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On non-Newtonian measure for $\alpha-$closed sets

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Abstract: In this paper, we define the non-Newtonian measure for $\alpha-$closed sets and study on its some basic properties.

Keywords: Non-Newtonian measure, non-Newtonian series, geometric calculus.

1 Introduction

Non-Newtonian calculus was created by Katz and Grossman as an alternative to classic calculus between 1967-1970 [1]. The first arithmetic calculus is defined as geometric, harmonic and quadratic calculus. Grossman also studied some properties of derivatives and integrals in non-Newtonian calculus [2]. Bashirov et. al. have recently studied some basic properties of derivatives and integrals in multiplicative calculus and gave the results with applications [3]. Later, Duyar, Sağır and Oğur gave some basic topological properties of non- Newtonian calculus [4]. Recently, Duyar and Sağır [5] introduced the concepts of the non-Newtonian measure for open sets. For more details see [7], [8], [9], [10].

Let α be a generator, α is a one-to-one function whose domain is real numbers and whose range is a subset A of $\mathbb{R}$. We know that each generator produces exatly one arithmetic and conversely, each arithmetic is produced by one generator. For instance, the identity function I generates the classic arithmetic and the exponential function exp generates geometric arithmetic. Let take a generator α such that have the following basic algebraic operations $[1-5]$:

$\alpha-$addition	$x\dot{+}y=\alpha\left\{\alpha^{-1}(x)+\alpha^{-1}(y)\right\}$
$\alpha-$subtraction	$x\dot{-}y=\alpha\left\{\alpha^{-1}(x)-\alpha^{-1}(y)\right\}$
$\alpha-$multiplicative	$x\dot{\times}y=\alpha\left\{\alpha^{-1}(x)\times\alpha^{-1}(y)\right\}$
$\alpha-$division	$x\dot{/}y=\alpha\left\{\alpha^{-1}(x)/\alpha^{-1}(y)\right\}$
$\alpha-$order	$x\dot{\leq}y\Leftrightarrow\alpha^{-1}(x)\leq\alpha^{-1}(y)$

for every $x,y\in A$.

The set of non-Newtonian numbers is defined as $\mathbb{R}(N)=\{\alpha(x):x\in\mathbb{R}\}$. A $\alpha-$ closed interval on $\mathbb{R}(N)$ can be represented by

$$[a,b]_N=\left\{x\in\mathbb{R}(N):a\dot{\leq}x\dot{\leq}b\right\}=\left\{x\in\mathbb{R}(N):\alpha^{-1}(a)\leq\alpha^{-1}(x)\leq\alpha^{-1}(b)\right\}=\alpha\left\{\left[\alpha^{-1}(a),\alpha^{-1}(b)\right]\right\}.$$

Definition 1. *Let F and S be two point sets. If $F\subset S$, then the set $S-F$ is called to complement of the set F with respect to the set S and denoted by the symbol C_S^F.*

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Theorem 1. *Let F be a non-void bounded $\alpha-$closed set and let S be the smallest $\alpha-$closed interval containing the set F. Then the set C_S^F $\alpha-$open [5].*

Definition 2. *The measure $m_N(a,b)_N$ in $\mathbb{R}(N)$ is defined by*

$$m_N(a,b)_N = \alpha\left\{m\left(\alpha^{-1}(a),\alpha^{-1}(b)\right)\right\}$$

[5].

Definition 3. *The measure $m_N G$ of a non-void bounded open set G in $\mathbb{R}(N)$ is the sum of the measures of all its component intervals δ_k :*

$$m_N G = {}_N\sum_k m_N \delta_k.$$

Here it should be noted that

$$m_N G = {}_N\sum_k m_N(a_k,b_k)_N = {}_N\sum_k b_k \dot{-} a_k$$

where $\delta_k = (a_k,b_k)_N$ [5].

Theorem 2. *Let G_1 and G_2 be two bounded open set in $\mathbb{R}(N)$. If $G_1 \subset G_2$, then*

$$m_N G_1 \dot{\leq} m_N G_2$$

[5].

In this paper, we define and study on non-Newtonian measure of bounded closed sets as a generalization of known results in real analysis.

