times V \to\mathbb{R}$ Satisfying the following three axioms for all vectors $x,y,z\in V$ and all scalars $a\in\mathbb{R}$*

1.Symmetry:$\langle x,y\rangle=\langle y,x\rangle$.
2. Linearity in the first argument:

$$\langle ax,y\rangle=a\langle x,y\rangle,\langle x+y,z\rangle=\langle x,z\rangle+\langle y,z\rangle.$$

3. Positive definiteness :$\langle x,x\rangle>0$ for all $x\neq 0$.

The following theorem defines a real inner product on space $\mathbb{C}^{m\times n}$ over the field $\mathbb{R}$.

Theorem 1. *[13] In the space $\mathbb{C}^{m\times n}$ over the field $\mathbb{R}$, an inner product can be defined as*

$$\langle A,B\rangle=Re[tr(A^HB)]. \tag{1}$$

3 Main Results

In this section, we propose two finite iterative algorithms to obtain the reflexive (Hermitian reflexive) solutions, respectively, and their convergence analysis is also given to the system of matrix equations of the form

$$\begin{aligned} A_1V^HB_1+C_1W^HD_1+A_2V^TB_2+C_2W^TD_2=E_1,\\ A_3V^HB_3+C_3W^HD_3+A_4V^TB_4+C_4W^TD_4=E_2, \end{aligned} \tag{2}$$

where $A_1,A_2,A_3,A_4\in\mathbb{C}^{m\times s}$, $B_1,B_2,B_3,B_4\in\mathbb{C}^{s\times n}$, $C_1,C_2,C_3,C_4\in\mathbb{C}^{m\times q}$, $D_1,D_2,D_3,D_4\in\mathbb{C}^{q\times n}$ and $E_1,E_2\in\mathbb{C}^{m\times n}$ are given matrices, while $V\in\mathbb{C}_r^{s\times s}(P)$ and $W\in\mathbb{C}_r^{q\times q}(Q)$ ($V\in$H $\mathbb{C}_r^{s\times s}(P)$ and $W\in$H $\mathbb{C}_r^{q\times q}(Q)$) are matrices to be determined.

Let $f(V,W)=A_1V^HB_1+C_1W^HD_1+A_2V^TB_2+C_2W^TD_2$, and $g(V,W)=A_3V^HB_3+C_3W^HD_3+A_4V^TB_4+C_4W^TD_4$.

We introduce the following finite iterative algorithm to obtain the reflexive solution to the system of matrix equations (2).

Algorithm I

1.Input matrices $A_1,A_2,A_3,A_4 \in\mathbb{C}^{m\times s}$,$B_1,B_2,B_3,B_4 \in\mathbb{C}^{s\times n}$, $C_1,C_2,C_3,C_4 \in\mathbb{C}^{m\times q}$, $D_1,D_2,D_3,D_4 \in\mathbb{C}^{q\times n}$ and $E_1,E_2 \in\mathbb{C}^{m\times n}$;
2.Choose arbitrary initial reflexive matrix pair $[V_1,W_1]$ with $V_1 \in \mathbb{C}_r^{s\times s}(P)$ and $W_1 \in \mathbb{C}_r^{q\times q}(Q)$;
3.Set

$$R_1 = diag(E_1 - f(V_1,W_1),E_2 - g(V_1,W_1));$$

$$\begin{aligned}S_1 = \tfrac{1}{2}[&B_1(E_1 - f(V_1,W_1))^H A_1 + \overline{B_2}(E_1 - f(V_1,W_1))^T \overline{A_2} + B_3(E_2 - g(V_1,W_1))^H A_3\\ &+\overline{B_4}(E_2 - g(V_1,W_1))^T \overline{A_4} + PB_1(E_1 - f(V_1,W_1))^H A_1P + P\overline{B_2}(E_1 - f(V_1,W_1))^T \overline{A_2}P\\ &+PB_3(E_2 - g(V_1,W_1))^H A_3P + P\overline{B_4}(E_2 - g(V_1,W_1))^T \overline{A_4}P];\end{aligned}$$

$$\begin{aligned}T_1 = \tfrac{1}{2}[&D_1(E_1 - f(V_1,W_1))^H C_1 + \overline{D_2}(E_1 - f(V_1,W_1))^T \overline{C_2} + D_3(E_2 - g(V_1,W_1))^H C_3\\ &+\overline{D_4}(E_2 - g(V_1,W_1))^T \overline{C_4} + QD_1(E_1 - f(V_1,W_1))^H C_1Q + Q\overline{D_2}(E_1 - f(V_1,W_1))^T \overline{C_2}Q\\ &+QD_3(E_2 - g(V_1,W_1))^H C_3Q + Q\overline{D_4}(E_2 - g(V_1,W_1))^T \overline{C_4}Q];\end{aligned}$$

$$k := 1;$$

4.If $R_k = 0$, then stop and V_k, W_k are the solutions ; else let $k := k+1$ go to STEP 5
5.Compute

$$V_{k+1} = V_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} S_k;$$

$$W_{k+1} = W_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} T_k;$$

$$\begin{aligned}R_{k+1} &= diag(E_1 - f(V_{k+1},W_{k+1}),E_2 - g(V_{k+1},W_{k+1}))\\ &= R_k - \tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} diag(f(S_k,T_k),g(S_k,T_k));\end{aligned}$$

$$\begin{aligned}S_{k+1} = \tfrac{1}{2}[&B_1(E_1 - f(V_{k+1},W_{k+1}))^H A_1 + \overline{B_2}(E_1 - f(V_{k+1},W_{k+1}))^T \overline{A_2} + B_3(E_2 - g(V_{k+1},W_{k+1}))^H A_3\\ &+\overline{B_4}(E_2 - g(V_{k+1},W_{k+1}))^T \overline{A_4} + PB_1(E_1 - f(V_{k+1},W_{k+1}))^H A_1P + P\overline{B_2}(E_1 - f(V_{k+1},W_{k+1}))^T \overline{A_2}P\\ &+PB_3(E_2 - g(V_{k+1},W_{k+1}))^H A_3P + P\overline{B_4}(E_2 - g(V_{k+1},W_{k+1}))^T \overline{A_4}P] + \tfrac{\|R_{k+1}\|^2}{\|R_k\|^2} S_k;\end{aligned}$$

$$\begin{aligned}T_{k+1} = \tfrac{1}{2}[&D_1(E_1 - f(V_{k+1},W_{k+1}))^H C_1 + \overline{D_2}(E_1 - f(V_{k+1},W_{k+1}))^T \overline{C_2} + D_3(E_2 - g(V_{k+1},W_{k+1}))^H C_3\\ &+\overline{D_4}(E_2 - g(V_{k+1},W_{k+1}))^T \overline{C_4} + QD_1(E_1 - f(V_{k+1},W_{k+1}))^H C_1Q + Q\overline{D_2}(E_1 - f(V_{k+1},W_{k+1}))^T \overline{C_2}Q\\ &+QD_3(E_2 - g(V_{k+1},W_{k+1}))^H C_3Q + Q\overline{D_4}(E_2 - g(V_{k+1},W_{k+1}))^T \overline{C_4}Q] + \tfrac{\|R_{k+1}\|^2}{\|R_k\|^2} T_k;\end{aligned}$$

6.If $R_{k+1} = 0$, then stop and V_k, W_k are the solutions; else let $k = k+1$ go to STEP 5.

We introduce the following finite iterative algorithm to obtain the Hermitian reflexive solution to the system of matrix equations (2).

Algorithm II

1.Input matrices $A_1,A_2,A_3,A_4 \in\mathbb{C}^{m\times s}$,$B_1,B_2,B_3,B_4 \in\mathbb{C}^{s\times n}$, $C_1,C_2,C_3,C_4 \in\mathbb{C}^{m\times q}$, $D_1,D_2,D_3,D_4 \in\mathbb{C}^{q\times n}$ and $E_1,E_2 \in\mathbb{C}^{m\times n}$;
2.Choose arbitrary initial Hermitian reflexive matrix pair $[V_1,W_1]$ with $V_1 \in \mathrm{H}\,\mathbb{C}_r^{s\times s}(P)$ and $W_1 \in \mathrm{H}\,\mathbb{C}_r^{q\times q}(Q)$;

3.Set

$$R_1 = diag(E_1 - f(V_1,W_1), E_2 - g(V_1,W_1));$$

$$\begin{aligned}S_1 = \tfrac{1}{2}[&B_1(E_1 - f(V_1,W_1))^H A_1 + \overline{B_2}(E_1 - f(V_1,W_1))^T \overline{A_2} + B_3(E_2 - g(V_1,W_1))^H A_3\\ &+\overline{B_4}(E_2 - g(V_1,W_1))^T \overline{A_4} + A_1^H(E_1 - f(V_1,W_1))B_1^H + A_2^T\overline{(E_1 - f(V_1,W_1))}B_2^T\\ &+A_3^H(E_2 - g(V_1,W_1))B_3^H + A_4^T\overline{(E_2 - g(V_1,W_1))}B_4^T + PB_1(E_1 - f(V_1,W_1))^H A_1 P\\ &+P\overline{B_2}(E_1 - f(V_1,W_1))^T\overline{A_2}P + PB_3(E_2 - g(V_1,W_1))^H A_3 P + P\overline{B_4}(E_2 - g(V_1,W_1))^T\overline{A_4}P\\ &+PA_1^H(E_1 - f(V_1,W_1))B_1^H P + PA_2^T\overline{(E_1 - f(V_1,W_1))}B_2^T P + PA_3^H(E_2 - g(V_1,W_1))B_3^H P\\ &+PA_4^T\overline{(E_2 - g(V_1,W_1))}B_4^T P];\end{aligned}$$

$$\begin{aligned}T_1 = \tfrac{1}{2}[&D_1(E_1 - f(V_1,W_1))^H C_1 + \overline{D_2}(E_1 - f(V_1,W_1))^T\overline{C_2} + D_3(E_2 - g(V_1,W_1))^H C_3\\ &+\overline{D_4}(E_2 - g(V_1,W_1))^T\overline{C_4} + C_1^H(E_1 - f(V_1,W_1))D_1^H + C_2^T\overline{(E_1 - f(V_1,W_1))}D_2^T\\ &+C_3^H(E_2 - g(V_1,W_1))D_3^H + C_4^T\overline{(E_2 - g(V_1,W_1))}D_4^T + QD_1(E_1 - f(V_1,W_1))^H C_1 Q\\ &+Q\overline{D_2}(E_1 - f(V_1,W_1))^T\overline{C_2}Q + QD_3(E_2 - g(V_1,W_1))^H C_3 Q + Q\overline{D_4}(E_2 - g(V_1,W_1))^T\overline{C_4}Q\\ &+QC_1^H(E_1 - f(V_1,W_1))D_1^H Q + QC_2^T\overline{(E_1 - f(V_1,W_1))}D_2^T Q + QC_3^H(E_2 - g(V_1,W_1))D_3^H Q\\ &+QC_4^T\overline{(E_2 - g(V_1,W_1))}D_4^T Q];\end{aligned}$$

$$k := 1;$$

4.If $R_k = 0$, then stop and V_k, W_k are the solutions; else let $k := k+1$ go to STEP 5

