Contra $P_p$-continuous functions

Shadya M. Mershkhan
Department of Mathematics, Faculty of Science, University of Zakho, Iraq

Received: 30 October 2018, Accepted: 29 May 2019
Published online: 30 June 2019.

Abstract: In this paper, we apply the notion of $P_p$-open sets in topological spaces to present and study a new class of functions called contra $P_p$-continuous functions which lies between classes of contra $\theta$-continuous functions and contra-precontinuous functions. It is shown that contra $P_p$-continuous is weaker than contra $\theta$-continuous, but it is stronger than contra-precontinuous and weakly $P_p$-continuous. Furthermore, we obtain basic properties and preservation theorems of contra $P_p$-continuity.

Keywords: $P_p$-open, preopen, contra-continuous; contra $P_p$-continuous, Contra-precontinuous.

1 Introduction

In 1996, Dontchev [3] introduced and investigated a new notion of continuity called contra-continuity. Following this, many authors introduced many types of new generalizations of contra-continuity called as contra $\theta$-continuous [2], perfectly continuous [14] and contra-precontinuous [6]. Long and Herrington [9] have introduced a new class of functions called strongly $\theta$-continuous function. Noiri and Popa [15] have introduced and studied quasi $\theta$-continuous function. In this direction, we will introduce and investigate the concept of contra $P_p$-continuous function via the notion of $P_p$-open set and study some properties of contra $P_p$-continuous.

2 Preliminaries

Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ stand for topological spaces with no separation axioms assumed unless otherwise stated. For a subset $A$ of $X$, the closure of $A$ and the interior of $A$ will be denoted by $Cl(A)$ and $Int(A)$, respectively.

**Definition 1.** A subset $A$ of a space $X$ is said to be

1. *preopen* [10] if $A \subset Int(Cl(A))$.
2. *$\alpha$-open* [13] if $A \subset Int(Cl(Int(A)))$.

The complement of a preopen (resp., $\alpha$-open and regular open) set is preclosed (resp., $\alpha$-closed and regular closed). The family of all preopen of $X$ is denoted by $PO(X)$. In 1968, Velicko [20] defined the concept of $\theta$-open set in $X$ which is denoted by $\theta O(X)$. A subset $A$ of a space $X$ is called $\theta$-open set if for each $x \in A$, there exists an open set $G$ such that $x \in G \subset Cl(G) \subset A$. The complement of $\theta$-open set is said to be $\theta$-closed set.

**Definition 2.** [8] A subset $A$ of a space $X$ is called $P_p$-open, if for each $x \in A \in PO(X)$, there exists a preclosed set $F$ such that $x \in F \subseteq A$. The complement of a $P_p$-open set is $P_p$-closed. The family of all $P_p$-open subsets of a topological space $(X, \tau)$ is denoted by $P_pO(X, \tau)$ or $P_pO(X)$. The intersection of all $P_p$-closed sets of $X$ containing $A$ is called the $P_p$-closure.
of A and is denoted by $P_p\text{Cl}(A)$. The union of all $P_p$-open sets of $X$ contained in $A$ is called the $P_p$-interior of $A$ and is denoted by $P_p\text{Int}(A)$.

**Definition 3.** A function $f : X \rightarrow Y$ is called

1. contra-continuous [3] if $f^{-1}(V)$ is closed in $X$ for each open set $V$ of $Y$.
2. contra-$\vartheta$-continuous [2] if $f^{-1}(V)$ is $\vartheta$-closed in $X$ for each open set $V$ of $Y$.
3. contra-precontinuous [6] if $f^{-1}(V)$ is preclosed in $X$ for each open set $V$ of $Y$.
4. perfectly continuous [14] if $f^{-1}(V)$ is clopen in $X$ for each open set $V$ of $Y$.
5. strongly $\vartheta$-continuous [9] if $f^{-1}(V)$ is $\vartheta$-open in $X$ for each open set $V$ of $Y$.
6. quasi $\vartheta$-continuous [15] at a point $x \in X$ if for each $\vartheta$-open $V$ of $Y$ containing $f(x)$, there exists a $\vartheta$-open $U$ of $X$ containing $x$ such that $f(U) \subseteq \text{Cl}(V)$.
7. weakly $P_p$-continuous [11] at a point $x \in X$ if for each open set $V$ of $Y$ containing $f(x)$, there exists a $P_p$-open $U$ of $X$ containing $x$ such that $f(U) \subseteq V$.

**Theorem 1.** [11] Let $f : X \rightarrow Y$ be a function. If the inverse image of each regular open set of $Y$ is $P_p$-closed in $X$, then $f$ is weakly $P_p$-continuous.