2 Main results

Definition 4. *In $\mathbb{R}(N)$, the measure of a non-void bounded $\alpha-$closed set F is defined as follows*

$$m_N F = \alpha\left\{m\left(\alpha^{-1}(A),\alpha^{-1}(B)\right) - m\left(\alpha^{-1}\left(C_S^F\right)\right)\right\}$$

where $S = [A,B]_N$ is the smallest $\alpha-$closed interval containing the set F.

We can restate the above relation as follows; since C_S^F is a $\alpha-$open set, it can be written in the form $C_S^F = \cup_k (a_k,b_k)_N$. Thus, we get

$$\begin{aligned}
m_N F &= \alpha\left\{m\left(\alpha^{-1}(A),\alpha^{-1}(B)\right) - m\left(\alpha^{-1}\left(C_S^F\right)\right)\right\} \\
&= \alpha\left\{m\left(\alpha^{-1}(A),\alpha^{-1}(B)\right) - m\left(\alpha^{-1}\left(\cup_k (a_k,b_k)_N\right)\right)\right\} \\
&= \alpha\left\{m\left(\alpha^{-1}(A),\alpha^{-1}(B)\right) - m\left(\cup_k\left(\alpha^{-1}(a_k),\alpha^{-1}(b_k)\right)\right)\right\} \\
&= \alpha\left\{m\left(\alpha^{-1}(A),\alpha^{-1}(B)\right) - \sum_k\left(\alpha^{-1}(b_k) - \alpha^{-1}(a_k)\right)\right\} \\
&= \alpha\left\{\alpha^{-1}(B) - \alpha^{-1}(A) - \sum_k\left(\alpha^{-1}(b_k) - \alpha^{-1}(a_k)\right)\right\} \\
&= \alpha\alpha^{-1}\left(\alpha\left(\alpha^{-1}(B) - \alpha^{-1}(A)\right)\right) - \alpha^{-1}\left(\alpha\left(\sum_k \alpha^{-1}\left(\alpha\left(\alpha^{-1}(b_k) - \alpha^{-1}(a_k)\right)\right)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha \left\{ \alpha^{-1} \left(B \dot{-} A \right) - \alpha^{-1} \left({}_N \sum_k \alpha \left(\alpha^{-1} (b_k) - \alpha^{-1} (a_k) \right) \right) \right\} \\
&= \alpha \left\{ \alpha^{-1} \left(B \dot{-} A \right) - \alpha^{-1} \left({}_N \sum_k m_N (a_k, b_k)_N \right) \right\} \\
&= \alpha \left\{ \alpha^{-1} \left(B \dot{-} A \right) - \alpha^{-1} \left(m_N C_S^F \right) \right\} \\
&= B \dot{-} A \dot{-} m_N \left(C_S^F \right)
\end{aligned}
$$

Remark. If $F = [a,b]_N$, then $S = [a,b]_N$ and $C_S^F = \varnothing$, so that $m_N F = b \dot{-} a$. If F is the union of a finite number of pairwise disjoint closed intervals in $\mathbb{R}(N)$, namely $F = [a_1,b_1]_N \cup [a_2,b_2]_N \cup \ldots \cup [a_n,b_n]_N$, then $m_N F = {}_N \sum_{k=1}^{n} b_k \dot{-} a_k$.

Example 1. Let take geometric calculus and let $F = [a_1,b_1]_N \cup [a_2,b_2]_N$. Then, we have $S = [a_1,b_2]_N$ and $C_S^F = (b_1,a_2)_N$. Thus, the measure of $\alpha-$closed set is

$$
m_N F = \exp\{(\ln b_2 - \ln a_1) - (\ln a_2 - \ln b_1)\} = \exp\left\{ \ln \frac{b_2 b_1}{a_1 a_2} \right\} = \frac{b_1 b_2}{a_1 a_2}.
$$

Theorem 3. *The non-Newtonian measure of a bounded $\alpha-$closed set F is non-negative.*

