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Some properties of an operation on $g\alpha$-open sets

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Abstract: The paper introduces an operation γ on the collection of $g\alpha$-open subsets of a topological space. Then γ is used to study the concepts of $g\alpha\gamma$-open and $g\alpha\gamma$-generalized closed sets. Moreover, the separation axioms called $g\alpha\gamma$-T_i for $i = 0, 1/2, 1, 2$, are given and their properties are obtained.

Keywords: γ-operation on $\tau_{g\alpha}$, $g\alpha\gamma$-open sets, $g\alpha\gamma$-T_i spaces ($i = 0, 1/2, 1, 2$).

1 Introduction and preliminaries

In 1965, Njastad [17] introduced the concept of α-open subsets of a topological space (X,τ). In 1993, Maki, Devi and Balachandran [16] used α-open sets to studied generalized α-closed sets. Kasahara [14] introduced an α operation on τ and study α-closed graphs of α-continuous functions and α-compact spaces. Later, Jankovic [13] used α operation to introduced α-closure of a set in X and gave some characterizations on α-closed graph of functions. Then, Ogata [18] defined γ-open sets to study operation-functions and operation-separation. Lately, many types of γ operations on different classes of sets in X have been defined. Asaad et al. [10] introduced the notion of γ-extremally disconnected spaces. Asaad et al. [8] studied further characterizations of γ-extremally disconnected spaces and investigated some relations of functions of γ-extremally disconnected spaces. An et al. [6] introduced a γ operation on preopen subsets of (X,τ). They, also, defined pre-γ-open sets and built up their properties. Krishnan et al. [15] gave a γ operation on semi-open sets in (X,τ), and studied semi γ-open sets. After this, Carpintero et al. [12] considered a γ operation on b-open sets in (X,τ) to investigate b-γ-open sets. Tahiliani [20] studied a γ operation on β-open sets of (X,τ) to define β-γ-open sets. Asaad [7] defined a γ operation on generalized open sets in X and studied its applications. In 2017-2018, Ahmad and Asaad ([1], [9]) introduced an operation γ on semi generalized open subsets of X and discussed some types of separation axioms, functions and closed spaces with respect to γ. Al-shami [4] investigated some separation axioms via supra topological spaces and he [2] introduced a concept of supra semi open sets. He [3] also studied somewhere dense sets on topological spaces and obtained interesting properties. El-Shafei et al. [11] defined a type of generalized supra open sets and studied some of its applications.

The goal of the present research is to define a γ operation on $\tau_{g\alpha}$ and then use it to analyze $g\alpha\gamma$-open sets of (X,τ). Furthermore, $g\alpha\gamma$-T_i spaces where $i = 0, 1, 2$, are studied. Then, the collection of $g\alpha\gamma$-generalized closed sets is defined to investigate $g\alpha\gamma$-$T_{\frac{1}{2}}$ spaces.

Throughout the study, a space (X,τ) represents a non-empty topological space on which no any other topological property is supposed except otherwise mentioned. Let $A \subset (X,\tau)$, $Int(A)$ and $Cl(A)$ refer to the interior and the closure

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of A, respectively. A set A is called α-open [17] if $A \subseteq Int(Cl(Int(A)))$. The complement of α-open is α-closed [19]. We denote the collection of α-open subsets of X by τ_α. The α-closure of A, denoted by $\alpha Cl(A)$, is defined to be the intersection of all α-closed supersets of A [19]. A set A is said to be generalized α-closed (in short $g\alpha$-closed) [16] if $\alpha Cl(A) \subseteq V$ for each $V \in \tau_\alpha$ with $A \subseteq V$. Its complement is $g\alpha$-open. The collection $\tau_{g\alpha}$ denotes $g\alpha$-open sets in X. It is well-known that each α-closed set is $g\alpha$-closed, but not conversely.

2 $g\alpha\gamma$-open sets

A mapping $\gamma\colon \tau_{g\alpha} \to P(X)$ is an operation γ on $\tau_{g\alpha}$ such that $V \subseteq \gamma(V)$ for every $V \in \tau_{g\alpha}$. Provided that for all operation $\gamma\colon \tau_{g\alpha} \to P(X)$ we have $\gamma(X) = X$.

Definition 1. *A non-empty set A of X is said to be $g\alpha\gamma$-open if for each $x \in A$, there exists $g\alpha$-open V containing x such that $\gamma(V) \subseteq A$. The complement of a $g\alpha\gamma$-open set of X is $g\alpha\gamma$-closed. Suppose that the empty set ϕ is also $g\alpha\gamma$-open for any operation $\gamma\colon \tau_{g\alpha} \to P(X)$. The class of all $g\alpha\gamma$-open subsets of a space (X,τ) is denoted by $\tau_{g\alpha\gamma}$.*

Theorem 1. *The union of any collection of $g\alpha\gamma$-open sets in a space X is $g\alpha\gamma$-open.*

Proof. Let $x \in \bigcup_{\delta\in\Delta}\{A_\delta\}$, where $\{A_\delta\}_{\delta\in\Delta}$ be a class of $g\alpha\gamma$-open sets in X. Then $x \in A_\delta$ for some $\delta \in \Delta$. Since A_δ is $g\alpha\gamma$-open in X, then there exists a $g\alpha$-open set V such that $x \in V \subseteq \gamma(V) \subseteq A_\delta \subseteq \bigcup_{\delta\in\Delta}\{A_\delta\}$. Therefore, $\bigcup_{\delta\in\Delta}\{A_\delta\}$ is $g\alpha\gamma$-open in X.