5.Compute

$$V_{k+1} = V_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} S_k;$$

$$W_{k+1} = W_k + \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} T_k;$$

$$\begin{aligned}R_{k+1} &= diag(E_1 - f(V_{k+1},W_{k+1}), E_2 - g(V_{k+1},W_{k+1}))\\ &= R_k - \tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} diag(f(S_k,T_k), g(S_k,T_k));\end{aligned}$$

$$\begin{aligned}S_{k+1} = \tfrac{1}{2}[&B_1(E_1 - f(V_{k+1},W_{k+1}))^H A_1 + \overline{B_2}(E_1 - f(V_{k+1},W_{k+1}))^T\overline{A_2} + B_3(E_2 - g(V_{k+1},W_{k+1}))^H A_3\\ &+\overline{B_4}(E_2 - g(V_{k+1},W_{k+1}))^T\overline{A_4} + A_1^H(E_1 - f(V_{k+1},W_{k+1}))B_1^H + A_2^T\overline{(E_1 - f(V_{k+1},W_{k+1}))}B_2^T\\ &+A_3^H(E_2 - g(V_{k+1},W_{k+1}))B_3^H + A_4^T\overline{(E_2 - g(V_{k+1},W_{k+1}))}B_4^T + PB_1(E_1 - f(V_{k+1},W_{k+1}))^H A_1 P\\ &+P\overline{B_2}(E_1 - f(V_{k+1},W_{k+1}))^T\overline{A_2}P + PB_3(E_2 - g(V_{k+1},W_{k+1}))^H A_3 P + P\overline{B_4}(E_2 - g(V_{k+1},W_{k+1}))^T\overline{A_4}P\\ &+PA_1^H(E_1 - f(V_{k+1},W_{k+1}))B_1^H P + PA_2^T\overline{(E_1 - f(V_{k+1},W_{k+1}))}B_2^T P + PA_3^H(E_2 - g(V_{k+1},W_{k+1}))B_3^H P\\ &+PA_4^T\overline{(E_2 - g(V_{k+1},W_{k+1}))}B_4^T P] + \tfrac{\|R_{k+1}\|^2}{\|R_k\|^2} S_k;\end{aligned}$$

$$\begin{aligned}T_{k+1} = \tfrac{1}{2}[&D_1(E_1 - f(V_{k+1},W_{k+1}))^H C_1 + \overline{D_2}(E_1 - f(V_{k+1},W_{k+1}))^T\overline{C_2} + D_3(E_2 - g(V_{k+1},W_{k+1}))^H C_3\\ &+\overline{D_4}(E_2 - g(V_{k+1},W_{k+1}))^T\overline{C_4} + C_1^H(E_1 - f(V_{k+1},W_{k+1}))D_1^H + C_2^T\overline{(E_1 - f(V_{k+1},W_{k+1}))}D_2^T\\ &+C_3^H(E_2 - g(V_{k+1},W_{k+1}))D_3^H + C_4^T\overline{(E_2 - g(V_{k+1},W_{k+1}))}D_4^T + QD_1(E_1 - f(V_{k+1},W_{k+1}))^H C_1 Q\\ &+Q\overline{D_2}(E_1 - f(V_{k+1},W_{k+1}))^T\overline{C_2}Q + QD_3(E_2 - g(V_{k+1},W_{k+1}))^H C_3 Q\\ &+Q\overline{D_4}(E_2 - g(V_{k+1},W_{k+1}))^T\overline{C_4}Q + QC_1^H(E_1 - f(V_{k+1},W_{k+1}))D_1^H Q\\ &+QC_2^T\overline{(E_1 - f(V_{k+1},W_{k+1}))}D_2^T Q + QC_3^H(E_2 - g(V_{k+1},W_{k+1}))D_3^H Q\\ &+QC_4^T\overline{(E_2 - g(V_{k+1},W_{k+1}))}D_4^T Q] + \tfrac{\|R_{k+1}\|^2}{\|R_k\|^2} T_k;\end{aligned}$$

6.If $R_{k+1} = 0$, then stop and V_k, W_k are the solutions; else let $k = k+1$ go to STEP 5.

To prove the convergence property of Algorithm I, we first establish the following basic properties.

Lemma 1. *Suppose that the system of matrix equations (2) is consistent and let V^*, W^* be its reflexive solutions. Then for any initial reflexive matrix pair $[V_1, W_1]$ with $V_1 \in \mathbb{C}_r^{s\times s}(P)$ and $W_1 \in \mathbb{C}_r^{q\times q}(Q)$, we have*

$$\langle S_i, V^* - V_i\rangle + \langle T_i, W^* - W_i\rangle = \|R_i\|^2, \tag{3}$$

where the sequences $\{V_i\}, \{S_i\}, \{W_i\}, \{T_i\}$ and $\{R_i\}$are generated by Algorithm I for $i = 1, 2, \ldots$

Proof. We prove this lemma by mathematical induction. When $i = 1$, from Algorithm I, we have

$$\begin{aligned}
&\langle S_1, V^* - V_1\rangle + \langle T_1, W^* - W_1\rangle = Re\{tr[(V^* - V_1)^H(\tfrac{1}{2}[B_1(E_1 - f(V_1,W_1))^H A_1 + \overline{B_2}(E_1 - f(V_1,W_1))^T\overline{A_2} \\
&+B_3(E_2 - g(V_1,W_1))^H A_3 + \overline{B_4}(E_2 - g(V_1,W_1))^T\overline{A_4} + PB_1(E_1 - f(V_1,W_1))^H A_1 P \\
&+P\overline{B_2}(E_1 - f(V_1,W_1))^T\overline{A_2}P + PB_3(E_2 - g(V_1,W_1))^H A_3 P + P\overline{B_4}(E_2 - g(V_1,W_1))^T\overline{A_4}P]) \\
&+(W^* - W_1)^H(\tfrac{1}{2}[D_1(E_1 - f(V_1,W_1))^H C_1 + \overline{D_2}(E_1 - f(V_1,W_1))^T\overline{C_2} + D_3(E_2 - g(V_1,W_1))^H C_3 \\
&+\overline{D_4}(E_2 - g(V_1,W_1))^T\overline{C_4} + QD_1(E_1 - f(V_1,W_1))^H C_1 Q + Q\overline{D_2}(E_1 - f(V_1,W_1))^T\overline{C_2}Q \\
&+QD_3(E_2 - g(V_1,W_1))^H C_3 Q + Q\overline{D_4}(E_2 - g(V_1,W_1))^T\overline{C_4}Q])]\}
\end{aligned}$$

$$\begin{aligned}
&= Re\{tr[(E_1 - f(V_1,W_1))^H A_1(V^* - V_1)^H B_1 + \overline{(E_1 - f(V_1,W_1))^T\overline{A_2}(V^* - V_1)^H\overline{B_2}} \\
&+(E_2 - g(V_1,W_1))^H A_3(V^* - V_1)^H B_3 + \overline{(E_2 - g(V_1,W_1))^T\overline{A_4}(V^* - V_1)^H\overline{B_4}} \\
&+(E_1 - f(V_1,W_1))^H C_1(W^* - W_1)^H D_1 + \overline{(E_1 - f(V_1,W_1))^T\overline{C_2}(W^* - W_1)^H\overline{D_2}} \\
&+(E_2 - g(V_1,W_1))^H C_3(W^* - W_1)^H D_3 + \overline{(E_2 - g(V_1,W_1))^T\overline{C_4}(W^* - W_1)^H\overline{D_4}}]\}
\end{aligned}$$

$$\begin{aligned}
&= Re\{tr[(E_1 - f(V_1,W_1))^H(A_1V^{*H}B_1 - A_1V_1^H B_1 + A_2V^{*T}B_2 - A_2V_1^T B_2 + C_1W^{*H}D_1 - C_1W_1^H D_1 \\
&+C_2W^{*T}D_2 - C_2W_1^T D_2) + (E_2 - g(V_1,W_1))^H(A_3V^{*H}B_3 - A_3V_1^H B_3 + A_4V^{*T}B_4 - A_4V_1^T B_4 \\
&+C_3W^{*H}D_3 - C_3W_1^H D_3 + C_4W^{*T}D_4 - C_4W_1^T D_4)]\}
\end{aligned}$$

$$\begin{aligned}
&= Re\{tr[(E_1 - f(V_1,W_1))^H(A_1V^{*H}B_1 + A_2V^{*T}B_2 + C_1W^{*H}D_1 + C_2W^{*T}D_2 - (A_1V_1^H B_1 + A_2V_1^T B_2 \\
&+C_1W_1^H D_1 + C_2W_1^T D_2)) + (E_2 - g(V_1,W_1))^H(A_3V^{*H}B_3 + A_4V^{*T}B_4 + C_3W^{*H}D_3 + C_4W^{*T}D_4 \\
&-(A_3V_1^H B_3 + A_4V_1^T B_4 + C_3W_1^H D_3 + C_4W_1^T D_4))]\}
\end{aligned}$$

$$= Re\{tr[(E_1 - f(V_1,W_1))^H(E_1 - f(V_1,W_1)) + (E_2 - g(V_1,W_1))^H(E_2 - g(V_1,W_1))]\}$$

$$= Re\{tr[\begin{bmatrix} E_1 - f(V_1,W_1) & 0 \\ 0 & E_2 - g(V_1,W_1)\end{bmatrix}^H \begin{bmatrix} E_1 - f(V_1,W_1) & 0 \\ 0 & E_2 - g(V_1,W_1)\end{bmatrix}]\} = Re\{tr[R_1^H R_1]\} = \|R_1\|^2$$