Proof. Let F be a bounded $\alpha-$closed set and let $S = [a,b]_N$ be the smallest $\alpha-$closed interval containing the set F.Then

$$
m_N F = \alpha \left\{ m \left(\alpha^{-1}(a), \alpha^{-1}(b) \right) - m \left(\alpha^{-1} \left(C_S^F \right) \right) \right\} = \alpha \left\{ \alpha^{-1}(b) - \alpha^{-1}(a) - m \left(\alpha^{-1} \left(C_S^F \right) \right) \right\} \dot{\geqslant} \alpha \left(\dot{0} \right).
$$

Lemma 1. *Let F be a bounded $\alpha-$closed set and let Δ be an $\alpha-$open interval containing F. Then $m_N F = m_N \Delta \dot{-} m_N \left(C_\Delta^F \right)$.*

Proof. Let $\Delta = (A,B)_N$ and let $S = [a,b]_N$ be the smallest $\alpha-$closed interval containing the set F. Then, we have

$$
\begin{aligned}
m_N F &= \alpha \left\{ m \left(\alpha^{-1}(a), \alpha^{-1}(b) \right) - m \left(\alpha^{-1} \left(C_S^F \right) \right) \right\} \\
&= \alpha \left\{ \alpha^{-1}(b) - \alpha^{-1}(a) - m \left(\alpha^{-1} \left(C_S^F \right) \right) \right\} \\
&= \alpha \left\{ \alpha^{-1}(B) - \alpha^{-1}(A) - m \left(\alpha^{-1} \left(C_\Delta^F \right) \right) \right\} \\
&= \alpha \left\{ \alpha^{-1}(B) - \alpha^{-1}(A) - \alpha^{-1} \left(\alpha \left(m \left(\alpha^{-1} \left(C_\Delta^F \right) \right) \right) \right) \right\} \\
&= \alpha \left\{ \alpha^{-1}(B) - \alpha^{-1}(A) - \alpha^{-1} \left(m_N \left(C_\Delta^F \right) \right) \right\} \\
&= B \dot{-} A \dot{-} m_N \left(C_\Delta^F \right) \\
&= m_N \Delta \dot{-} m_N \left(C_\Delta^F \right).
\end{aligned}
$$

Theorem 4. *Let F_1 and F_2 be two non-void bounded $\alpha-$closed sets in $\mathbb{R}(N)$. If $F_1 \subset F_2$, then*

$m_N F_1 \dot{\leq} m_N F_2$.

Proof. Let $S = (a,b)_N$ be an $\alpha-$open interval containing the set F_2. We can easily see that

$$
\begin{aligned}
m_N F_1 &= \alpha \left\{ m \left(\alpha^{-1}(a), \alpha^{-1}(b) \right) - m \left(\alpha^{-1} \left(C_S^{F_1} \right) \right) \right\} \\
&= \alpha \left\{ \alpha^{-1}(b) - \alpha^{-1}(a) - m \left(\alpha^{-1} \left(C_S^{F_1} \right) \right) \right\} \\
&\dot{\leq} \alpha \left\{ \alpha^{-1}(b) - \alpha^{-1}(a) - m \left(\alpha^{-1} \left(C_S^{F_2} \right) \right) \right\} \\
&= \alpha \left\{ m \left(\alpha^{-1}(a), \alpha^{-1}(b) \right) - m \left(\alpha^{-1} \left(C_S^{F_2} \right) \right) \right\} \\
&= m_N F_2.
\end{aligned}
$$

Theorem 5. *Let F be an $\alpha-$closed set and let G be a bounded $\alpha-$open set in $\mathbb{R}(N)$. If $F \subset G$, then $m_N F \dot{\leq} m_N G$.*

Proof. Let $S = (a,b)_N$ be an $\alpha-$open interval containing the set $G = \cup_k (a_k, b_k)_N$. We can easily see that

$$\begin{aligned} m_N F &= \alpha \left\{ m\left(\alpha^{-1}(a), \alpha^{-1}(b)\right) - m\left(\alpha^{-1}\left(C_S^F\right)\right) \right\} \\ &= \alpha \left\{ \alpha^{-1}(b) - \alpha^{-1}(a) - m\left(\alpha^{-1}\left(C_S^F\right)\right) \right\} \\ &\dot{\leq} \alpha \left\{ \sum_k \left(\alpha^{-1}(b_k) - \alpha^{-1}(a_k)\right) \right\} \\ &=_N \sum_k \alpha\left(\alpha^{-1}(b_k) - \alpha^{-1}(a_k)\right) \\ &=_N \sum_k b_k \dot{-} a_k \\ &= m_N G. \end{aligned}$$