**Definition 4.** [7] A subset $A$ of a space $X$ is called preclopen, if $A$ is both preopen and preclosed.

**Definition 5.** [12] Let $A \subseteq X$. The set $\{ U \in \tau : A \subseteq U \}$ is called the kernel of $A$ and is denoted by $\text{ker}(A)$.

**Lemma 1.** [5] The following properties hold for subsets $A$ and $B$ of a space $X$:

1. $x \in \text{ker}(A)$ if and only if $A \cap F \neq \emptyset$ for any closed subset $F$ of $X$ containing $x$.
2. $A \subseteq \text{ker}(A)$ and $A = \text{ker}(A)$ if $A$ is open in $X$.
3. If $A \subseteq B$, then $\text{ker}(A) \subseteq \text{ker}(B)$.

**Proposition 1.** [8] For any subset $A$ of a space $(X, \tau)$. The following statements are equivalent:

1. $A$ is clopen.
2. $A$ is $P_p$-open and closed.
3. $A$ is preopen and closed.

**Definition 6.** A space $X$ is said to be:

1. Locally indiscrete [4] if every open subset of $X$ is closed.
2. Pre-$R_0$ [11] if $U$ is a preopen and $x \in U$, then $\text{PCI}([x]) \subseteq U$.
3. Pre-$T_1$ [7] if for each pair of distinct points $x, y$ of $X$, there exist two preopen sets one containing $x$ but not $y$ and the other containing $y$ but not $x$.
4. $P_p$-$T_1$ [11] if for each pair of distinct points $x, y$ of $X$, there exist two disjoint $P_p$-open sets $U$ and $V$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.
5. $P_p$-$T_2$ [11] if for each pair of distinct points $x, y$ of $X$, there exist two disjoint $P_p$-open sets $U$ and $V$ containing $x$ and $y$ respectively.

**Definition 7.** [16] A space $X$ is said to be pre-regular if for each preclosed $F$ and each point $x \notin F$, there exist disjoint preopen sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

**Proposition 2.** The following statements are true:

1. If a space $X$ is pre-$T_1$, then $PO(X) = P_pO(X)$ [8].
2. If a space $X$ is pre-$R_0$, then $PO(X) = P_pO(X)$ [11].
3. If a space $X$ is pre-regular, then $\tau \subseteq P_pO(X)$ [8].
If a space \((X, \tau)\) is locally indiscrete, then \(\tau \subseteq P_pO(X)\) [8].

If a space \((X, \tau)\) is locally indiscrete, then \(PO(X) = P_pO(X)\) [8].

**Corollary 1.** [8] Let \(A\) and \(B\) be any subsets of a space \(X\). If \(A \in P_pO(X)\) and \(B\) is both \(\alpha\)-open and preclosed subset of \(X\), then \(A \cap B \in P_pO(X)\).

**Proposition 3.** [8] Let \((Y, \tau_Y)\) be a subspace of a space \((X, \tau)\) and \(A \subset Y\). If \(A \in P_pO(Y, \tau_Y)\) and \(Y\) is preclopen, then \(A \in P_pO(X, \tau)\).

**Definition 8.** A topological space \((X, \tau)\) is said to be

1. Ultra Hausdorff [18] if for each pair of distinct points \(x, y\) of \(X\), there exist two clopen sets \(U\) and \(V\) such that \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\).
2. Ultra normal [18] if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.
3. Weakly Hausdorff [17] if each element of \(X\) is an intersection of regular closed sets.

**Proposition 4.** [8] Let \(X\) and \(Y\) be two topological spaces and \(X \times Y\) be the product topology. If \(A \in P_pO(X)\) and \(B \in P_pO(Y)\), then \(A \times B \in P_pO(X \times Y)\).

**Theorem 2.** [8] For a function \(f : X \rightarrow Y\), the following statements are equivalent:

1. \(f\) is \(P_p\)-continuous.
2. \(f^{-1}(V)\) is \(P_p\)-open set in \(X\), for each open set \(V\) of \(Y\).
3. \(f^{-1}(F)\) is \(P_p\)-closed set in \(X\), for each closed set \(F\) of \(Y\).

**Corollary 2.** [8] Every quasi \(\theta\)-continuous function is a \(P_p\)-continuous function.

### 3 Contra \(P_p\)-continuous functions

**Definition 9.** A function \(f : X \rightarrow Y\) is called contra \(P_p\)-continuous if \(f^{-1}(V)\) is \(P_p\)-closed in \(X\) for each open set \(V\) of \(Y\).

**Lemma 2.** Every contra \(\theta\)-continuous function is contra \(P_p\)-continuous and every contra \(P_p\)-continuous function is contra-precontinuous.

