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Product of composition and differentiation operators on a space of entire functions

Pawan Kumar[1] *and Mohd Arief*[2]

[1] Department of Mathematics, Govt. Degree College Kathua, India
[2] Department of Mathematics, Central University of Jammu, India

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Abstract: The product of composition operator C_φ and differentiation operator D is written as $C_\varphi D$ and DC_φ which are defined as $C_\varphi Df = f'o\varphi$ and $DC_\varphi f = (fo\varphi)'$ respectively. In this paper, we characterize the continuity of the operators $C_\varphi D$ and DC_φ on $\mathscr{E}$, the space of entire functions.

Keywords: Composition operator, differentiation operator, entire functions.

1 Introduction

Let X be a non-empty set and $V(X)$ be a vector space of complex valued functions on X. If $\varphi : X \to X$ is a mapping such that $fo\varphi \in V(X)$ for all $f \in V(X)$, then the composition transformation $C_\varphi : V(X) \to V(X)$ is defined as

$$C_\varphi f = fo\varphi \ \ \forall \ \ f \in V(X)$$

If $V(X)$ is a topological vector space and C_φ is continuous on $V(X)$, then we call C_φ as composition operator induced by φ. Further, let $\psi : X \to \mathbb{C}$ be a function, then the multiplication transformation $M_\psi : V(X) \to V(X)$ defined as

$$M_\psi f = \psi.f \ \ \forall \ \ f \in V(X)$$

If $V(X)$ is a topological vector space and M_ψ is continuous on $V(X)$, then M_ψ is called the multiplication operator induced by ψ. Let D be the differentiation operator defined on $V(X)$ as $Df = f'$. The generalized composition operators $C_\varphi D$ and DC_φ on $V(X)$ are defined as $C_\varphi Df = f'o\varphi$ and $DC_\varphi f = (fo\varphi)'$ for all $f \in V(X)$ respectively. A complex valued function $f : \mathbb{C} \to \mathbb{C}$ is called entire function if it is analytic in the whole complex plane $\mathbb{C}$. If f is an entire function, then the power series representation of f can be written as

$$f(z) = \sum_{n=0}^{\infty} \hat{f}_n z^n \tag{1}$$

where $\{\hat{f}_n\}$ a sequence of complex numbers such that $\lim_{n\to\infty} |\hat{f}_n|^{\frac{1}{n}} = 0$. Conversely on every sequence $\{\hat{f}_n\}$ of complex numbers such that $\lim_{n\to\infty} |\hat{f}_n|^{\frac{1}{n}} = 0$, there is an entire function f represented by (1.1). A metric d on the class of entire functions is defined by $d(f,g) = sup\{|\hat{f}_0 - \hat{g}_0|, |\hat{f}_n - \hat{g}_n|^{\frac{1}{n}} : n \geq 1\}$. The class of entire functions topologized by this metric is denoted by $\mathscr{E}$. It has been shown in Iyer [8] that $\mathscr{E}$ is a non-normable complex metrizable locally convex

* Corresponding author e-mail: pawan811@yahoo.in; ariefcuj15@gmail.com

topological vector space. In the space $\mathscr{E}$ of entire functions, the convergence of a sequence of entire functions is equivalent to the uniform convergence of entire functions in any circle of finite radius and this convergence is called the strong convergence in $\mathscr{E}$.