Remark. The intersection of any two $g\alpha\gamma$-open sets in (X,τ) is generally not $g\alpha\gamma$-open as shown by the following example.

Example 1. Consider the space $X = \{1,2,3\}$ and $\tau = P(X) = \tau_{g\alpha}$. Let $\gamma\colon \tau_{g\alpha} \to P(X)$ be an operation on $\tau_{g\alpha}$ defined as follows. For every $A \in \tau_{g\alpha}$

$$\gamma(A) = \begin{cases} A & \text{if } A \neq \{2\} \\ \{2,3\} & \text{if } A = \{2\} \end{cases}.$$

Then, $\tau_{g\alpha\gamma} = P(X)\backslash\{2\}$. Then $\{1,2\} \in \tau_{g\alpha\gamma}$ and $\{2,3\} \in \tau_{g\alpha\gamma}$, but $\{1,2\} \cap \{2,3\} = \{2\} \notin \tau_{g\alpha\gamma}$.

Remark. Notice that $g\alpha$-open and $g\alpha\gamma$-open sets are not equal because, generally, the (even finite) union of $g\alpha$-open sets are not $g\alpha$-open. For instance, the singleton $\{2\}$, in Example 1, is $g\alpha$-open, but not $g\alpha\gamma$-open. Also, since every α-open set is $g\alpha$-open, then α-open and $g\alpha\gamma$-open are not equal.

Definition 2. *A space (X,τ) with an operation γ on $\tau_{g\alpha}$ is said to be $g\alpha\gamma$-regular if for each $x \in X$ and for each $g\alpha$-open set V containing x, there exists a $g\alpha$-open set U such that $x \in U$ and $\gamma(U) \subseteq V$.*

Theorem 2. *Let (X,τ) be a topological space and let $\gamma\colon \tau_{g\alpha} \to P(X)$ be an operation on $\tau_{g\alpha}$. Then the following statements are equivalent:*

(1) $\tau_{g\alpha} \subseteq \tau_{g\alpha\gamma}$.
(2) *X is $g\alpha\gamma$-regular.*
(3) *For every $x \in X$ and for every $g\alpha$-open set V of (X,τ) containing x, there exists a $g\alpha\gamma$-open set U of (X,τ) containing x such that $U \subseteq V$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and let V be a $g\alpha$-open set in X containing x. It follows from assumption that V is a $g\alpha\gamma$-open set. This implies that there exists a $g\alpha$-open set U such that $x \in U$ and $\gamma(U) \subseteq V$. Therefore, the space (X,τ) is $g\alpha\gamma$-regular.

(2) $\Rightarrow$ (3) Let $x \in X$ and let V be a $g\alpha$-open set in (X,τ) containing x. Then by (2), there is a $g\alpha$-open set U such that $x \in U \subseteq \gamma(U) \subseteq V$. Again, by using (2) for the set U, it is shown that U is $g\alpha\gamma$-open. Hence U is a $g\alpha\gamma$-open set containing x such that $U \subseteq V$.

(3) $\Rightarrow$ (1) By using the part (3) and Theorem 1, it is clear that every $g\alpha$-open set of X is $g\alpha\gamma$-open in X. Hence, $\tau_{g\alpha} \subseteq \tau_{g\alpha\gamma}$.

Definition 3. *An operation γ on $\tau_{g\alpha}$ is said to be $g\alpha$-regular if for each $x \in X$ and for every pair of $g\alpha$-open sets V_1 and V_2 such that both containing x, there exists a $g\alpha$-open set U containing x such that $\gamma(U) \subseteq \gamma(V_1) \cap \gamma(V_2)$.*

Lemma 1. *Let a mapping γ be $g\alpha$-regular operation on $\tau_{g\alpha}$.Then the following statements hold:*

(1) *If the subsets A and B are $g\alpha\gamma$-open in (X,τ), then $A\cap B$ is also $g\alpha\gamma$-open in (X,τ).*
(2) *$\tau_{g\alpha\gamma}$ forms a topology on (X,τ).*

Proof. (1) Suppose $x \in A\cap B$ for any $g\alpha\gamma$-open subsets A and B in (X,τ). Then there exist $g\alpha$-open sets V_1 and V_2 such that $x \in V_1 \subseteq A$ and $x \in V_2 \subseteq B$. Since γ is a $g\alpha$-regular operation on $\tau_{g\alpha}$, then there exists a $g\alpha$-open set U containing x such that $\gamma(U) \subseteq \gamma(V_1)\cap\gamma(V_2) \subseteq A\cap B$. Therefore, $A\cap B$ is $g\alpha\gamma$-open in (X,τ).