**Proof.** Follows directly from their definitions.

**Theorem 3.** If a function \(f : X \rightarrow Y\) is contra \(P_p\)-continuous, then \(f\) is weakly \(P_p\)-continuous.

**Proof.** Let \(V\) be any regular open of \(Y\), then \(V\) is open. Since \(f\) is contra \(P_p\)-continuous, then \(f^{-1}(V)\) is \(P_p\)-closed of \(X\). Therefore, by Theorem 1, \(f\) is weakly \(P_p\)-continuous.

By Lemma 2 and Theorem 3, the following diagram is obtained:

![Diagram 1](image)

In the sequel, we shall show that none of the implications that concerning contra \(P_p\)-continuity in Diagram 1 is reversible.
Example 31 Let \( X = \{a, b, c, d\} \) with the two topologies \( \tau = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, X\} \) and \( \sigma = \{\phi, \{b\}, \{a, d\}, \{a, b, d\}, X\} \). Let \( f : (X, \tau) \to (X, \sigma) \) be the identity function. Then \( f \) is contra \( P_p \)-continuous but not contra \( \theta \)-continuous, since \( \{b\} \in \sigma \) but \( f^{-1}(\{b\}) = \{b\} \) is not \( \theta \)-closed in \( (X, \tau) \).

Example 32 Let \( X = \{a, b, c, d\} \) with the two topologies \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\phi, \{b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\} \). Let \( f : (X, \tau) \to (X, \sigma) \) be the identity function. Then \( f \) is contra-precontinuous but not contra \( P_p \)-continuous, since \( \{b\} \in \sigma \) but \( f^{-1}(\{b\}) = \{b\} \) is not \( P_p \)-closed in \( (X, \tau) \).

Example 33 Let \( X = \{a, b, c, d\} \) with the two topologies \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\phi, \{a\}, \{a, d\}, \{a, b, d\}, X\} \). Let \( f : (X, \tau) \to (X, \sigma) \) be the identity function. Then \( f \) is weakly \( P_p \)-continuous but not contra \( P_p \)-continuous, since \( \{a, b, d\} \in \sigma \) but \( f^{-1}(\{a, b, d\}) = \{a, b, d\} \) is not \( P_p \)-closed in \( (X, \tau) \).

**Theorem 4.** For a function \( f : X \to Y \), the following statements are equivalent:

1. \( f \) is contra \( P_p \)-continuous.
2. For every closed subset \( F \) of \( Y \), \( f^{-1}(F) \in P_pO(X) \).
3. For each \( x \in X \) and each closed set \( F \) of \( Y \) containing \( f(x) \), there exists a \( P_p \)-open \( U \) of \( X \) containing \( x \) such that \( f(U) \subset F \).
4. \( f(P_pCl(A)) \subset ker(f(A)) \) for each \( A \subset X \).
5. \( P_pCl(f^{-1}(B)) \subset f^{-1}(ker(B)) \) for each \( B \subset Y \).

**Proof.** The implications (1) \( \Leftrightarrow \) (2) and (2) \( \Rightarrow \) (3) are obvious.

(3) \( \Rightarrow \) (2) Let \( F \) be any closed set of \( Y \) and \( x \in f^{-1}(F) \). Then \( f(x) \in F \) and by (3) there exists \( U \in P_pO(X) \) containing \( x \) such that \( f(U) \subset F \). Therefore, we obtain that \( f^{-1}(F) = \bigcup \{U : x \in f^{-1}(F)\} \in P_pO(X) \).

(2) \( \Rightarrow \) (4) Let \( A \) be any subset of \( X \). Suppose that \( y \notin ker(f(A)) \). Then by Lemma 1(1), there exists a closed set \( F \) of \( Y \) containing \( y \) such that \( f(A) \cap F = \phi \). Thus, we have \( A \cap f^{-1}(F) = \phi \). Therefore, we have \( f(P_pCl(A)) \cap f^{-1}(F) = \phi \) which implies that \( f(P_pCl(A)) \cap F = \phi \) and hence \( y \notin f(P_pCl(A)) \). Therefore, we obtain that \( f(P_pCl(A)) = ker(f(A)) \).

(4) \( \Rightarrow \) (5) Let \( B \) be any subset of \( Y \). By (4) and Lemma 1, we have \( f(P_pCl(f^{-1}(B))) \subset ker(f(B)) \) and \( P_pCl(f^{-1}(B)) \subset ker(B) \).

(5) \( \Rightarrow \) (1) Let \( V \) be any open set of \( Y \). By (5) and Lemma 1, we have \( P_pCl(f^{-1}(V)) \subset f^{-1}(ker(V)) = f^{-1}(V) \) and \( P_pCl(f^{-1}(V)) = f^{-1}(V) \). This shows that \( f^{-1}(V) \) is \( P_p \)-closed in \( X \). Therefore, \( f \) is contra \( P_p \)-continuous.