Theorem 6. *The non-Newtonian measure of a bounded $\alpha-$open set G is the least upper bound of the mesure of all $\alpha-$closed sets contained in G.*

Proof. By the preceding theorem, $m_N G$ is an upper bound for the measures of $\alpha-$closed sets $F \subset G$. Let $G = \bigcup_k (\lambda_k, \mu_k)_N$. Since $m_N G = {}_N\sum_k \mu_k \dot{-} \lambda_k$, we have

$$\begin{aligned} \alpha^{-1}(m_N G) &= \alpha^{-1}\left({}_N\sum_k \mu_k \dot{-} \lambda_k \right) \\ &= \alpha^{-1}\left(\alpha\left(\sum_k \alpha^{-1}\left(\mu_k \dot{-} \lambda_k\right) \right) \right) \\ &= \sum_k \alpha^{-1}\left(\mu_k \dot{-} \lambda_k\right) \\ &= \sum_k \alpha^{-1}\left(\alpha\left(\alpha^{-1}(\mu_k) - \alpha^{-1}(\lambda_k)\right)\right) \\ &= \sum_k \alpha^{-1}(\mu_k) - \alpha^{-1}(\lambda_k). \end{aligned}$$

Take an arbitrary $\varepsilon \dot{>} \dot{0}$ and find a natural number n so large that

$$\sum_{k=1}^{n} \alpha^{-1}(\mu_k) - \alpha^{-1}(\lambda_k) > \alpha^{-1}(m_N G) - \frac{\alpha^{-1}(\varepsilon)}{2}.$$

Therefore, we have

$$\alpha^{-1}\left(\alpha\left(\sum_{k=1}^{n} \alpha^{-1}(\mu_k) - \alpha^{-1}(\lambda_k) \right) \right) > \alpha^{-1}\left(\alpha\left(\alpha^{-1}(m_N G) - \frac{\alpha^{-1}(\varepsilon)}{2} \right) \right)$$

and so

$$\alpha\left(\sum_{k=1}^{n} \alpha^{-1}(\mu_k) - \alpha^{-1}(\lambda_k) \right) \dot{>} \alpha\left(\alpha^{-1}(m_N G) - \frac{\alpha^{-1}(\varepsilon)}{2} \right)$$

which gives

$${}_N\sum_{k=1}^{n} = \mu_k \dot{-} \lambda_k \dot{>} m_N G \dot{-} \frac{\varepsilon}{\dot{2}}.$$

For every k $(k = 1,2,...,n)$, we choose a $\alpha-$closed inerval $[a_k, b_k]_N$ so that $[a_k, b_k]_N \subset (\lambda_k, \mu_k)_N$. Thus, we get

$$\alpha\left[\alpha^{-1}(a_k), \alpha^{-1}(b_k)\right] \subset \alpha\left(\alpha^{-1}(\lambda_k), \alpha^{-1}(\mu_k)\right)$$

and so

$$\left[\alpha^{-1}(a_k), \alpha^{-1}(b_k)\right] \subset \left(\alpha^{-1}(\lambda_k), \alpha^{-1}(\mu_k)\right).$$

Therefore, we have

$$\alpha^{-1}(b_k) - \alpha^{-1}(a_k) > \alpha^{-1}(\mu_k) - \alpha^{-1}(\lambda_k) - \frac{\alpha^{-1}(\varepsilon)}{2n}$$

and so

$$\alpha^{-1}\left(\alpha\left(\alpha^{-1}(b_k) - \alpha^{-1}(a_k)\right)\right) > \alpha^{-1}\left(\alpha\left(\alpha^{-1}(\mu_k) - \alpha^{-1}(\lambda_k) - \frac{\alpha^{-1}(\varepsilon)}{2n}\right)\right).$$

Then, we get, by inequlity above

$$\alpha\left(\alpha^{-1}(b_k) - \alpha^{-1}(a_k)\right) \dot{>} \alpha\left(\alpha^{-1}(\mu_k) - \alpha^{-1}(\lambda_k) - \frac{\alpha^{-1}(\varepsilon)}{2n}\right)$$

which means

$$m_N [a_k, b_k]_N \dot{>} m_N (\lambda_k, \mu_k)_N \dot{-} \frac{\varepsilon}{\alpha(2n)}.$$

Let define $F_0 = \bigcup_{k=1}^{n} [a_k, b_k]_N$. It is clear that $F_0 \subset G$ and F_0 is $\alpha-$closed set.