The continuous linear functional F on $\mathscr{E}$ is given by $F(f) = \sum_{n=0}^{\infty} f_n a_n$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\{f_n\}$ be a sequence of complex numbers such that $\{|\hat{f}_n|^{\frac{1}{n}}\}$ be a bounded sequence. The set of all bounded linear continuous functional on $\mathscr{E}$ is denoted by $\mathscr{E}^{\star}$. For each $n \in \mathbb{N}$, we define $e_n : \mathbb{C} \to \mathbb{C}$ as $e_n(z) = z^n \;\; \forall \;\; z \in \mathbb{C}$. Then the sequence $\{e_n : n \in \mathbb{N}\}$ is called a basis for $\mathscr{E}$. A sequence $\{\alpha_n\}$ in $\mathscr{E}$ is called a basis for $\mathscr{E}$ if for each $\alpha \in \mathscr{E}$, there exists a unique sequence $\{f_n(\alpha)\}$ of complex nos such that $\alpha = \sum_{n=0}^{\infty} f_n(\alpha).\alpha_n$. For $R > 0$, we denote by $\mathbb{D}_R$, the open unit disc in $\mathbb{C}$ defined as $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < 1\}$. The space $\mathscr{E}$ of entire functions has been studied extensively by Iyer [8,9,10] and [11].

In this paper, we initiated the study of generalized composition operators on the space $\mathscr{E}$ of entire functions. Much of the work on composition operators is done on Hardy space. For more about composition operator on Hardy space, we refer to Schwartz [19] and Shapiro [20] .

This paper is organised as follows. In the first section, we give introduction of the work done here. We study the boundedness of the operator $C_\varphi D$ in the second section and in the third section, we study the boundedness of the operator DC_φ on the space $\mathscr{E}$.

2 Boundedness of the operator $C_\varphi D$

In this section, we shall characterize the boundedness of generalized composition operator $C_\varphi D$ on space $\mathscr{E}$ of entire functions. For this purpose, we need the following Lemma.

Lemma 1. *Let $f \in \mathscr{E}$. Then for each $z \in \mathbb{D}_R$*

$$|f'(z)| \leq \frac{M(R,f).R}{(R-|z|)^2}$$

Proof. By the Cauchy integral formula for derivative, we have

$$f'(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{(w-z)^2}dw \quad \text{where} \;\; C_R : |z| = R.$$

This implies that

$$|f'(z)| \leq \frac{1}{2\pi}\int_{C_R} \frac{|f(w)|}{|w-z|^2}|dw| \leq \frac{M(R,f)}{2\pi}\frac{1}{(R-|z|)^2}\int_{C_R}|dw| = \frac{M(R,f)}{2\pi}\frac{1}{(R-|z|)^2}2\pi R = \frac{M(R,f).R}{(R-|z|)^2}$$

Therefore

$$|f'(z)| \leq \frac{M(R,f).R}{(R-|z|)^2}, \quad \forall \;\; z \in \mathbb{D}_R.$$

Theorem 1. *Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a mapping and $D : \mathscr{E} \to \mathscr{E}$ be the differentiation operator. Then the generalized composition operator $C_\varphi D : \mathscr{E} \to \mathscr{E}$ is continuous (bounded) iff φ is an entire function.*

Proof. Assume that the operator $C_\varphi D : \mathscr{E} \to \mathscr{E}$ is continuous. Then $C_\varphi Df = f' o\varphi$ is an entire function. In particular for $f = \frac{e_2}{2} \in \mathscr{E}$, where $e_2(z) = z^2$, we have $f' o\varphi = e_1 o\varphi = \varphi$ is an entire function.

Conversely, assume that φ is an entire function. In order to prove that $C_\varphi D$ is a continuous operator, it is sufficient to show that $C_\varphi D$ is continuous at origin.

Let $R > 0$ be given, then $\overline{\mathbb{D}}_R$ is a compact subset of $\mathbb{C}$, but φ is a continuous map, therefore $\varphi(\overline{\mathbb{D}}_R)$ is compact subset of $\mathbb{C}$ and so we can find $K > M(R,\varphi)$ such that $\varphi(\overline{\mathbb{D}}_R) \subset \mathbb{D}_K$. Now, convergence in $\mathscr{E}$ is equivalent to the uniform convergence in any circle of finite radius. Let $\{f_n\}$ be a sequence in $\mathscr{E}$ s.t $f_n \to 0$. Then for each $\varepsilon > 0$, there exists $n_0 \in N$ such that $M(K, f_n) < \varepsilon.\frac{K_0^2}{K}$ for $n \geq n_0$, where $K_0 = K - M(K,\varphi)$.