**Theorem 5.** A function \( f : X \to Y \) is contra \( P_p \)-continuous if and only if \( f \) is contra-precontinuous and for each \( x \in X \) and each closed set \( F \) of \( Y \) containing \( f(x) \), there exists a preclosed \( E \) in \( X \) containing \( x \) such that \( f(E) \subset F \).

**Proof.** Necessity. Let \( x \in X \) and \( U \) be any closed set of \( Y \) containing \( f(x) \). Since \( f \) is contra \( P_p \)-continuous, then by Theorem 4, there exists a \( P_p \)-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subset F \). Since \( U \) is \( P_p \)-open set. Then for each \( x \in U \), there exists a preclosed \( E \) of \( X \) such that \( x \in E \subset U \). Therefore, we have \( f(E) \subset F \). Hence, contra \( P_p \)-continuous always implies contra-precontinuous.

Sufficiency. Let \( F \) be any closed set of \( Y \). We have to show that \( f^{-1}(F) \) is \( P_p \)-open set in \( X \). Since \( f \) is contra-precontinuous, then \( f^{-1}(F) \) is preopen in \( X \). Let \( x \in f^{-1}(F) \), then \( f(x) \in F \). By hypothesis, there exists a preclosed \( E \) of \( X \) containing \( x \) such that \( f(E) \subset F \), which implies that \( x \in E \subset f^{-1}(F) \). Therefore, \( f^{-1}(F) \) is \( P_p \)-open set in \( X \). Hence, by Theorem 4, \( f \) is contra \( P_p \)-continuous.

**Theorem 6.** If a function \( f : X \to Y \) is contra \( P_p \)-continuous and \( Y \) is regular, then \( f \) is \( P_p \)-continuous.

**Proof.** Let \( x \) be any arbitrary point of \( X \) and \( V \) be an open set of \( Y \) containing \( f(x) \). Since \( Y \) is regular, there exists an open set \( G \) of \( Y \) containing \( f(x) \) such that \( Cl(G) \subset V \). Since \( f \) is contra \( P_p \)-continuous, so by Theorem 4, there exists a \( P_p \)-open \( U \) of \( X \) containing \( x \) such that \( f(U) \subset Cl(G) \). Then \( f(U) \subset Cl(G) \subset V \). Hence, \( f \) is \( P_p \)-continuous.
Corollary 3. If a function \( f : X \rightarrow Y \) is contra \( P_p \)-continuous and \( Y \) is regular, then \( f \) is quasi \( \theta \)-continuous.

Proof. Follows from Corollary 2.

Theorem 7. The following statements are equivalent for a function \( f : X \rightarrow Y \):

1. \( f \) is perfectly continuous.
2. \( f \) is contra \( P_p \)-continuous and continuous.
3. \( f \) is contra-precontinuous and continuous.

Proof. This is an immediate consequence of Proposition 1.

Corollary 4. Let \( f : X \rightarrow Y \) be a function and \( X \) be a pre-\( T_1 \) space. \( f \) is contra \( P_p \)-continuous if and only if \( f \) is contra-precontinuous.

Proof. Follows from Proposition 2(1).

Corollary 5. Let \( f : X \rightarrow Y \) be a function and \( X \) be a pre-\( R_0 \) space. \( f \) is contra \( P_p \)-continuous if and only if \( f \) is contra-precontinuous.

Proof. Follows from Proposition 2(2).

Corollary 6. Let \( f : X \rightarrow Y \) be a function and \( X \) be a pre-regular space. If \( f \) is contra-continuous, then \( f \) is contra \( P_p \)-continuous.

Proof. Follows from Proposition 2(3).

Corollary 7. Let \( f : X \rightarrow Y \) be a function and \( X \) be a locally indiscrete space. If \( f \) is contra-continuous, then \( f \) is contra \( P_p \)-continuous.

Proof. Follows from Proposition 2(4).

Corollary 8. Let \( f : X \rightarrow Y \) be a function and \( X \) be a locally indiscrete space. \( f \) is contra-precontinuous if and only if \( f \) is contra \( P_p \)-continuous.

Proof. Follows from Proposition 2(5).

Corollary 9. If \( X \) is both pre-\( T_1 \) and \( X \) locally indiscrete space, the following statements are equivalent for a function \( f : X \rightarrow Y \):

1. \( f \) is contra \( P_p \)-continuous.
2. \( f \) is contra-precontinuous.

Proof. Follows from Corollary 4 and Corollary 8.