This implies that (3) holds for $i = 1$. Now, assume that (3) holds for $i = k$. That is,

$$\langle S_k, V^* - V_k\rangle + \langle T_k, W^* - W_k\rangle = \|R_k\|^2$$

Then we have to prove that the conclusion holds for $i = k+1$. It follows from Algorithm I that

$$\begin{aligned}
&\langle S_{k+1},V^*-V_{k+1}\rangle+\langle T_{k+1},W^*-W_{k+1}\rangle = Re\{tr[(V^*-V_{k+1})^H(\tfrac{1}{2}[B_1(E_1-f(V_{k+1},W_{k+1}))^HA_1\\
&+\overline{B_2}(E_1-f(V_{k+1},W_{k+1}))^T\overline{A_2}+B_3(E_2-g(V_{k+1},W_{k+1}))^HA_3+\overline{B_4}(E_2-g(V_{k+1},W_{k+1}))^T\overline{A_4}\\
&+PB_1(E_1-f(V_{k+1},W_{k+1}))^HA_1P+P\overline{B_2}(E_1-f(V_{k+1},W_{k+1}))^T\overline{A_2}P\\
&+PB_3(E_2-g(V_{k+1},W_{k+1}))^HA_3P+P\overline{B_4}(E_2-g(V_{k+1},W_{k+1}))^T\overline{A_4}P]+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}S_k)\\
&+(W^*-W_{k+1})^H(\tfrac{1}{2}[D_1(E_1-f(V_{k+1},W_{k+1}))^HC_1+\overline{D_2}(E_1-f(V_{k+1},W_{k+1}))^T\overline{C_2}\\
&+D_3(E_2-g(V_{k+1},W_{k+1}))^HC_3+\overline{D_4}(E_2-g(V_{k+1},W_{k+1}))^T\overline{C_4}+QD_1(E_1-f(V_{k+1},W_{k+1}))^HC_1Q\\
&+Q\overline{D_2}(E_1-f(V_{k+1},W_{k+1}))^T\overline{C_2}Q+QD_3(E_2-g(V_{k+1},W_{k+1}))^HC_3Q\\
&+Q\overline{D_4}(E_2-g(V_{k+1},W_{k+1}))^T\overline{C_4}Q]+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}T_k)]\}
\end{aligned}$$

$$\begin{aligned}
&=Re\{tr[(E_1-f(V_{k+1},W_{k+1}))^HA_1(V^*-V_{k+1})^HB_1+\overline{(E_1-f(V_{k+1},W_{k+1}))^T\overline{A_2}(V^*-V_{k+1})^H\overline{B_2}}\\
&+(E_2-g(V_{k+1},W_{k+1}))^HA_3(V^*-V_{k+1})^HB_3+\overline{(E_2-g(V_{k+1},W_{k+1}))^T\overline{A_4}(V^*-V_{k+1})^H\overline{B_4}}\\
&+(E_1-f(V_{k+1},W_{k+1}))^HC_1(W^*-W_{k+1})^HD_1+\overline{(E_1-f(V_{k+1},W_{k+1}))^T\overline{C_2}(W^*-W_{k+1})^H\overline{D_2}}\\
&+(E_2-g(V_{k+1},W_{k+1}))^HC_3(W^*-W_{k+1})^HD_3+\overline{(E_2-g(V_{k+1},W_{k+1}))^T\overline{C_4}(W^*-W_{k+1})^H\overline{D_4}}]\}\\
&+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}Re\{tr[(V^*-V_{k+1})^HS_k+(W^*-W_{k+1})^HT_k]\}
\end{aligned}$$

$$\begin{aligned}
&=Re\{tr[(E_1-f(V_{k+1},W_{k+1}))^H(A_1V^{*H}B_1-A_1V_{k+1}^HB_1+A_2V^{*T}B_2-A_2V_{k+1}^TB_2+C_1W^{*H}D_1\\
&-C_1W_{k+1}^HD_1+C_2W^{*T}D_2-C_2W_{k+1}^TD_2)+(E_2-g(V_{k+1},W_{k+1}))^H(A_3V^{*H}B_3-A_3V_{k+1}^HB_3\\
&+A_4V^{*T}B_4-A_4V_{k+1}^TB_4+C_3W^{*H}D_3-C_3W_{k+1}^HD_3+C_4W^{*T}D_4-C_4W_{k+1}^TD_4)]\}\\
&+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}Re\{tr[(V^*-V_k-\tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2}S_k)^HS_k+(W^*-W_k-\tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2}T_k)^HT_k]\}
\end{aligned}$$

$$\begin{aligned}
&=Re\{tr[(E_1-f(V_{k+1},W_{k+1}))^H(A_1V^{*H}B_1+A_2V^{*T}B_2+C_1W^{*H}D_1+C_2W^{*T}D_2-(A_1V_{k+1}^HB_1+A_2V_{k+1}^TB_2\\
&+C_1W_{k+1}^HD_1+C_2W_{k+1}^TD_2))+(E_2-g(V_{k+1},W_{k+1}))^H(A_3V^{*H}B_3+A_4V^{*T}B_4+C_3W^{*H}D_3+C_4W^{*T}D_4\\
&-(A_3V_{k+1}^HB_3+A_4V_{k+1}^TB_4+C_3W_{k+1}^HD_3+C_4W_{k+1}^TD_4))]\}\\
&+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}Re\{tr[(V^*-V_k)^HS_k+(W^*-W_k)^HT_k]\}-\frac{\|R_{k+1}\|^2}{\|R_k\|^2}\frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2}Re\{tr[S_k^HS_k+T_k^HT_k]\}
\end{aligned} \tag{4}$$

In view that V^*,W^* are solutions of the system of matrix equations (2), with relation (4) one has

$$\begin{aligned}
&\langle S_{k+1},V^*-V_{k+1}\rangle+\langle T_{k+1},W^*-W_{k+1}\rangle == Re\{tr[(E_1-f(V_{k+1},W_{k+1}))^H(E_1-f(V_{k+1},W_{k+1}))\\
&+(E_2-g(V_{k+1},W_{k+1}))^H(E_2-g(V_{k+1},W_{k+1}))]\}+\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}\|R_k\|^2-\tfrac{\|R_{k+1}\|^2}{\|R_k\|^2}\tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2}(\|S_k\|^2+\|T_k\|^2)\\
&=Re\{tr[\begin{bmatrix}E_1-f(V_{k+1},W_{k+1}) & 0\\ 0 & E_2-g(V_{k+1},W_{k+1})\end{bmatrix}^H\begin{bmatrix}E_1-f(V_{k+1},W_{k+1}) & 0\\ 0 & E_2-g(V_{k+1},W_{k+1})\end{bmatrix}]\}\\
&+\|R_{k+1}\|^2-\|R_{k+1}\|^2
\end{aligned}$$

$$=Re\{tr(R_{k+1}^HR_{k+1})\}=\|R_{k+1}\|^2$$

So, (3) holds for $i=k+1$. Hence relation (3) holds by the principle of induction.

Lemma 2. *Suppose that system of matrix equations (2) is consistent and the sequences* $\{S_i\}$, $\{T_i\}$ *and* $\{R_i\}$ *are generated by Algorithm I , such that* $R_i \neq 0$*for all* $i = 1,2,\dots,$ *then*

$$\langle R_i, R_j \rangle = 0 \tag{5}$$

and

$$\langle S_i, S_j \rangle + \langle T_i, T_j \rangle = 0, fori, j = 1,2,\dots,k, i \neq j. \tag{6}$$

Proof. We prove the conclusion by induction,

Step 1: We prove

$$\langle R_i, R_{i+1} \rangle = 0 \tag{7}$$

$$\langle S_i, S_{i+1} \rangle + \langle T_i, T_{i+1} \rangle = 0, for\ i = 1,2,\dots. \tag{8}$$

First from Algorithm I, we have

$$R_{k+1} = diag(E_1 - f(V_{k+1}, W_{k+1}), E_2 - g(V_{k+1}, W_{k+1}))$$

$$= diag(E_1 - A_1 V_{k+1}^H B_1 - C_1 W_{k+1}^H D_1 - A_2 V_{k+1}^T B_2 - C_2 W_{k+1}^T D_2, E_2 - A_3 V_{k+1}^H B_3 - C_3 W_{k+1}^H D_3 - A_4 V_{k+1}^T B_4 - C_4 W_{k+1}^T D_4)$$

$$= diag(E_1 - A_1(V_k + \tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} S_k)^H B_1 - C_1(W_k + \tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} T_k)^H D_1 - A_2(V_k + \tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} S_k)^T B_2$$
$$-C_2(W_k + \tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} T_k)^T D_2, E_2 - A_3(V_k + \tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} S_k)^H B_3 - C_3(W_k + \tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} T_k)^H D_3$$
$$-A_4(V_k + \tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} S_k)^T B_4 - C_4(W_k + \tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} T_k)^T D_4)$$

$$= diag(E_1 - A_1 V_k^H B_1 - C_1 W_k^H D_1 - A_2 V_k^T B_2 - C_2 W_k^T D_2, E_2 - A_3 V_k^H B_3 - C_3 W_k^H D_3 - A_4 V_k^T B_4 - C_4 W_k^T D_4)$$
$$-\tfrac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} diag(A_1 S_k^H B_1 + C_1 T_k^H D_1 + A_2 S_k^T B_2 + C_2 T_k^T D_2, A_3 S_k^H B_3 + C_3 T_k^H D_3 + A_4 S_k^T B_4 + C_4 T_k^T D_4)$$

$$= diag(E_1 - f(V_k, W_k), E_2 - g(V_k, W_k)) - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} diag(f(S_k, T_k), g(S_k, T_k))$$

$$= R_k - \frac{\|R_k\|^2}{\|S_k\|^2 + \|T_k\|^2} diag(f(S_k, T_k), g(S_k, T_k)). \tag{9}$$

For $i = 1$, it follows from (9) that

$$\langle R_1, R_2 \rangle = Re\{tr[R_2^H R_1]\}$$

$$= Re\{tr[(R_1 - \frac{\|R_1\|^2}{\|S_1\|^2 + \|T_1\|^2} diag(f(S_1, T_1), g(S_1, T_1)))^H R_1]\}$$

$$= Re\{tr[R_1^H R_1]\} - \frac{\|R_1\|^2}{\|S_1\|^2 + \|T_1\|^2} Re\{tr[\begin{bmatrix} f(S_1,T_1) & 0 \\ 0 & g(S_1,T_1) \end{bmatrix}^H \cdot \begin{bmatrix} E_1 - f(V_1,W_1) & 0 \\ 0 & E_2 - g(V_1,W_1) \end{bmatrix}]\}$$

$$= \|R_1\|^2 - \tfrac{\|R_1\|^2}{\|S_1\|^2+\|T_1\|^2} Re\{tr[(A_1 S_1^H B_1 + C_1 T_1^H D_1 + A_2 S_1^T B_2 + C_2 T_1^T D_2)^H (E_1 - f(V_1, W_1))$$
$$+(A_3 S_1^H B_3 + C_3 T_1^H D_3 + A_4 S_1^T B_4 + C_4 T_1^T D_4)^H (E_2 - g(V_1, W_1))]\}$$

$$= \|R_1\|^2 - \tfrac{\|R_1\|^2}{\|S_1\|^2+\|T_1\|^2} Re\{tr[S_1^H B_1 (E_1 - f(V_1, W_1))^H A_1 + T_1^H D_1 (E_1 - f(V_1, W_1))^H C_1$$
$$+\overline{S_1^T B_2 (E_1 - f(V_1, W_1))^H A_2} + \overline{T_1^T D_2 (E_1 - f(V_1, W_1))^H C_2} + S_1^H B_3 (E_2 - g(V_1, W_1))^H A_3$$
$$+T_1^H D_3 (E_2 - g(V_1, W_1))^H C_3 + \overline{S_1^T B_4 (E_2 - g(V_1, W_1))^H A_4} + \overline{T_1^T D_4 (E_2 - g(V_1, W_1))^H C_4}]\}$$