From Lemma 1, we have

$$|f_n'(\varphi(z))| \leq \frac{M(K, f_n)K}{(K-|\varphi(z)|)^2} \leq \frac{M(K, f_n)K}{(K-M(K,\varphi))^2} < \varepsilon, \ \ \forall \ z \in \mathbb{D}_R, \ \ n \geq n_0.$$

Therefore

$$C_\varphi D f_n = f_n' o\varphi \to 0 \ \ as \ \ n \to \infty$$

This proves that the operator $C_\varphi D$ is continuous.

Theorem 2. *Let $T \in C(\mathscr{E})$. Then T is a generalized composition operator of the type $C_\varphi D$ for some entire function $\varphi : \mathbb{C} \to \mathbb{C}$ iff*

$$Te_n = n\left[T\frac{e_2}{2}\right]^{n-1}.$$

Proof. Suppose, there exists an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $T = C_\varphi D$. Now

$$Te_n = C_\varphi De_n = e_n' o\varphi = n\varphi^{n-1} = n\left[e_1 o\varphi\right]^{n-1} = n\left[C_\varphi D\frac{e_2}{2}\right]^{n-1} = n\left[T(\frac{e_2}{2})\right]^{n-1}, \ \ \forall \ n \in \mathbb{N}$$

Conversely, assume that $Te_n = n[T(\frac{e_2}{2})]^{n-1}$.

Setting $T(\frac{e_2}{2}) = \varphi$, then φ is an entire function and so $C_\varphi D$ is a generalized composition operator. Now

$$\begin{aligned} Tf &= T\left[\sum_{n=0}^{\infty} \hat{f}_n e_n\right] = \sum_{n=0}^{\infty} \hat{f}_n Te_n = \sum_{n=0}^{\infty} \hat{f}_n.n\left[T(\frac{e_2}{2})\right]^{n-1} = \sum_{n=0}^{\infty} \hat{f}_n.n\varphi^{n-1} = \sum_{n=0}^{\infty} \hat{f}_n.e_n' o\varphi = \sum_{n=0}^{\infty} \hat{f}_n.C_\varphi De_n \\ &= C_\varphi D\left[\sum_{n=0}^{\infty} \hat{f}_n e_n\right] = C_\varphi Df, \ \ \forall \ f \in \mathscr{E} \end{aligned}$$

Therefore, $T = C_\varphi D$ and so T is a generalized composition operator.

Theorem 3. *Let $T \in C(\mathscr{E})$. Then T is generalized composition operator of the type $C_\varphi D$ iff $T^\star A \subset B$, where $A = \{E_z : z \in \mathbb{C}\}$ and $B = \{E_z oD : z \in \mathbb{C}\}$*

Proof. Firstly, suppose that T be a generalized composition operator. Then $\exists$ an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $T = C_\varphi D$.
Now, for $z \in \mathbb{C}$, $E_z \in \mathscr{E}^\star$, we have

$$\begin{aligned} &(T^\star E_z)f = E_z(Tf) = E_z(C_\varphi Df) = E_z(f' o\varphi) = (f' o\varphi)(z) = f'(\varphi(z)) = E_{\varphi(z)} f' = (E_{\varphi(z)} oD)f, \ \textit{for all} \ f \in \mathscr{E} \ \textit{and} \ z \in \mathbb{C}. \\ &\Rightarrow T^\star E_z f = (E_{\varphi(z)} oD)f = (E_w oD)f, \ \ \textit{where} \ \varphi : \mathbb{C} \to \mathbb{C} \ \textit{is defined by} \ \varphi(z) = w. \end{aligned}$$

Hence $T^\star E_z = E_w oD$, for some $w \in \mathbb{C}$.