Definition 10. A space \((X; \tau)\) is said to be \( P_p \)-space (resp., locally \( P_p \)-indiscrete) if every \( P_p \)-open set is open (resp., closed) in \( X \).

Theorem 8. If a function \( f : X \rightarrow Y \) is contra \( P_p \)-continuous and \( X \) is \( P_p \)-space, then \( f \) is contra-continuous.

Proof. Let \( F \) be a closed set in \( Y \). Since \( f \) is contra \( P_p \)-continuous, \( f^{-1}(F) \) is \( P_p \)-open in \( X \). Since \( X \) is \( P_p \)-space, \( f^{-1}(F) \) is open in \( X \). Hence \( f \) is contra-continuous.

Theorem 9. Let \( X \) be locally \( P_p \)-indiscrete. If a function \( f : X \rightarrow Y \) is contra \( P_p \)-continuous, then \( f \) is continuous.
Proof. Let $F$ be a closed set in $Y$. Since $f$ is contra $P$-continuous, $f^{-1}(F)$ is $P$-open in $X$. Since X is locally $P$-indiscrete, $f^{-1}(F)$ is closed in X. Hence $f$ is continuous.

**Theorem 10.** Let $f : X \to Y$ be a contra $P$-continuous function. If $A$ is an open and preclosed subset of $X$, then $f \mid A : A \to Y$ is contra $P$-continuous in the subspace $A$.

**Proof.** Let $F$ be any closed set of $Y$. Since $f$ is contra $P$-continuous, then by Theorem 4, $f^{-1}(F)$ is $P$-open in $X$. Since $A$ is open and preclosed subset of $X$, then by Corollary 1, $(f \mid A)^{-1} = f^{-1}(F) \cap A$ is a $P$-open subspace of $A$. Therefore, by Theorem 4, $f \mid A : A \to Y$ is contra $P$-continuous.

**Theorem 11.** A function $f : X \to Y$ is contra $P$-continuous, if for each $x \in X$, there exists a preclopen $A$ of $X$ containing $x$ such that $f \mid A : A \to Y$ is contra $P$-continuous in the subspace $A$.

**Proof.** Let $x \in X$, then by hypothesis, there exists a preclopen $A$ containing $x$ such that $f \mid A : A \to Y$ is contra $P$-continuous. Let $F$ be any closed subset of $Y$ containing $f(x)$. By Theorem 4, there exists a $P$-open $V$ in $A$ containing $x$ such that $(f \mid A)(U) \subset F$. Since $A$ is preclopen, then by Proposition 3, $U$ is $P$-open in $X$ and hence $f(U) \subset V$. Therefore, by Theorem 4, $f$ is contra $P$-continuous.

**Theorem 12.** If $X = R \cup S$, where $R$ and $S$ are preclopen sets, and $f : X \to Y$ is a function such that both $f \mid R$ and $f \mid S$ are contra $P$-continuous, then $f$ is contra $P$-continuous.

**Proof.** Let $F$ be any closed subset of $Y$. Then $f^{-1}(F) = (f \mid R)^{-1}(F) \cup (f \mid S)^{-1}(F)$. Since $f \mid R$ and $f \mid S$ are contra $P$-continuous, then by Theorem 4 $(f \mid R)^{-1}$ and $(f \mid S)^{-1}$ are $P$-open sets in $R$ and $S$, respectively. Since $R$ and $S$ are preclopen sets in $X$, then by Proposition 3 $(f \mid R)^{-1}$ and $(f \mid S)^{-1}$ are $P$-open sets in $X$. Therefore, by Proposition 3, $f$ is contra $P$-continuous.

**Theorem 13.** If $X$ is a topological space and for each pair of distinct points $x_1$ and $x_2$ in $X$, there exists a function $f$ of $X$ into Urysohn topological space $Y$ such that $f(x_1) \neq f(x_2)$ and $f$ is contra $P$-continuous at $x_1$ and $x_2$, then $X$ is a $P,T_2$ space.

**Proof.** Let $x_1$ and $x_2$ be any distinct points in $X$. By hypothesis, there is a Urysohn space $Y$ and a function $f : X \to Y$ such that $f(x_1) \neq f(x_2)$ and $f$ is contra $P$-continuous at $x_1$ and $x_2$. Let $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since $Y$ is Urysohn, there exist open sets $U_{y_1}$ and $U_{y_2}$ containing $y_1$ and $y_2$ respectively in $Y$ such that $Cl(U_{y_1}) \cap Cl(U_{y_2}) = \emptyset$. Since $f$ is contra $P$-continuous at $x_1$ and $x_2$, there exist $P$-open sets $V_{x_1}$ and $V_{x_2}$ containing $x_1$ and $x_2$ respectively in $X$ such that $f(V_{x_i}) \subset Cl(U_{y_i})$ for $i = 1, 2$. Hence, we have $V_{x_1} \cap V_{x_2} = \emptyset$. Therefore, $X$ is a $P,T_2$ space.