Thus, we have

$$m_N F_0 =_N \sum_{k=1}^{n} b_k \dot{-} a_k = \alpha\left\{\sum_{k=1}^{n} \alpha^{-1}\left(b_k \dot{-} a_k\right)\right\} = \alpha\left\{\sum_{k=1}^{n} \alpha^{-1}(b_k) - \alpha^{-1}(a_k)\right\}.$$

Thus, we obtain

$$\begin{aligned}
\alpha\left\{\sum_{k=1}^{n} \alpha^{-1}(b_k) - \alpha^{-1}(a_k)\right\} &> \alpha\left(\sum_{k=1}^{n}\left(\alpha^{-1}(\mu_k) - \alpha^{-1}(\lambda_k) - \frac{\alpha^{-1}(\varepsilon)}{2n}\right)\right) \\
&= \alpha\left\{\sum_{k=1}^{n}\left(\alpha^{-1}(\mu_k) - \alpha^{-1}(\lambda_k)\right) - \frac{\alpha^{-1}(\varepsilon)}{2n}\right\} \\
&=_N \sum_{k=1}^{n} \mu_k \dot{-} \lambda_k \dot{-} \frac{\varepsilon}{2} > m_N G \dot{-} \frac{\varepsilon}{2} \dot{-} \frac{\varepsilon}{2} = m_N G \dot{-} \varepsilon.
\end{aligned}$$

Theorem 7. *Let F be a bounded $\alpha-$closed set. Then, the non-Newtonian measure of F is the greatest lower bound of the measure of all possible $\alpha-$open sets containing F.*

Proof. Let Δ be an $\alpha-$open interval containing the set F. Then, we have

$$m_N F = m_N \Delta \dot{-} m_N (C_\Delta^F).$$

By Theorem 5, we can find an $\alpha-$closed set Φ such that $\Phi \subset C_\Delta^F$. By Theorem 6, we have

$$m_N \Phi \dot{>} m_N C_\Delta^F \dot{-} \varepsilon$$

for every $\varepsilon \dot{>} \dot{0}$. Let define $G_0 = C_\Delta^\Phi$. It is clear that G_0 is an $\alpha-$open set containing F. Also, we have

$$m_N G_0 = m_N C_\Delta^\Phi = m_N \Delta \dot{-} m_N \Phi \dot{<} m_N \Delta \dot{-} m_N C_\Delta^F \dot{+} \varepsilon.$$

Thus, we get

$$m_N G_0 \dot{<} m_N F \dot{+} \varepsilon.$$

Theorem 8. *Let the bounded $\alpha-$closed set F be the union of a finite number of pairwise disjoint $\alpha-$closed sets, i.e. $F = \bigcup_{k=1}^{n} F_k$, where $F_k \cap F_l = \varnothing$ for $k \neq l$. Then*

$$m_N F =_N \sum_{k=1}^{n} m_N F_k.$$

Proof. Since F is $\alpha-$closed set, we have $\alpha^{-1}(F) = \bigcup_{k=1}^{n} \alpha^{-1}(F_k)$ is closed set. Then, by the properties of Lebesgue measure of bounded closed set in real numbers, we have

$$m\left(\alpha^{-1}(F)\right) = m\left(\bigcup_{k=1}^{n} \alpha^{-1}(F_k)\right) = \sum_{k=1}^{n} m\left(\alpha^{-1}(F_k)\right).$$

Thus, we get

$$m_N F = \alpha\left(\sum_{k=1}^{n} m\left(\alpha^{-1}(F_k)\right)\right) = \alpha\left(\sum_{k=1}^{n} \alpha^{-1}\left(\alpha\left(m\left(\alpha^{-1}(F_k)\right)\right)\right)\right) =_N \sum_{k=1}^{n} m_N F_k.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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