$$\therefore T^\star A \subset B.$$

Conversely, suppose that $T^{\star}A \subset B$, where $A = \{E_z : z \in \mathbb{C}\}$ and $B = \{E_z oD : z \in \mathbb{C}\}$. Now for $f \in \mathscr{E}$ and $z \in \mathbb{C}$, we have

$$(Tf)(z) = E_z(Tf) = (T^{\star}E_z)f = (E_w oD)f \quad for\ some\ w \in \mathbb{C}$$

Now define $\varphi : \mathbb{C} \to \mathbb{C}$ as $\varphi(z) = w$. Then

$$(Tf)(z)E_{\varphi(z)}f' = f'(\varphi(z)) = (C_{\varphi}Df)(z)$$

This implies that $T = C_{\varphi}D$. Hence T is a generalized composition operator.

Theorem 4. *Let $T = C_{\varphi}D \in C(\mathscr{E})$. Then $T^{\star} : \mathscr{E}^{\star} \to \mathscr{E}^{\star}$ is a generalized composition operator if $\varphi(z) = \alpha z$.*

Proof. Let $F \in \mathscr{E}^{\star}$, $f \in \mathscr{E}$ and $\varphi(z) = \alpha z$. Define $\psi : \mathbb{C} \to \mathbb{C}$ by $\psi(z) = \alpha z$. Then

$$F(z) = \sum_{n=0}^{\infty} F_n z^n, \qquad f(z) = \sum_{n=0}^{\infty} \hat{f}_n z^n.$$

Therefore

$$F'(z) = \sum_{n=1}^{\infty} nF_n z^{n-1}, \quad f'(z) = \sum_{n=1}^{\infty} n\hat{f}_n z^{n-1}.$$

Now

$$(f'o\varphi)(z) = \sum_{n=0}^{\infty} (\widehat{f'o\varphi})(n).z^n = \sum_{n=1}^{\infty} (\widehat{f'o\varphi})(n-1)z^{n-1} \tag{2}$$

and

$$(f'o\varphi)(z) = f'(\varphi(z)) = \sum_{n=1}^{\infty} n\hat{f}_n(\varphi(z))^{n-1} = \sum_{n=1}^{\infty} n\hat{f}_n \alpha^{n-1} z^{n-1}. \tag{3}$$

From (1) and (2), we get

$$(\widehat{f'o\varphi})(n-1) = n\hat{f}_n \alpha^{n-1} = nz^{n-1}\alpha^{n-1} \quad where \quad \hat{f}_n = z^{n-1}.$$

Also

$$F'(\psi(z)) = \sum_{n=1}^{\infty} nF_n(\psi(z))^{n-1} = \sum_{n=1}^{\infty} nF_n \alpha^{n-1} z^{n-1}.$$

Now

$$\begin{aligned}(C_{\varphi}D^{\star}F)(f) &= F[C_{\varphi}Df] = F(f'o\varphi) = \sum_{n=0}^{\infty} F_n(f'o\varphi)(n) \\ &= F_0(f'o\varphi)(0) + \sum_{n=1}^{\infty} F_n(f'o\varphi)(n-1) \quad [\because \ F_0(f'o\varphi)(0) = 0] \\ &= \sum_{n=1}^{\infty} nF_n.\hat{f}_n \alpha^{n-1} = F'(\psi(f)) = (C_{\psi}DF)(f).\end{aligned}$$

Therefore $C_{\varphi}D^{\star} = C_{\psi}D$ for some entire function ψ.

3 Boundedness of the operator DC_φ.

In this section, we characterize the boundedness of the generalized composition operators DC_φ on the space $\mathscr{E}$ of entire functions.

Theorem 5. *Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a mapping such that φ' is bounded and $D : \mathscr{E} \to \mathscr{E}$ be the differentiation operator. Then the generalized composition operator $DC_\varphi : \mathscr{E} \to \mathscr{E}$ is continuous iff φ' is constant.*

Proof. Suppose that the operator $DC_\varphi : \mathscr{E} \to \mathscr{E}$ is continuous. Then $DC_\varphi f = (fo\varphi)'$ is entire for all $f \in \mathscr{E}$.