**Corollary 10.** If $f$ is contra $P$-continuous injection of a topological space $X$ into a Urysohn space $Y$, then $X$ is a $P,T_2$ space.

**Proof.** Let $x_1$ and $x_2$ be any distinct points in $X$. Then by hypothesis, $f$ is contra $P$-continuous of $X$ into a Urysohn space $Y$ such that $f(x_1) \neq f(x_2)$ because $f$ is injective. Hence, by Theorem 13, $X$ is a $P,T_2$ space.

**Proposition 5.** Let $f : X_1 \to Y$ and $g : X_2 \to Y$ be two contra $P$-continuous functions. If $Y$ is Urysohn, then the set $E = \{(x_1, x_2) : x_1 \times x_2 \in X_1 \times X_2 : f(x_1) = g(x_2)\}$ is $P$-closed in the product space $X_1 \times X_2$.

**Proof.** In order to show that $E$ is $P$-closed, we show that $(X_1 \times X_2) \setminus E$ is $P$-open. Let $(x_1, x_2) \notin E$. Then $f(x_1) \neq g(x_2)$. Since $Y$ is Urysohn, there exist open sets $U_1$ and $U_2$ of $Y$ containing $f(x_1)$ and $g(x_2)$ respectively, such that $Cl(U_1) \cap Cl(U_2) = \emptyset$. Since $f$ and $g$ are contra $P$-continuous, $f^{-1}(Cl(U_1))$ and $g^{-1}(Cl(U_2))$ are $P$-open sets containing $x_1$ and $x_2$ in $X_i (i = 1, 2)$. Hence, by Proposition 4, $f^{-1}(Cl(U_1)) \times g^{-1}(Cl(U_2))$ is $P$-open. Further $(x_1, x_2) \in f^{-1}(Cl(U_1)) \times g^{-1}(Cl(U_2)) \subset (X_1 \times X_2) \setminus E$. It follows that $(X_1 \times X_2) \setminus E$ is $P$-open. Thus, $E$ is $P$-closed in the product space $X_1 \times X_2$. 

© 2019 BISKA Bibliotek Technology
Corollary 11. If \( f : X \rightarrow Y \) is a contra \( P_\sigma \)-continuous function and \( Y \) is a Urysohn space, then \( E = \{(x_1, x_2) \mid f(x_1) = f(x_2)\} \) is \( P_\sigma \)-closed in the product space \( X \times X \).

Corollary 12. Let \( f : X_1 \rightarrow Y \) and \( g : X_2 \rightarrow Y \) be two contra \( P_\sigma \)-continuous functions. If \( Y \) is ultra Hausdorff, then the set \( E = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1) = g(x_2)\} \) is \( P_\sigma \)-closed in the product space \( X_1 \times X_2 \).

Corollary 13. If \( f : X \rightarrow Y \) is a contra \( P_\sigma \)-continuous function and \( Y \) is ultra Hausdorff, then \( E = \{(x_1, x_2) \mid f(x_1) = f(x_2)\} \) is \( P_\sigma \)-closed in \( X \).

Theorem 14. Let \( f : X \rightarrow Y \) be a contra \( P_\sigma \)-continuous injection function. If \( Y \) is an ultra Hausdorff space, then \( X \) is a \( P_\sigma ; T_2 \) space.

Proof. Let \( x_1 \) and \( x_2 \) be any distinct points in \( X \), then \( f(x_1) \neq f(x_2) \) and there exist clopen sets \( U \) and \( V \) containing \( f(x_1) \) and \( f(x_2) \) respectively, such that \( U \cap V = \emptyset \). Since \( f \) is contra \( P_\sigma \)-continuous, then \( f^{-1}(U) \in P_\sigma O(X) \) and \( f^{-1}(V) \in P_\sigma O(X) \) such that \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \). Hence, \( X \) is a \( P_\sigma ; T_2 \) space.

Proposition 6. If \( f_i : X_i \rightarrow Y_i \) is a contra \( P_\sigma \)-continuous function for each \( i = 1, 2 \). Let \( f : X_1 \times Y_2 \rightarrow X_1 \times Y_2 \) be a function defined as follows: \( f(x_1, y_2) = (f_1(x_1), f_2(y_2)) \). Then \( f \) is contra \( P_\sigma \)-continuous.