$$\begin{aligned}&=\|R_1\|^2-\frac{\|R_1\|^2}{\|S_1\|^2+\|T_1\|^2}Re\{tr[S_1^H(B_1(E_1-f(V_1,W_1))^HA_1+\overline{B_2}(E_1-f(V_1,W_1))^T\overline{A_2}\\&+B_3(E_2-g(V_1,W_1))^HA_3+\overline{B_4}(E_2-g(V_1,W_1))^T\overline{A_4})+T_1^H(D_1(E_1-f(V_1,W_1))^HC_1\\&+\overline{D_2}(E_1-f(V_1,W_1))^T\overline{C_2}+D_3(E_2-g(V_1,W_1))^HC_3+\overline{D_4}(E_2-g(V_1,W_1))^T\overline{C_4})]\}\end{aligned}$$

$$\begin{aligned}&=\|R_1\|^2-\frac{\|R_1\|^2}{\|S_1\|^2+\|T_1\|^2}Re\{tr[S_1^H(\tfrac{1}{2}[B_1(E_1-f(V_1,W_1))^HA_1+\overline{B_2}(E_1-f(V_1,W_1))^T\overline{A_2}\\&+B_3(E_2-g(V_1,W_1))^HA_3+\overline{B_4}(E_2-g(V_1,W_1))^T\overline{A_4}+PB_1(E_1-f(V_1,W_1))^HA_1P\\&+P\overline{B_2}(E_1-f(V_1,W_1))^T\overline{A_2}P+PB_3(E_2-g(V_1,W_1))^HA_3P+P\overline{B_4}(E_2-g(V_1,W_1))^T\overline{A_4}P])\\&+T_1^H(\tfrac{1}{2}[D_1(E_1-f(V_1,W_1))^HC_1+\overline{D_2}(E_1-f(V_1,W_1))^T\overline{C_2}+D_3(E_2-g(V_1,W_1))^HC_3\\&+\overline{D_4}(E_2-g(V_1,W_1))^T\overline{C_4}+QD_1(E_1-f(V_1,W_1))^HC_1Q+Q\overline{D_2}(E_1-f(V_1,W_1))^T\overline{C_2}Q\\&+QD_3(E_2-g(V_1,W_1))^HC_3Q+Q\overline{D_4}(E_2-g(V_1,W_1))^T\overline{C_4}Q])]\}\end{aligned}$$

$$\begin{aligned}&=\|R_1\|^2-\frac{\|R_1\|^2}{\|S_1\|^2+\|T_1\|^2}Re\{tr[S_1^HS_1+T_1^HT_1]\}\\&=\|R_1\|^2-\frac{\|R_1\|^2}{\|S_1\|^2+\|T_1\|^2}[\|S_1\|^2+\|T_1\|^2]=0\end{aligned}$$

This implies that (7) is satisfied for $i=1$. From Algorithm I, we also have

$$\langle S_1,S_2\rangle+\langle T_1,T_2\rangle=Re\{tr[S_2^HS_1+T_2^HT_1]\}$$

$$\begin{aligned}&=Re\{tr[(\tfrac{1}{2}[B_1(E_1-f(V_2,W_2))^HA_1+\overline{B_2}(E_1-f(V_2,W_2))^T\overline{A_2}+B_3(E_2-g(V_2,W_2))^HA_3\\&+\overline{B_4}(E_2-g(V_2,W_2))^T\overline{A_4}+PB_1(E_1-f(V_2,W_2))^HA_1P+P\overline{B_2}(E_1-f(V_2,W_2))^T\overline{A_2}P\\&+PB_3(E_2-g(V_2,W_2))^HA_3P+P\overline{B_4}(E_2-g(V_2,W_2))^T\overline{A_4}P]+\frac{\|R_2\|^2}{\|R_1\|^2}S_1)^HS_1\\&+(\tfrac{1}{2}[D_1(E_1-f(V_2,W_2))^HC_1+\overline{D_2}(E_1-f(V_2,W_2))^T\overline{C_2}+D_3(E_2-g(V_2,W_2))^HC_3\\&+\overline{D_4}(E_2-g(V_2,W_2))^T\overline{C_4}+QD_1(E_1-f(V_2,W_2))^HC_1Q+Q\overline{D_2}(E_1-f(V_2,W_2))^T\overline{C_2}Q\\&+QD_3(E_2-g(V_2,W_2))^HC_3Q+Q\overline{D_4}(E_2-g(V_2,W_2))^T\overline{C_4}Q]+\frac{\|R_2\|^2}{\|R_1\|^2}T_1)^HT_1]\}\end{aligned}$$

$$\begin{aligned}&=Re\{tr[(E_1-f(V_2,W_2))^HA_1S_1^HB_1+\overline{(E_1-f(V_2,W_2))^T\overline{A_2}S_1^H\overline{B_2}}+(E_2-g(V_2,W_2))^HA_3S_1^HB_3\\&+\overline{(E_2-g(V_2,W_2))^T\overline{A_4}S_1^H\overline{B_4}}+(E_1-f(V_2,W_2))^HC_1T_1^HD_1+\overline{(E_1-f(V_2,W_2))^T\overline{C_2}T_1^H\overline{D_2}}\\&+(E_2-g(V_2,W_2))^HC_3T_1^HD_3+\overline{(E_2-g(V_2,W_2))^T\overline{C_4}T_1^H\overline{D_4}}]\}+\frac{\|R_2\|^2}{\|R_1\|^2}Re\{tr[S_1^HS_1+T_1^HT_1]\}\end{aligned}$$

$$\begin{aligned}&=Re\{tr[(E_1-f(V_2,W_2))^H(A_1S_1^HB_1+A_2S_1^TB_2+C_1T_1^HD_1+C_2T_1^TD_2)\\&+(E_2-g(V_2,W_2))^H(A_3S_1^HB_3+A_4S_1^TB_4+C_3T_1^HD_3+C_4T_1^TD_4)]\}\\&+\frac{\|R_2\|^2}{\|R_1\|^2}Re\{tr[S_1^HS_1+T_1^HT_1]\}\end{aligned}$$

$$\begin{aligned}&=Re\{tr[\begin{bmatrix}E_1-f(V_2,W_2)&0\\0&E_2-g(V_2,W_2)\end{bmatrix}^H\begin{bmatrix}f(S_1,T_1)&0\\0&g(S_1,T_1)\end{bmatrix}]\}\\&+\frac{\|R_2\|^2}{\|R_1\|^2}Re\{tr[S_1^HS_1+T_1^HT_1]\}\end{aligned}$$

$$\begin{aligned}&=\frac{\|S_1\|^2+\|T_1\|^2}{\|R_1\|^2}[Re\{tr[R_2^H(R_1-R_2)]\}+\frac{\|R_2\|^2}{\|R_1\|^2}(\|S_1\|^2+\|T_1\|^2)\\&=-\frac{\|S_1\|^2+\|T_1\|^2}{\|R_1\|^2}[\|R_2\|^2]+\frac{\|R_2\|^2}{\|R_1\|^2}(\|S_1\|^2+\|T_1\|^2)=0\end{aligned}$$

Thus, (8) satisfied for $i=1$.

Assume (7) and (8) hold for $i = k-1$. From (9) and applying mathematical assumption, from Algorithm I one has

$$\langle R_k, R_{k+1}\rangle = Re\{tr[R_{k+1}^H R_k]\}$$

$$= Re\{tr[(R_k - \frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} diag(f(S_k,T_k), g(S_k,T_k)))^H R_k]\}$$

$$= Re\{tr[R_k^H R_k]\} - \frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} Re\{tr[\begin{bmatrix} f(S_k,T_k) & 0 \\ 0 & g(S_k,T_k)\end{bmatrix}^H \cdot \begin{bmatrix} E_1 - f(V_k,W_k) & 0 \\ 0 & E_2 - g(V_k,W_k)\end{bmatrix}]\}$$

$$= \|R_k\|^2 - \frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} Re\{tr[(A_1 S_k^H B_1 + C_1 T_k^H D_1 + A_2 S_k^T B_2 + C_2 T_k^T D_2)^H (E_1 - f(V_k,W_k)) + (A_3 S_k^H B_3 + C_3 T_k^H D_3 + A_4 S_k^T B_4 + C_4 T_k^T D_4)^H (E_2 - g(V_k,W_k))]\}$$

$$= \|R_k\|^2 - \frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} Re\{tr[S_k^H B_1 (E_1 - f(V_k,W_k))^H A_1 + T_k^H D_1 (E_1 - f(V_k,W_k))^H C_1 + \overline{S_k^T B_2 (E_1 - f(V_k,W_k))^H A_2} + \overline{T_k^T D_2 (E_1 - f(V_k,W_k))^H C_2} + S_k^H B_3 (E_2 - g(V_k,W_k))^H A_3 + T_k^H D_3 (E_2 - g(V_k,W_k))^H C_3 + \overline{S_k^T B_4 (E_2 - g(V_k,W_k))^H A_4} + \overline{T_k^T D_4 (E_2 - g(V_k,W_k))^H C_4}]\}$$

$$= \|R_k\|^2 - \frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} Re\{tr[S_k^H (B_1 (E_1 - f(V_k,W_k))^H A_1 + \overline{B_2} (E_1 - f(V_k,W_k))^T \overline{A_2} + B_3 (E_2 - g(V_k,W_k))^H A_3 + \overline{B_4} (E_2 - g(V_k,W_k))^T \overline{A_4}) + T_k^H (D_1 (E_1 - f(V_k,W_k))^H C_1 + \overline{D_2} (E_1 - f(V_k,W_k))^T \overline{C_2} + D_3 (E_2 - g(V_k,W_k))^H C_3 + \overline{D_4} (E_2 - g(V_k,W_k))^T \overline{C_4})]\}$$

$$= \|R_k\|^2 - \frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} Re\{tr[S_k^H (\tfrac{1}{2}[B_1 (E_1 - f(V_{k+1},W_{k+1}))^H A_1 + \overline{B_2} (E_1 - f(V_{k+1},W_{k+1}))^T \overline{A_2} + B_3 (E_2 - g(V_{k+1},W_{k+1}))^H A_3 + \overline{B_4} (E_2 - g(V_{k+1},W_{k+1}))^T \overline{A_4} + P B_1 (E_1 - f(V_{k+1},W_{k+1}))^H A_1 P + P\overline{B_2} (E_1 - f(V_{k+1},W_{k+1}))^T \overline{A_2} P + P B_3 (E_2 - g(V_{k+1},W_{k+1}))^H A_3 P + P\overline{B_4} (E_2 - g(V_{k+1},W_{k+1}))^T \overline{A_4} P]) + T_k^H (\tfrac{1}{2}[D_1 (E_1 - f(V_{k+1},W_{k+1}))^H C_1 + \overline{D_2} (E_1 - f(V_{k+1},W_{k+1}))^T \overline{C_2} + D_3 (E_2 - g(V_{k+1},W_{k+1}))^H C_3 + \overline{D_4} (E_2 - g(V_{k+1},W_{k+1}))^T \overline{C_4} + Q D_1 (E_1 - f(V_{k+1},W_{k+1}))^H C_1 Q + Q\overline{D_2} (E_1 - f(V_{k+1},W_{k+1}))^T \overline{C_2} Q + Q D_3 (E_2 - g(V_{k+1},W_{k+1}))^H C_3 Q + Q\overline{D_4} (E_2 - g(V_{k+1},W_{k+1}))^T \overline{C_4} Q])]\}$$

$$= \|R_k\|^2 - \frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} Re\{tr[S_k^H (S_k - \frac{\|R_k\|^2}{\|R_{k-1}\|^2} S_{k-1}) + T_k^H (T_k - \frac{\|R_k\|^2}{\|R_{k-1}\|^2} T_{k-1})]\}$$

$$= \|R_k\|^2 - \frac{\|R_k\|^2}{\|S_k\|^2+\|T_k\|^2} [\|S_k\|^2 + \|T_k\|^2 - \frac{\|R_k\|^2}{\|R_{k-1}\|^2} Re\{tr[S_k^H S_{k-1} + T_k^H T_{k-1}]\} = 0.$$

Thus, (7) holds for $i = k$.