In particular for $f = z \in \mathscr{E}$, we have $DC_\varphi f = f'(\varphi).\varphi' = 1.\varphi' = \varphi'$ is an entire function. Therefore φ' being a bounded entire function must be constant.

Conversely, suppose that φ' is constant. Then φ is differentiable and hence continuous. To prove that DC_φ is continuous in $\mathscr{E}$, it is enough to prove that DC_φ is continuous at origin. Let $R > 0$ be given, then $\overline{\mathbb{D}}_R$ is a compact subset of $\mathbb{C}$, but φ is a continuous map, therefore $\varphi(\overline{\mathbb{D}}_R)$ is compact subset of $\mathbb{C}$ and so we can find $K > M(R, \varphi)$ such that $\varphi(\overline{\mathbb{D}}_R) \subset \mathbb{D}_K$. Now, convergence in $\mathscr{E}$ is equivalent to the uniform convergence in any circle of finite radius.

Let $\{f_n\}$ be a sequence in $\mathscr{E}$ s.t $f_n \to 0$. Then for each $\varepsilon > 0$, there exists $n_0 \in N$ such that $M(K, f_n) < \varepsilon.\frac{K_0^2}{K|\varphi'(z)|}$, where $K_0 = K - M(K, \varphi)$ for $n \geq n_0$.

From Lemma (1), we have

$$|f_n'(\varphi(z)).\varphi'(z)| = |f_n'(\varphi(z))|.|\varphi'(z)| \leq \frac{KM(K, f_n).|\varphi'(z)|}{(K - |\varphi(z)|)^2} < \varepsilon, \quad \forall\ z \in \mathbb{D}_R,\ n \geq n_0.$$

Hence $DC_\varphi f_n = (f_n o\varphi)' \to 0$ *as* $n \to \infty$.

Theorem 6. *Let $T \in C(\mathscr{E})$. Then T be a generalized composition operator of type DC_φ iff*

$$Te_n = Te_1^n \ \ for \ \ n = 0, 1, 2, 3...$$

Proof. Let T be a generalized composition operator of the type DC_φ. Then $\exists$ an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $T = DC_\varphi$. Now

$$Te_n = DC_\varphi e_n = (e_n o\varphi)' = [\varphi^n]' = [(e_1 o\varphi)^n]' = [e_1^n o\varphi]' = DC_\varphi e_1^n = Te_1^n \ \ for \ \ n = 0, 1, 2, 3, ...$$

Conversely, suppose that $Te_n = Te_1^n$. Then set $Te_1^n = (\varphi^n)'$. Clearly φ is an entire function. Now

$$\begin{aligned} Tf =& T\Big[\sum_{n=0}^{\infty} \hat{f}_n e_n\Big] = \sum_{n=0}^{\infty} \hat{f}_n Te_n = \sum_{n=0}^{\infty} \hat{f}_n Te_1^n = \sum_{n=0}^{\infty} \hat{f}_n (\varphi^n)' = \sum_{n=0}^{\infty} \hat{f}_n (e_n o\varphi)' = \sum_{n=0}^{\infty} \hat{f}_n DC_\varphi e_n \\ =& DC_\varphi\Big[\sum_{n=0}^{\infty} \hat{f}_n e_n\Big] = (DC_\varphi) f, \quad for \ \ every \ \ f \in \mathscr{E}. \end{aligned}$$

Therefore $T = DC_\varphi$ and so T be generalized composition operator.

Theorem 7. *Let $T \in C(\mathscr{E})$. Then T be a generalized composition operator of the type DC_φ iff $T^\star A \subset B$, where $A = \{E_z : z \in \mathbb{C}\}$ and $B = \{E_w DC_\varphi : w \in \mathbb{C}$ and φ an entire function$\}$*

Proof. First suppose that $T \in C(\mathscr{E})$ be a generalized composition operator. Then $\exists$ an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ such that $T = DC_\varphi$. Now

$$(T^\star E_z)f = E_z(Tf) = E_z(DC_\varphi f) = E_z(fo\varphi)' = (E_zD)(fo\varphi) = (E_zD)(C_\varphi f) = (E_zDC_\varphi)f.$$

Thus $T^\star A \subset B$.