Proof. Let \( R_1 \times R_2 \subset Y_1 \times Y_2 \), where \( R_i \) is open set in \( Y_i \) for each \( i = 1, 2 \). Then \( f^{-1}(R_1 \times R_2) = f_1^{-1}(R_1) \times f_2^{-1}(R_2) \). Since \( f_i \) is contra \( P_\sigma \)-continuous for \( i = 1, 2 \), then by Proposition 4, \( f^{-1}(R_1 \times R_2) \) is \( P_\sigma \)-closed in \( X_1 \times X_2 \).

Definition 11. The graph \( G(f) \) of a function \( f : X \rightarrow Y \) is contra \( P_\sigma \)-closed in \( X \times Y \) if for each \( (x, y) \in (X \times Y) \setminus G(f) \), there exists \( U \in P_\sigma (X, x) \) and \( V \) \( P_\sigma \)-closed in \( Y \) containing \( y \) such that \( (U \times V) \cap G(f) = \emptyset \).

Lemma 3. The graph \( G(f) \) of a function \( f : X \rightarrow Y \) is contra \( P_\sigma \)-closed in \( X \times Y \) if and only if for each \( (x, y) \in (X \times Y) \setminus G(f) \), there exists an \( U \in P_\sigma O(X) \) containing \( x \) and \( V \in C(Y) \) containing \( y \) such that \( f(U) \cap V = \emptyset \).

Theorem 15. If \( f : X \rightarrow Y \) is a contra \( P_\sigma \)-continuous function and \( Y \) is Urysohn, then \( G(f) \) is contra \( P_\sigma \)-closed in \( X \times Y \).

Proof. Let \( (x, y) \in (X \times Y) \setminus G(f) \). It follows that \( f(x) \neq y \). Since \( Y \) is Urysohn, there exist open sets \( V \) and \( W \) such that \( f(x) \in V, y \in W \) and \( C(V) \cap C(W) = \emptyset \). Since \( f \) is contra \( P_\sigma \)-continuous, there exists \( U \in P_\sigma O(X, x) \) such that \( f(U) \subset C(V) \) and \( f(U) \cap C(W) = \emptyset \). Hence, \( G(f) \) is contra \( P_\sigma \)-closed in \( X \times Y \).

Theorem 16. If \( f : X \rightarrow Y \) is a \( P_\sigma \)-continuous function and \( Y \) is \( T_1 \), then \( G(f) \) is contra \( P_\sigma \)-closed in \( X \times Y \).

Proof. Let \( (x, y) \in (X \times Y) \setminus G(f) \). Then \( y \neq f(x) \) and there exists an open set \( V \) of \( Y \), such that \( f(x) \in V \) and \( y \notin V \). Since \( f \) is \( P_\sigma \)-continuous, there exists \( U \in P_\sigma O(X, x) \) such that \( f(U) \subset V \). Therefore, \( f(U) \cap V = \emptyset \) and \( Y \setminus V \in C(Y, y) \). This shows that \( G(f) \) is contra \( P_\sigma \)-closed in \( X \times Y \).

Theorem 17. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function and \( g : X \rightarrow X \times Y \) be a graph function of \( f \) defined by \( g(x) = (x, f(x)) \) for every \( x \in X \). If \( g \) is contra \( P_\sigma \)-continuous, then \( f \) is contra \( P_\sigma \)-continuous.

Proof. Let \( V \) be an open set in \( Y \). Then \( X \times V \) is an open set in \( X \times Y \). Since \( g \) is contra \( P_\sigma \)-continuous, \( g^{-1}(X \times V) \) is \( P_\sigma \)-closed in \( X \). Also \( g^{-1}(X \times V) = f^{-1}(V) \) which is \( P_\sigma \)-closed in \( X \). Hence, \( f \) is contra \( P_\sigma \)-continuous.

Theorem 18. Let \( f : X \rightarrow Y \) has a contra \( P_\sigma \)-closed graph. If \( f \) is injective, then \( X \) is \( P_\sigma ; T_1 \).

Proof. Let \( x_1 \) and \( x_2 \) be any two distinct points of \( X \). Then, we have \( (x_1, f(x_2)) \in (X \times Y) \setminus G(f) \). Then, there exist \( P_\sigma \)-open \( U \) in \( X \) containing \( x_1 \) and \( F \in C(Y, f(x_2)) \) such that \( f(U) \cap F = \emptyset \). Hence, \( U \cap f^{-1}(F) = \emptyset \). Therefore, we have \( x_2 \notin U \). This implies that \( X \) is \( P_\sigma ; T_1 \).

Definition 12. A topological space \( X \) is said to be \( P_\sigma \)-normal if each pair of disjoint closed sets can be separated by disjoint \( P_\sigma \)-open sets.
Theorem 19. If a function \( f : X \to Y \) is contra \( P_p \)-continuous, closed injection and \( Y \) is ultra normal, then \( X \) is \( P_p \)-normal.