Also, from Algorithm I one also has

$$\langle S_k, S_{k+1}\rangle + \langle T_k, T_{k+1}\rangle = Re\{tr[S_{k+1}^H S_k + T_{k+1}^H T_k]\}$$

$$= Re\{tr[(\tfrac{1}{2}[B_1 (E_1 - f(V_{k+1},W_{k+1}))^H A_1 + \overline{B_2} (E_1 - f(V_{k+1},W_{k+1}))^T \overline{A_2} + B_3 (E_2 - g(V_{k+1},W_{k+1}))^H A_3 + \overline{B_4} (E_2 - g(V_{k+1},W_{k+1}))^T \overline{A_4} + P B_1 (E_1 - f(V_{k+1},W_{k+1}))^H A_1 P + P\overline{B_2} (E_1 - f(V_{k+1},W_{k+1}))^T \overline{A_2} P + P B_3 (E_2 - g(V_{k+1},W_{k+1}))^H A_3 P + P\overline{B_4} (E_2 - g(V_{k+1},W_{k+1}))^T \overline{A_4} P] + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} S_k)^H S_k + (\tfrac{1}{2}[D_1 (E_1 - f(V_{k+1},W_{k+1}))^H C_1 + \overline{D_2} (E_1 - f(V_{k+1},W_{k+1}))^T \overline{C_2} + D_3 (E_2 - g(V_{k+1},W_{k+1}))^H C_3 + \overline{D_4} (E_2 - g(V_{k+1},W_{k+1}))^T \overline{C_4} + Q D_1 (E_1 - f(V_{k+1},W_{k+1}))^H C_1 Q + Q\overline{D_2} (E_1 - f(V_{k+1},W_{k+1}))^T \overline{C_2} Q + Q D_3 (E_2 - g(V_{k+1},W_{k+1}))^H C_3 Q + Q\overline{D_4} (E_2 - g(V_{k+1},W_{k+1}))^T \overline{C_4} Q] + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} T_k)^H T_k]\}$$

$$= Re\{tr[(E_1 - f(V_{k+1},W_{k+1}))^H A_1 S_k^H B_1 + \overline{(E_1 - f(V_{k+1},W_{k+1}))^T \overline{A_2} S_k^H \overline{B_2}} + (E_2 - g(V_{k+1},W_{k+1}))^H A_3 S_k^H B_3$$
$$+ \overline{(E_2 - g(V_{k+1},W_{k+1}))^T \overline{A_4} S_k^H \overline{B_4}} + (E_1 - f(V_{k+1},W_{k+1}))^H C_1 T_k^H D_1 + \overline{(E_1 - f(V_{k+1},W_{k+1}))^T \overline{C_2} T_k^H \overline{D_2}}$$
$$+ (E_2 - g(V_{k+1},W_{k+1}))^H C_3 T_k^H D_3 + \overline{(E_2 - g(V_{k+1},W_{k+1}))^T \overline{C_4} T_k^H \overline{D_4}}]\} + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} Re\{tr[S_k^H S_k + T_k^H T_k]\}$$

$$= Re\{tr[(E_1 - f(V_{k+1},W_{k+1}))^H (A_1 S_k^H B_1 + A_2 S_k^T B_2 + C_1 T_k^H D_1 + C_2 T_k^T D_2)$$
$$+ (E_2 - g(V_{k+1},W_{k+1}))^H (A_3 S_k^H B_3 + A_4 S_k^T B_4 + C_3 T_k^H D_3 + C_4 T_k^T D_4)]\}$$
$$+ \frac{\|R_{k+1}\|^2}{\|R_k\|^2} (\|S_k\|^2 + \|T_k\|^2)$$

$$= Re\{tr[\begin{bmatrix} E_1 - f(V_{k+1},W_{k+1}) & 0 \\ 0 & E_2 - g(V_{k+1},W_{k+1}) \end{bmatrix}^H \begin{bmatrix} f(S_k,T_k) & 0 \\ 0 & g(S_k,T_k) \end{bmatrix}]\}$$
$$+ \frac{\|R_{k+1}\|^2}{\|R_k\|^2} (\|S_k\|^2 + \|T_k\|^2)$$

$$= \frac{\|S_k\|^2 + \|T_k\|^2}{\|R_k\|^2} [Re\{tr[R_{k+1}^H (R_k - R_{k+1})]\} + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} (\|S_k\|^2 + \|T_k\|^2)$$
$$= -\frac{\|S_k\|^2 + \|T_k\|^2}{\|R_k\|^2} [\|R_{k+1}\|^2] + \frac{\|R_{k+1}\|^2}{\|R_k\|^2} (\|S_k\|^2 + \|T_k\|^2)$$

This implies that (7) and (8) hold for $i = k$. Hence, the relation (7) and (8) hold for all $1 \le i \le k$.

Step2: We want to show that,

$$\langle R_i, R_{i+l} \rangle = 0, \tag{10}$$

and

$$\langle S_i, S_{i+l} \rangle + \langle T_i, T_{i+l} \rangle = 0, \tag{11}$$

hold for integer $l \ge 1$. We prove these two equations given by (10) and (11) by using induction. The case of $l = 1$ is proved in Step 1. Now we assume that (10) and (11) hold for $l \le q, q \ge 1$ the aim is to show

$$\langle R_i, R_{i+q+1} \rangle = 0, \tag{12}$$

and

$$\langle S_i, S_{i+q+1} \rangle + \langle T_i, T_{i+q+1} \rangle = 0. \tag{13}$$

First, we prove the following,

$$\langle R_0, R_{q+1} \rangle = 0, \tag{14}$$

and

$$\langle S_0, S_{q+1} \rangle + \langle T_0, T_{q+1} \rangle = 0. \tag{15}$$

According Algorithm I, from (9) and induction assumption one has

$$\langle R_0, R_{q+1} \rangle = Re\{tr[R_{q+1}^H R_0]\}$$

$$= Re\{tr[(R_q - \frac{\|R_q\|^2}{\|S_q\|^2 + \|T_q\|^2} diag(f(S_q,T_q), g(S_q,T_q)))^H R_0]\}$$

$$= Re\{tr[R_q^H R_0]\} - \frac{\|R_q\|^2}{\|S_q\|^2 + \|T_q\|^2} Re\{tr[\begin{bmatrix} f(S_q,T_q) & 0 \\ 0 & g(S_q,T_q) \end{bmatrix}^H \cdot \begin{bmatrix} E_1 - f(V_0,W_0) & 0 \\ 0 & E_2 - g(V_0,W_0) \end{bmatrix}]\}$$

$$=-\frac{\|R_q\|^2}{\|S_q\|^2+\|T_q\|^2}Re\{tr[(A_1S_q^HB_1+C_1T_q^HD_1+A_2S_q^TB_2+C_2T_q^TD_2)^H(E_1-f(V_0,W_0)) +(A_3S_q^HB_3+C_3T_q^HD_3+A_4S_q^TB_4+C_4T_q^TD_4)^H(E_2-g(V_0,W_0))]\}$$

$$=-\frac{\|R_q\|^2}{\|S_q\|^2+\|T_q\|^2}Re\{tr[S_q^HB_1(E_1-f(V_0,W_0))^HA_1+T_q^HD_1(E_1-f(V_0,W_0))^HC_1 +\overline{S_q^TB_2(E_1-f(V_0,W_0))^HA_2}+\overline{T_q^TD_2(E_1-f(V_0,W_0))^HC_2}+S_q^HB_3(E_2-g(V_0,W_0))^HA_3 +T_q^HD_3(E_2-g(V_0,W_0))^HC_3+\overline{S_q^TB_4(E_2-g(V_0,W_0))^HA_4}+\overline{T_q^TD_4(E_2-g(V_0,W_0))^HC_4}]\}$$

$$=-\frac{\|R_q\|^2}{\|S_q\|^2+\|T_q\|^2}Re\{tr[S_q^H(B_1(E_1-f(V_0,W_0))^HA_1+\overline{B_2}(E_1-f(V_0,W_0))^T\overline{A_2} +B_3(E_2-g(V_0,W_0))^HA_3+\overline{B_4}(E_2-g(V_0,W_0))^T\overline{A_4})+T_q^H(D_1(E_1-f(V_0,W_0))^HC_1 +\overline{D_2}(E_1-f(V_0,W_0))^T\overline{C_2}+D_3(E_2-g(V_0,W_0))^HC_3+\overline{D_4}(E_2-g(V_0,W_0))^T\overline{C_4})]\}$$

$$=-\frac{\|R_q\|^2}{\|S_q\|^2+\|T_q\|^2}Re\{tr[S_q^H(\tfrac{1}{2}[B_1(E_1-f(V_0,W_0))^HA_1+\overline{B_2}(E_1-f(V_0,W_0))^T\overline{A_2} +B_3(E_2-g(V_0,W_0))^HA_3+\overline{B_4}(E_2-g(V_0,W_0))^T\overline{A_4}+PB_1(E_1-f(V_0,W_0))^HA_1P +P\overline{B_2}(E_1-f(V_0,W_0))^T\overline{A_2}P+PB_3(E_2-g(V_0,W_0))^HA_3P+P\overline{B_4}(E_2-g(V_0,W_0))^T\overline{A_4}P]) +T_q^H(\tfrac{1}{2}[D_1(E_1-f(V_0,W_0))^HC_1+\overline{D_2}(E_1-f(V_0,W_0))^T\overline{C_2}+D_3(E_2-g(V_0,W_0))^HC_3 +\overline{D_4}(E_2-g(V_0,W_0))^T\overline{C_4}+QD_1(E_1-f(V_0,W_0))^HC_1Q+Q\overline{D_2}(E_1-f(V_0,W_0))^T\overline{C_2}Q +QD_3(E_2-g(V_0,W_0))^HC_3Q+Q\overline{D_4}(E_2-g(V_0,W_0))^T\overline{C_4}Q])]\}$$