Conversely, suppose that $T^\star A \subset B$. Now for $f \in \mathscr{E}$ and $z \in \mathbb{C}$, we have

$$(Tf)(z) = E_z(Tf) = T^\star(E_z f) = (T^\star E_z)(f) = (E_w DC_{\varphi_1})(f),$$

where $w \in \mathbb{C}$ *and* φ_1 an entire function. Now define $\varphi_2 : \mathbb{C} \to \mathbb{C}$ as $\varphi_2(z) = w$. Then

$$\begin{aligned}(Tf)(z) =&(E_{\varphi_2(z)}DC_{\varphi_1})(f) = E_{\varphi_2(z)}(DC_{\varphi_1} f) = (DC_{\varphi_1} f)(\varphi_2(z)) = (Dfo\varphi_1 o\varphi_2)(z)\\ =&D(fo\varphi)(z), \ where\ \varphi = \varphi_1 o \varphi_2\ is\ an\ entire\ function.\\ =&(DC_\varphi f)(z) \Rightarrow T = DC_\varphi\end{aligned}$$

This completes the proof.

Theorem 8. *Let $T = DC_\varphi \in C(\mathscr{E})$. Then $T^\star : \mathscr{E}^\star \to \mathscr{E}^\star$ be a generalized composition operator if $\varphi(z) = z$.*

Proof. Let $\varphi : \mathbb{C} \to \mathbb{C}$ is defined by $\varphi(z) = z$. Now, let $F \in \mathscr{E}^\star$, $f \in \mathscr{E}$. Then we have

$$F(z) = \sum_{n=0}^{\infty} F_n z^n, \quad f(z) = \sum_{n=0}^{\infty} \hat{f}_n z^n, \quad F'(z) = \sum_{n=1}^{\infty} nF_n z^{n-1}, \quad f'(z) = \sum_{n=1}^{\infty} n\hat{f}_n z^{n-1}.$$

Define $\psi : \mathbb{C} \to \mathbb{C}$ by $\psi(z) = z$. Then clearly ψ is an entire function. Now

$$\begin{aligned}(fo\varphi)'(z) &= f'(\varphi(z))\varphi'(z) = \sum_{n=1}^{\infty} n\hat{f}_n(\varphi(z))^{n-1}1 = \sum_{n=1}^{\infty} n\hat{f}_n z^{n-1} \quad and\\ (fo\varphi)'(z) &= \sum_{n=1}^{\infty} (\widehat{fo\varphi})'(n)z^n = \sum_{n=1}^{\infty} (\widehat{fo\varphi})'(n-1)z^{n-1}.\end{aligned}$$

Since $(fo\varphi)'(z)$ has a unique representation. Therefore, we have

$$(\widehat{fo\varphi})'(n-1) = n\hat{f}_n = nz^{n-1}, \quad where \quad \hat{f}_n = z^{n-1}.$$

Also, we have

$$F'(\psi(z))\psi'(z) = \sum_{n=1}^{\infty} nF_n(\psi(z))^{n-1}1 = \sum_{n=1}^{\infty} nF_n z^{n-1}.$$

Now

$$\begin{aligned}(T^\star F)(f) =&F(Tf) = F(DC_\varphi f) = F(fo\varphi)' = \sum_{n=0}^{\infty} (\widehat{fo\varphi})'(n)F_n = \sum_{n=1}^{\infty} (\widehat{fo\varphi})'(n-1)F_n = \sum_{n=1}^{\infty} n.F_n.z^{n-1}\\ =&F'(\psi(z))\psi'(z) = (DC_\varphi F)(f) = (TF)(f).\end{aligned}$$

Therefore, $T^\star = T$ and so $T^\star$ be a generalized composition operator.

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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