Proof. Let \( F_1 \) and \( F_2 \) be disjoint closed subsets of \( X \). Since \( f \) is closed injective, \( f(F_1) \) and \( f(F_2) \) are disjoint closed subsets of \( Y \). Since \( Y \) is ultra normal, \( f(F_1) \) and \( f(F_2) \) are separated by disjoint clopen sets \( V_1 \) and \( V_2 \), respectively. Hence, \( F_i \subset f^{-1}(V_i), f^{-1}(V_1) \in P_p(X) \) for \( i = 1, 2 \) and \( f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset \). Thus \( X \) is \( P_p \)-normal.

Theorem 20. If a function \( f : X \to Y \) is contra \( P_p \)-continuous injection and \( Y \) is weakly Hausdorff, then \( X \) is \( P_p \)-T1.

Proof. Suppose that \( Y \) weakly Hausdorff. For any distinct points \( x_1 \) and \( x_2 \) in \( X \), there exist regular closed sets \( U \) and \( V \) in \( Y \) such that \( f(x_1) \in U, f(x_2) \notin U, f(x_1) \notin V \) and \( f(x_2) \in V \). Since \( f \) is contra \( P_p \)-continuous, \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( P_p \)-open subsets of \( X \) such that \( x_1 \in f^{-1}(U), x_2 \notin f^{-1}(U), x_1 \notin f^{-1}(V) \) and \( x_2 \in f^{-1}(V) \). This shows that \( X \) is \( P_p \)-T1.

Theorem 21. Let \( f : X \to Y \) be a contra \( P_p \)-continuous surjective function and \( A \) is \( \alpha \)-open and preclosed subset of \( X \). If \( f \) is a closed function, then the function \( g : A \to f(A) \), which is defined by \( g(x) = f(x) \) for each \( x \in A \), is contra \( P_p \)-continuous.

Proof. Putting \( H = f(A) \). Let \( x \in A \) and \( F \) be any closed set in \( H \) containing \( g(x) \). Since \( H \) is closed in \( Y \) and \( F \) is closed in \( H \), then \( F \) is closed in \( Y \). Since \( f \) is contra \( P_p \)-continuous, then by Theorem 4, there exists a \( P_p \)-open \( U \) in \( X \) containing \( x \) such that \( f(U) \subset F \). Taking \( W = U \cap A \), since \( A \) is \( \alpha \)-open and preclosed subset of \( X \). Then by Corollary 1, \( W \) is \( P_p \)-open in \( A \) containing \( x \) and \( g(W) \subset F \cap H = F_H \). Then \( g(W) \subset F_H \). Therefore, by Theorem 4, \( g \) is contra \( P_p \)-continuous.

We shall obtain some conditions for the composition of two functions to be contra \( P_p \)-continuous.

Theorem 22. Let \( f : X \to Y \) and \( g : Y \to Z \) be functions. Then the composition function \( g \circ f : X \to Z \) is contra \( P_p \)-continuous if \( f \) and \( g \) satisfy one of the following conditions:

1. \( f \) is contra \( P_p \)-continuous and \( g \) is continuous.
2. \( f \) is \( P_p \)-continuous and \( g \) is contra-continuous.
3. \( f \) is contra \( P_p \)-continuous and \( g \) is a strongly \( \theta \)-continuous.
4. \( f \) is contra \( P_p \)-continuous and \( g \) is a quasi \( \theta \)-continuous.

Proof.

1. Let \( W \) be any open subset of \( Z \). Since \( g \) is continuous \( g^{-1}(W) \) is an open subset of \( Y \). Since \( f \) is contra \( P_p \)-continuous, then \( (g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \) is a \( P_p \)-closed subset in \( X \). Therefore, \( g \circ f \) is contra \( P_p \)-continuous.
2. Let \( W \) be any open subset of \( Z \). Since \( g \) is contra-continuous, then \( g^{-1}(W) \) is a closed subset of \( Y \). Since \( f \) is \( P_p \)-continuous, then by Theorem 2, \( (g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \) is a \( P_p \)-closed subset in \( X \). Therefore, \( g \circ f \) is contra \( P_p \)-continuous.
3. Let \( W \) be any open subset of \( Z \). In view of strongly \( \theta \)-continuity of \( g \), \( g^{-1}(W) \) is a \( \theta \)-open subset of \( Y \). Again, since \( f \) is contra \( P_p \)-continuous, \( (g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \) is a \( P_p \)-closed subset in \( X \). Therefore, \( g \circ f \) is contra \( P_p \)-continuous.
4. Obvious

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.
References