$$=-\frac{\|R_q\|^2}{\|S_q\|^2+\|T_q\|^2}Re\{tr[S_q^HS_0+T_q^HT_0]\}=0,$$

and

$$\langle S_0,S_{q+1}\rangle+\langle T_0,T_{q+1}\rangle=Re\{tr[S_{q+1}^HS_0+T_{q+1}^HT_0]\}$$

$$=Re\{tr[(\tfrac{1}{2}[B_1(E_1-f(V_{q+1},W_{q+1}))^HA_1+\overline{B_2}(E_1-f(V_{q+1},W_{q+1}))^T\overline{A_2}+B_3(E_2-g(V_{q+1},W_{q+1}))^HA_3 +\overline{B_4}(E_2-g(V_{q+1},W_{q+1}))^T\overline{A_4}+PB_1(E_1-f(V_{q+1},W_{q+1}))^HA_1P+P\overline{B_2}(E_1-f(V_{q+1},W_{q+1}))^T\overline{A_2}P +PB_3(E_2-g(V_{q+1},W_{q+1}))^HA_3P+P\overline{B_4}(E_2-g(V_{q+1},W_{q+1}))^T\overline{A_4}P]+\frac{\|R_{q+1}\|^2}{\|R_q\|^2}S_q)^HS_0 +(\tfrac{1}{2}[D_1(E_1-f(V_{q+1},W_{q+1}))^HC_1+\overline{D_2}(E_1-f(V_{q+1},W_{q+1}))^T\overline{C_2}+D_3(E_2-g(V_{q+1},W_{q+1}))^HC_3 +\overline{D_4}(E_2-g(V_{q+1},W_{q+1}))^T\overline{C_4}+QD_1(E_1-f(V_{q+1},W_{q+1}))^HC_1Q+Q\overline{D_2}(E_1-f(V_{q+1},W_{q+1}))^T\overline{C_2}Q +QD_3(E_2-g(V_{q+1},W_{q+1}))^HC_3Q+Q\overline{D_4}(E_2-g(V_{q+1},W_{q+1}))^T\overline{C_4}Q]+\frac{\|R_{q+1}\|^2}{\|R_q\|^2}T_q)^HT_0]\}$$

$$=Re\{tr[(E_1-f(V_{q+1},W_{q+1}))^HA_1S_0^HB_1+\overline{(E_1-f(V_{q+1},W_{q+1}))^T\overline{A_2}S_0^H\overline{B_2}}+(E_2-g(V_{q+1},W_{q+1}))^HA_3S_0^HB_3 +\overline{(E_2-g(V_{q+1},W_{q+1}))^T\overline{A_4}S_0^H\overline{B_4}}+(E_1-f(V_{q+1},W_{q+1}))^HC_1T_0^HD_1+(E_1-f(V_{q+1},W_{q+1}))^T\overline{C_2}T_0^H\overline{D_2} +(E_2-g(V_{q+1},W_{q+1}))^HC_3T_0^HD_3+\overline{(E_2-g(V_{q+1},W_{q+1}))^T\overline{C_4}T_0^H\overline{D_4}}]\}+\frac{\|R_{k+1}\|^2}{\|R_k\|^2}Re\{tr[S_q^HS_0+T_q^HT_0]\}$$

$$=Re\{tr[(E_1-f(V_{q+1},W_{q+1}))^H(A_1S_0^HB_1+A_2S_0^TB_2+C_1T_0^HD_1+C_2T_0^TD_2) +(E_2-g(V_{q+1},W_{q+1}))^H(A_3S_0^HB_3+A_4S_0^TB_4+C_3T_0^HD_3+C_4T_0^TD_4)]\}$$

$$=Re\{tr[\begin{bmatrix}E_1-f(V_{q+1},W_{q+1}) & 0\\ 0 & E_2-g(V_{q+1},W_{q+1})\end{bmatrix}^H\begin{bmatrix}f(S_0,T_0) & 0\\ 0 & g(S_0,T_0)\end{bmatrix}]\}$$

$$=\frac{\|S_0\|^2+\|T_0\|^2}{\|R_k\|^2}[Re\{tr[R_{q+1}^H(R_0-R_1)]\}=0.$$

Then (14) and (15) hold.

From Algorithm I and (9), induction assumption one has

$$\langle S_i, S_{i+q+1}\rangle + \langle T_i, T_{i+q+1}\rangle = Re\{tr[S_{i+q+1}^H S_i + T_{i+q+1}^H T_i]\}$$

$$= Re\{tr[(\tfrac{1}{2}[B_1(E_1 - f(V_{i+q+1}, W_{i+q+1}))^H A_1 + \overline{B_2}(E_1 - f(V_{i+q+1}, W_{i+q+1}))^T \overline{A_2} + B_3(E_2 - g(V_{i+q+1}, W_{i+q+1}))^H A_3$$
$$+ \overline{B_4}(E_2 - g(V_{i+q+1}, W_{i+q+1}))^T \overline{A_4} + PB_1(E_1 - f(V_{i+q+1}, W_{i+q+1}))^H A_1 P + P\overline{B_2}(E_1 - f(V_{i+q+1}, W_{i+q+1}))^T \overline{A_2} P$$
$$+ PB_3(E_2 - g(V_{i+q+1}, W_{i+q+1}))^H A_3 P + P\overline{B_4}(E_2 - g(V_{i+q+1}, W_{i+q+1}))^T \overline{A_4} P] + \frac{\|R_{i+q+1}\|^2}{\|R_{i+q}\|^2} S_{i+q})^H S_i$$
$$+ (\tfrac{1}{2}[D_1(E_1 - f(V_{i+q+1}, W_{i+q+1}))^H C_1 + \overline{D_2}(E_1 - f(V_{i+q+1}, W_{i+q+1}))^T \overline{C_2} + D_3(E_2 - g(V_{i+q+1}, W_{i+q+1}))^H C_3$$
$$+ \overline{D_4}(E_2 - g(V_{i+q+1}, W_{i+q+1}))^T \overline{C_4} + QD_1(E_1 - f(V_{i+q+1}, W_{i+q+1}))^H C_1 Q + Q\overline{D_2}(E_1 - f(V_{i+q+1}, W_{i+q+1}))^T \overline{C_2} Q$$
$$+ QD_3(E_2 - g(V_{i+q+1}, W_{i+q+1}))^H C_3 Q + Q\overline{D_4}(E_2 - g(V_{i+q+1}, W_{i+q+1}))^T \overline{C_4} Q] + \frac{\|R_{i+q+1}\|^2}{\|R_{i+q}\|^2} T_{i+q})^H T_i]\}$$

$$= Re\{tr[(E_1 - f(V_{i+q+1}, W_{i+q+1}))^H A_1 S_i^H B_1 + \overline{(E_1 - f(V_{i+q+1}, W_{i+q+1}))^T \overline{A_2} S_i^H \overline{B_2}}$$
$$+ (E_2 - g(V_{i+q+1}, W_{i+q+1}))^H A_3 S_i^H B_3 + \overline{(E_2 - g(V_{i+q+1}, W_{i+q+1}))^T \overline{A_4} S_i^H \overline{B_4}}$$
$$+ (E_1 - f(V_{i+q+1}, W_{i+q+1}))^H C_1 T_i^H D_1 + \overline{(E_1 - f(V_{i+q+1}, W_{i+q+1}))^T \overline{C_2} T_i^H \overline{D_2}}$$
$$+ (E_2 - g(V_{i+q+1}, W_{i+q+1}))^H C_3 T_i^H D_3 + \overline{(E_2 - g(V_{i+q+1}, W_{i+q+1}))^T \overline{C_4} T_i^H \overline{D_4}}]\} + \frac{\|R_{i+q+1}\|^2}{\|R_{i+q}\|^2} Re\{tr[S_{i+q}^H S_i + T_{i+q}^H T_i]\}$$

$$= Re\{tr[(E_1 - f(V_{i+q+1}, W_{i+q+1}))^H (A_1 S_i^H B_1 + A_2 S_i^T B_2 + C_1 T_i^H D_1 + C_2 T_i^T D_2)$$
$$+ (E_2 - g(V_{i+q+1}, W_{i+q+1}))^H (A_3 S_i^H B_3 + A_4 S_i^T B_4 + C_3 T_i^H D_3 + C_4 T_i^T D_4)]\} + \frac{\|R_{i+q+1}\|^2}{\|R_{i+q}\|^2} Re\{tr[S_{i+q}^H S_i + T_{i+q}^H T_i]\}$$

$$= Re\{tr[\begin{bmatrix} E_1 - f(V_{i+q+1}, W_{i+q+1}) & 0 \\ 0 & E_2 - g(V_{i+q+1}, W_{i+q+1}) \end{bmatrix}^H \begin{bmatrix} f(S_i, T_i) & 0 \\ 0 & g(S_i, T_i) \end{bmatrix}]\}$$

$$= \frac{\|S_i\|^2 + \|T_i\|^2}{\|R_i\|^2} Re\{tr[R_{i+q+1}^H (R_i - R_{i+1})]\}$$

$$= \frac{\|S_i\|^2 + \|T_i\|^2}{\|R_i\|^2} Re\{tr[R_{i+q+1}^H R_i]\}. \tag{16}$$

In addition, from (9) it can be shown that

$$\langle R_i, R_{i+q+1}\rangle = Re\{tr[R_{i+q+1}^H R_i]\} = Re\{tr[(R_{i+q} - \frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2 + \|T_{i+q}\|^2} diag(f(S_{i+q}, T_{i+q}), g(S_{i+q}, T_{i+q})))^H R_i]\}$$

$$= Re\{tr(R_{i+q}^H R_i)\} - \frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2 + \|T_{i+q}\|^2} Re\{tr[\begin{bmatrix} f(S_{i+q}, T_{i+q}) & 0 \\ 0 & g(S_{i+q}, T_{i+q}) \end{bmatrix}^H \cdot \begin{bmatrix} E_1 - f(V_i, W_i) & 0 \\ 0 & E_2 - g(V_i, W_i) \end{bmatrix}]\}$$

$$= -\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2 + \|T_{i+q}\|^2} Re\{tr[(A_1 S_{i+q}^H B_1 + C_1 T_{i+q}^H D_1 + A_2 S_{i+q}^T B_2 + C_2 T_{i+q}^T D_2)^H (E_1 - f(V_i, W_i))$$
$$+ (A_3 S_{i+q}^H B_3 + C_3 T_{i+q}^H D_3 + A_4 S_{i+q}^T B_4 + C_4 T_{i+q}^T D_4)^H (E_2 - g(V_i, W_i))]\}$$

$$= -\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2} Re\{tr[S_{i+q}^H B_1(E_1-f(V_i,W_i))^H A_1 + T_{i+q}^H D_1(E_1-f(V_i,W_i))^H C_1 + \overline{S_{i+q}^T B_2(E_1-f(V_i,W_i))^H A_2} + \overline{T_{i+q}^T D_2(E_1-f(V_i,W_i))^H C_2} + S_{i+q}^H B_3(E_2-g(V_i,W_i))^H A_3 + T_{i+q}^H D_3(E_2-g(V_i,W_i))^H C_3 + \overline{S_{i+q}^T B_4(E_2-g(V_i,W_i))^H A_4} + \overline{T_{i+q}^T D_4(E_2-g(V_i,W_i))^H C_4}]\}$$

$$= -\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2} Re\{tr[S_{i+q}^H (B_1(E_1-f(V_i,W_i))^H A_1 + \overline{B_2}(E_1-f(V_i,W_i))^T \overline{A_2} + B_3(E_2-g(V_i,W_i))^H A_3 + \overline{B_4}(E_2-g(V_i,W_i))^T \overline{A_4}) + T_{i+q}^H (D_1(E_1-f(V_i,W_i))^H C_1 + \overline{D_2}(E_1-f(V_i,W_i))^T \overline{C_2} + D_3(E_2-g(V_i,W_i))^H C_3 + \overline{D_4}(E_2-g(V_i,W_i))^T \overline{C_4})]\}$$

$$= -\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2} Re\{tr[S_{i+q}^H (\tfrac{1}{2}[B_1(E_1-f(E_1-f(V_i,W_i))^H A_1 + \overline{B_2}(E_1-f(V_i,W_i))^T \overline{A_2} + B_3(E_2-g(V_i,W_i))^H A_3 + \overline{B_4}(E_2-g(V_i,W_i))^T \overline{A_4} + PB_1(E_1-f(V_i,W_i))^H A_1 P + P\overline{B_2}(E_1-f(V_i,W_i))^T \overline{A_2} P + PB_3(E_2-g(V_i,W_i))^H A_3 P + P\overline{B_4}(E_2-g(V_i,W_i))^T \overline{A_4} P]) + T_{i+q}^H (\tfrac{1}{2}[D_1(E_1-f(V_i,W_i))^H C_1 + \overline{D_2}(E_1-f(V_i,W_i))^T \overline{C_2} + D_3(E_2-g(V_i,W_i))^H C_3 + \overline{D_4}(E_2-g(V_i,W_i))^T \overline{C_4} + QD_1(E_1-f(V_i,W_i))^H C_1 Q + Q\overline{D_2}(E_1-f(V_i,W_i))^T \overline{C_2} Q + QD_3(E_2-g(V_i,W_i))^H C_3 Q + Q\overline{D_4}(E_2-g(V_i,W_i))^T \overline{C_4} Q])]\}$$

$$= -\frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2} Re\{tr[S_{i+q}^H (S_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2} S_{i-1}) + T_{i+q}^H (T_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2} T_{i-1})]\}$$

$$= \frac{\|R_{i+q}\|^2}{\|S_{i+q}\|^2+\|T_{i+q}\|^2} \frac{\|R_i\|^2}{\|R_{i-1}\|^2} Re\{tr[S_{i+q}^H S_{i-1} + T_{i+q}^H T_{i-1}]\} = \langle S_{i-1}, S_{i+q}\rangle + \langle T_{i-1}, T_{i+q}\rangle \tag{17}$$

Repeating (16) and (17), we can obtain, for certain α and β

$$\langle S_i, S_{i+q+1}\rangle + \langle T_i, T_{i+q+1}\rangle = \alpha \langle S_0, S_{q+1}\rangle + \langle T_0, T_{q+1}\rangle,$$

and

$$\langle R_i, R_{i+q+1}\rangle = \beta \langle R_0, R_{q+1}\rangle.$$

Combining these two relations with (14) and (15) implies that (10) and (11) holds for $l = q+1$. From Step (1) and (2) the conclusion holds by the principle of induction.

Remark. Lemma 1 implies that if there exist a positive number i such that $S_i = 0$and $T_i = 0$but $R_i \neq 0$, then the system of matrix equations (2) is inconsistent.

With the above two lemmas, one has the following theorem:

Theorem 2. *[13] If the system of matrix equations (2) is consistent, then for any initial reflexive matrix pair $[V_1, W_1]$ with $V_1 \in \mathbb{C}_r^{s\times s}(P)$ and $W_1 \in \mathbb{C}_r^{q\times q}(Q)$ a reflexive solution pair can be obtained with a finite number of iteration steps by using Algorithm I.*

Proof. Suppose that $R_i \neq 0$ for $i = 1,2,3,\ldots,2mn$, by Lemma 1 and the previous remark, we have $S_i \neq 0$ or $T_i \neq 0$. Then we can compute $V_{2mn+1}, W_{2mn+1}, R_{2mn+1}$ by Algorithm I. Also, from Lemma 2 we have $\langle R_i, R_{2mn+1}\rangle = 0$ and $\langle R_i, R_j\rangle = 0$ for $i = 1,2,3,\ldots,2mn, i \neq j$ So the set of $R_1, R_2, \ldots, R_{2mn}$ is an orthogonal basis of the linear space Ω of dimension $2mn$ where $\Omega = \{U \,|U = diag(K_1,K_2) where\, K_1,K_2 \in \mathbb{C}^{m\times n}\}$ Which implies that $R_{2mn+1} = 0$ i.e. V_{2mn+1}, W_{2mn+1} is the solution of the system of matrix equations (2).

To prove the convergence property of Algorithm II, we first establish the following basic properties.

Lemma 3. *Suppose that the system of matrix equations (2) is consistent and let V^*, W^* be its Hermitian reflexive solutions. Then for any initial Hermitian reflexive matrix pair $[V_1, W_1]$ with $V_1 \in H\,\mathbb{C}_r^{s\times s}(P)$ and $W_1 \in H\,\mathbb{C}_r^{q\times q}(Q)$, we have*

$$\langle S_i, V^* - V_i\rangle + \langle T_i, W^* - W_i\rangle = \|R_i\|^2, \tag{18}$$

where the sequences $\{V_i\}, \{S_i\}, \{W_i\}$, $\{T_i\}$ and $\{R_i\}$are generated by Algorithm II for $i = 1, 2, \ldots$

The proof of Lemma 3 is similar to Lemma 1.

Lemma 4. *Suppose that system of matrix equations (2) is consistent and the sequences $\{S_i\}$, $\{T_i\}$ and $\{R_i\}$ are generated by Algorithm II , such that $R_i \neq 0$ for all $i = 1, 2, \ldots$, then*

$$\langle R_i, R_j\rangle = 0, \tag{19}$$

and

$$\langle S_i, S_j\rangle + \langle T_i, T_j\rangle = 0, for i, j = 1, 2, \ldots, k, i \neq j. \tag{20}$$

The proof of Lemma 4 is similar to Lemma 2.

Theorem 3. *[13] If the system of matrix equations (2) is consistent, then for any initial Hermitian reflexive matrix pair $[V_1, W_1]$ with $V_1 \in H\,\mathbb{C}_r^{s\times s}(P)$ and $W_1 \in H\,\mathbb{C}_r^{q\times q}(Q)$ a Hermitian reflexive solution pair can be obtained with a finite number of iteration steps by using Algorithm II.*

The proof of Theorem 3 is similar to Theorem 2.

4 Numerical Examples

In this section, we report two numerical examples to illustrate the application of our proposed iterative methods.

Example 1. In this example, we illustrate our theoretical results of Algorithm I for solving the system of matrix equations (2) where,

$$A_1 = \begin{bmatrix} -i & 0 & 2-i \\ 1 & -1 & -3i \\ 2+i & 3+i & -i \\ -3i & 1+2i & 4+i \end{bmatrix}, A_2 = \begin{bmatrix} -3i & 0 & 4 \\ 1+2i & 2+3i & 1+i \\ 0 & 3-i & 2i \\ -1+3i & 2i & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 4 & -3i & 0 \\ -1-i & -1 & 4+i \\ 2 & 2+i & 3+i \\ 2i & 0 & -3i \end{bmatrix}, A_4 = \begin{bmatrix} 1+2i & 4 & 3-i \\ 1 & i & 1+i \\ 3-i & 0 & 1 \\ i & 2i & -2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} i & -3i & 1+i \\ 1-i & 0 & 2 \\ 1+i & 3 & 0 \\ 2 & 3i & 3i \end{bmatrix}, C_2 = \begin{bmatrix} -1+i & 1 & 2 \\ 2+i & 0 & -2i \\ -3i & -1-i & 0 \\ 0 & 1 & i \end{bmatrix}, C_3 = \begin{bmatrix} 2i & 4 & -1-i \\ 1 & 2-i & 3+i \\ 2 & 1 & -3i \\ -i & -1-i & -2i \end{bmatrix}, C_4 = \begin{bmatrix} 2i & 4 & 5i \\ -3i & -1 & -2 \\ 1+i & -3i & 0 \\ i & 2-i & 3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1+3i & 1+2i & 1+i & -i \\ 2-i & -2 & 2-2i & -3i \\ 1+i & 1 & 3+i & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1-3i & 2i & -3i & 0 \\ 1 & 0 & 2+3i & 4i \\ 1-2i & 2 & 1+i & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 3-i & 1+i & 0 & -1 \\ 4+i & -i & 4i & 0 \\ 0 & 0 & 2+2i & -i \end{bmatrix}, B_4 = \begin{bmatrix} 0 & i & 1+2i & 1+3i \\ -3i & -1-i & -2i & -1-i \\ 2 & 2i & 0 & 0 \end{bmatrix},$$

$$D_1=\begin{bmatrix}-1+i & 2+i & 0 & -1+2i\\ 1-2i & i & -3+i & 0\\ 3i & 1+3i & 3i & 2i\end{bmatrix}, D_2=\begin{bmatrix}1 & 2i & 0 & -1+2i\\ -2+i & 0 & -1+3i & -i\\ 2 & i & 0 & 1\end{bmatrix}, D_3=\begin{bmatrix}i & 3+i & 1+2i & 2-i\\ 0 & 1+i & 3i & 2+i\\ 4+2i & 3-i & 1+2i & 1+i\end{bmatrix},$$

$$D_4=\begin{bmatrix}-3 & -1-i & 0 & 1+4i\\ 3+2i & 1+4i & 0 & 2\\ 3i & 1+2i & 1 & 2i\end{bmatrix}, P=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}, Q=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$E_1=\begin{bmatrix}63-26i & -16+5i & 15-25i & -9-9i\\ 25-4i & -6-18i & 1-2i & 1-7i\\ 10+26i & 8+8i & -30+18i & -6+24i\\ 30+34i & 44-20i & 65+7i & -20+14i\end{bmatrix}, E_2=\begin{bmatrix}36-87i & 16-41i & -28+10i & -112+38i\\ 14+17i & -3-9i & 13-4i & 23+28i\\ -33-77i & -17-59i & 38+12i & -51+30i\\ -24-40i & -5-26i & -2-17i & -32-i\end{bmatrix}.$$

When the initial matrices are chosen as $V_1=W_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. We apply Algorithm I to compute V_k, W_k. After iterating 22 steps we obtain

$$V_{22}=\begin{bmatrix}1+i & 0 & 2i\\ 0 & i & 0\\ 1-i & 0 & 1+i\end{bmatrix}, W_{22}=\begin{bmatrix}1+2i & -1+2i & 0\\ 2i & -1-2i & 0\\ 0 & 0 & 1+i\end{bmatrix},$$

which satisfy the system of matrix equations (2). Moreover, it can be verified that $PVP=V$ and $QWQ=W$. With the corresponding residual

$$R_{k+1}=diag(E_1-f(V_{k+1},W_{k+1}),E_2-g(V_{k+1},W_{k+1})),$$

$$\|R_{22}\|=\|diag(E_1-f(V_{22},W_{22}),E_2-g(V_{22},W_{22}))\|=6.0203\times10^{-12}.$$

The obtained results are presented in Figure 1, where $r_k=\|R_k\|$(Residual) $\delta_k=\frac{\|[V_k,W_k]-[V,W]\|}{\|[V,W]\|}$ (Relative error).

From Figure 1, it is clear that the error δ_k is becoming smaller and approaches zero as iteration number k increases. This indicates that the proposed algorithm is effective and convergent.

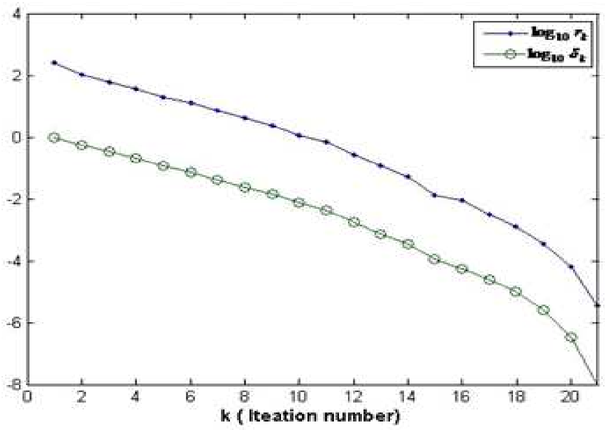

Figure 1. The residual and the relative error versus k (iteration number)

Example 2. In this example, we illustrate our theoretical results of Algorithm II for solving the matrix equation (2) where,

$$A_1=\begin{bmatrix}3+2i & 0 & i & 2i\\ 0 & -1+3i & 2 & 1\\ 1+i & 3i & -i & 0\end{bmatrix}, A_2=\begin{bmatrix}1+3i & 2+2i & 1+2i & i\\ -3i & -1-i & 0 & 1\\ 2i & 0 & 5 & 3\end{bmatrix}, A_3=\begin{bmatrix}-i & 0 & 1 & 2+3i\\ 1+4i & 3+i & 3 & 0\\ i & i & 1+2i & -1-i\end{bmatrix},$$

$$A_4=\begin{bmatrix}0 & 3 & 1+2i & 2-i\\ i & 1-2i & -1+2i & 3-i\\ 0 & 1+i & 2-i & 0\end{bmatrix}, C_1=\begin{bmatrix}1 & -2 & 2i\\ 3i & 1-2i & 0\\ 0 & -2+i & i\end{bmatrix}, C_2=\begin{bmatrix}3i & 1 & -2i\\ -2 & 2i & 0\\ 0 & 1-2i & -1\end{bmatrix},$$

$$C_3=\begin{bmatrix}0 & -2 & 0\\ -1 & 0 & 3i\\ -2i & 1+2i & -2+i\end{bmatrix}, C_4=\begin{bmatrix}-1+i & 1+2i & 1\\ 1-2i & 0 & 1-2i\\ i & 3i & 1+i\end{bmatrix}, B_1=\begin{bmatrix}-3i & 3 & 0\\ 1 & 2i & i\\ 2i & 0 & 4-i\\ 1 & 3+i & -4\end{bmatrix},$$

$$B_2=\begin{bmatrix}2+i & i & 1+i\\ i & 2 & 0\\ 3 & -i & 2+i\\ i & -1 & 0\end{bmatrix}, B_3=\begin{bmatrix}1 & 2+i & 2i\\ 1-2i & 0 & -1\\ 3i & 1 & i\\ 0 & -1 & -i\end{bmatrix}, B_4=\begin{bmatrix}-1 & 1-i & 3\\ 2i & -2 & -i\\ 3i & i & 0\\ 1 & -2+2i & 0\end{bmatrix},$$

$$D_1=\begin{bmatrix}i & i & 2-i\\ -1+2i & 3 & i\\ -2 & 0 & -3i\end{bmatrix}, D_2=\begin{bmatrix}1-2i & i & 2\\ 1 & 3i & 3+i\\ 3i & 0 & 0\end{bmatrix}, D_3=\begin{bmatrix}-3+2i & 0 & -2+i\\ 1 & 0 & -1-i\\ 2i & -1+3i & 0\end{bmatrix}, D_4=\begin{bmatrix}i & 0 & i\\ 3 & -1 & 0\\ 2i & 4i & 2\end{bmatrix}, Q=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

$$E_1=\begin{bmatrix}-85-23i & 19-9i & 39-34i\\ -29-62i & 10-6i & -4+10i\\ -50+33i & -38-68i & -12+48i\end{bmatrix}, E_2=\begin{bmatrix}2+44i & -65+22i & -40-7i\\ 40-32i & -5+38i & -1-71i\\ 42-78i & -40-4i & 33\end{bmatrix}, P=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix},$$

When the initial matrices are chosen as $V_1=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ and $W_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. We apply Algorithm II to compute V_k, W_k.

After iterating 15 steps we obtain

$$V_{15}=\begin{bmatrix}1 & -3 & 0 & 0\\ -3 & 3 & 0 & 0\\ 0 & 0 & -3 & 4\\ 0 & 0 & 4 & 2\end{bmatrix}, W_{15}=\begin{bmatrix}3 & 0 & 5\\ 0 & -2 & 0\\ 5 & 0 & 4\end{bmatrix}$$

which satisfy the system of matrix equations (2). Moreover, it can be verified that $PVP=V=V^H$ and $QWQ=W=W^H$ With the corresponding residual

$$R_{k+1}=diag(E_1-f(V_{k+1},W_{k+1}),E_2-g(V_{k+1},W_{k+1})),$$

$$\|R_{15}\|=\|diag(E_1-f(V_{15},W_{15}),E_2-g(V_{15},W_{15}))\|=8.0247\times10^{-13}.$$

The obtained results are presented in Figure 2, where $r_k=\|R_k\|$(Residual), $\delta_k=\frac{\|[V_k,W_k]-[V,W]\|}{\|[V,W]\|}$ (Relative error).

From Figure 2, it is clear that the error δ_k is becoming smaller and approaches zero as iteration number k increases. This indicates that the proposed algorithm is effective and convergent.

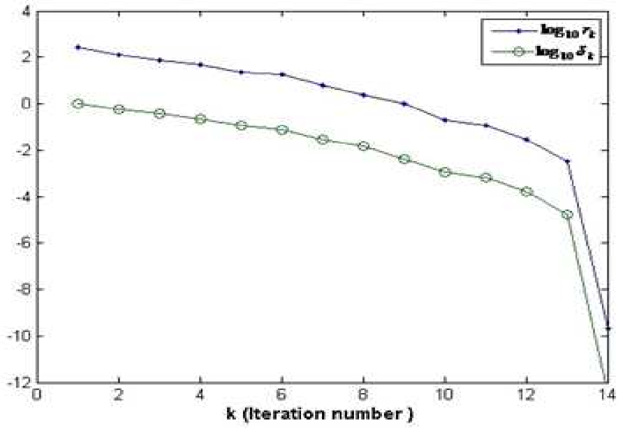

Figure 2. The residual and the relative error versus k (iteration number)

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5 Conclusion

Two finite iterative algorithms for finding reflexive and Hermitian reflexive solution to coupled complex of conjugate and transpose matrix equations (2) are presented. We proved that the iterative algorithms always converge to the solution for any initial reflexive and Hermitian reflexive matrices. We stated and proved some lemmas and theorems where the solutions are obtained. The obtained results show that the methods are very neat and efficient. The proposed methods are illustrated by two numerical examples.

References

[1] G.X. Huang, F. Yin, K. Guo, An iterative method for the skew-symmetric solution and the optimal approximate solution of the matrix equation, Journal of Computational and Applied Mathematics, 212(2) (2008), pp.231-244.

[2] X. Sheng, G. Chen, A finite iterative method for solving a pair of linear matrix equations, Applied Mathematics and Computation,189(2) (2007), pp. 1350-1358.

[3] A.G. Wu, B. Li, Y. Zhang, G.R. Duan, Finite iterative solutions to coupled Sylvester-conjugate matrix equations, Applied Mathematical Modelling, 35 (3) (2011), pp. 1065-1080.

[4] M. A. Ramadan, T. S. El-Danaf and A. M. E Bayoumi, Two iterative algorithms for the reflexive and Hermitian reflexive solutions of the generalized Sylvester matrix equation, Journal of Vibration and Control, 21 (3) (2013), pp.483-492.

[5] F. P. A. Beik, A modified iterative algorithm for the (Hermitian) reflexive solution of the generalized Sylvester matrix equation, Transactions of the Institute of Measurement and Control, 36(6) (2014), pp. 815-827.

[6] M. Dehghan, M. Hajarian, An iterative algorithm for the reflexive solutions of the generalized coupled Sylvester matrix equations and its optimal approximation, Applied Mathematics and Computation, 202 (2008), pp. 571-588.

[7] M. Dehghan, M. Hajarian, The reflexive and Hermitian reflexive solutions of the generalized Sylvester-conjugate matrix equation, Bulletin of the Belgian Mathematical Society - Simon Stevin, 20 (4) (2013), pp. 639-653.

[8] X. Y. Peng, X.Y. Hu, L. Zhang, The reflexive and anti-reflexive solutions of the matrix equation , Journal of Computational and Applied Mathematics, 200(2) (2007), pp. 749-760.

[9] A. G. Wu, G. Feng, G.R. Duan and W. J. Wu, Finite iterative solutions to a class of complex matrix equations with conjugate and transpose of the unknowns, Mathematical and Computer Modelling, 52 (9-10) (2010), pp. 1463-1478.
[10] M. Dehghan, M. Hajarian, The reflexive and anti-reflexive solutions of a linear matrix equation and systems of matrix equations, Rocky Mountain Journal of Mathematics, 40(3) (2010), pp. 825-848.
[11] M. Dehghan, M. Hajarian, Two algorithms for the Hermitian reflexive and skew-Hermitian solutions of Sylvester matrix equations, Applied Mathematics Letters, 24 (4) (2011), pp. 444-449.
[12] X. Zhang, matrix analysis and application, Beijing: Tsinghua University Press, 2004.
[13] A. G. Wu, L. Lv, M. Z. Hou, Finite iterative algorithms for extended Sylvester-conjugate matrix equation, Mathematical and Computer Modelling, 54 (2011), pp. 2363-2384.