(2) Follows from the part (1) above and Theorem 1.

Definition 4. *The point $x \in X$ is in the $g\alpha$-closure$_\gamma$ of a set A if $\gamma(V)\cap A \neq \phi$ for each $g\alpha$-open set V containing x. The set of all $g\alpha$-closure$_\gamma$ points of A is called $g\alpha$-closure$_\gamma$ of A and is denoted by $g\alpha Cl_\gamma(A)$.*

Definition 5. *Let A be any subset of a topological space (X,τ) and γ be an operation on $\tau_{g\alpha}$. The $g\alpha\gamma$-closure of A is defined as the intersection of all $g\alpha\gamma$-closed sets of X containing A and it is denoted by $g\alpha_\gamma Cl(A)$. That is,*

$$g\alpha_\gamma Cl(A) = \bigcap\{E : A \subseteq E,\, X\backslash E \in \tau_{g\alpha\gamma}\}.$$

Theorem 3. *Let A be any subset of a topological space (X,τ) and let γ be an operation on $\tau_{g\alpha}$. Then $x \in g\alpha_\gamma Cl(A)$ if and only if $A\cap V \neq \phi$ for every $g\alpha\gamma$-open set V of X containing x.*

Proof. Let $x \in g\alpha_\gamma Cl(A)$ and let $A\cap V = \phi$ for some $g\alpha\gamma$-open set V of X containing x. Then $A \subseteq X\backslash V$ and $X\backslash V$ is $g\alpha\gamma$-closed in X. So $g\alpha_\gamma Cl(A) \subseteq X\backslash V$. Thus, $x \in X\backslash V$. This is a contradiction. Hence $A\cap V \neq \phi$ for every $g\alpha\gamma$-open set V of X containing x.

Conversely, suppose that $x \notin g\alpha_\gamma Cl(A)$. So there exists a $g\alpha\gamma$-closed set E such that $A \subseteq E$ and $x \notin E$. Then $X\backslash E$ is a $g\alpha\gamma$-open set such that $x \in X\backslash E$ and $A\cap(X\backslash E) = \phi$. Contradiction of hypothesis. Therefore, $x \in g\alpha_\gamma Cl(A)$.

Lemma 2. *The following statements are true for any subsets A and B of a topological space (X,τ) with an operation γ on $\tau_{g\alpha}$.*

(1) *$g\alpha_\gamma Cl(A)$ is $g\alpha\gamma$-closed in X and $g\alpha Cl_\gamma(A)$ is $g\alpha$-closed in X.*
(2) *$A \subseteq g\alpha Cl_\gamma(A) \subseteq g\alpha_\gamma Cl(A)$.*
(3) *$g\alpha_\gamma Cl(\phi) = g\alpha Cl_\gamma(\phi) = \phi$ and $g\alpha_\gamma Cl(X) = g\alpha Cl_\gamma(X) = X$.*
(4) (a) *A is $g\alpha\gamma$-closed if and only if $g\alpha_\gamma Cl(A) = A$ and,*
    (b) *A is $g\alpha\gamma$-closed if and only if $g\alpha Cl_\gamma(A) = A$.*
(5) *If $A \subseteq B$, then $g\alpha_\gamma Cl(A) \subseteq g\alpha_\gamma Cl(B)$ and $g\alpha Cl_\gamma(A) \subseteq g\alpha Cl_\gamma(B)$.*
(6) (a) *$g\alpha_\gamma Cl(A\cap B) \subseteq g\alpha_\gamma Cl(A)\cap g\alpha_\gamma Cl(B)$ and,*
    (b) *$g\alpha Cl_\gamma(A\cap B) \subseteq g\alpha Cl_\gamma(A)\cap g\alpha Cl_\gamma(B)$.*
(7) (a) *$g\alpha_\gamma Cl(A)\cup g\alpha_\gamma Cl(B) \subseteq g\alpha_\gamma Cl(A\cup B)$ and,*

(b) $g\alpha Cl_\gamma(A) \cup g\alpha Cl_\gamma(B) \subseteq g\alpha Cl_\gamma(A \cup B)$.

(8) $g\alpha_\gamma Cl(g\alpha_\gamma Cl(A)) = g\alpha_\gamma Cl(A)$.

Proof. Straightforward.

Theorem 4. *For any subsets A, B of a topological space (X,τ), if γ is a $g\alpha$-regular operation on $\tau_{g\alpha}$, then*

(1) $g\alpha_\gamma Cl(A) \cup g\alpha_\gamma Cl(B) = g\alpha_\gamma Cl(A \cup B)$.

(2) $g\alpha Cl_\gamma(A) \cup g\alpha Cl_\gamma(B) = g\alpha Cl_\gamma(A \cup B)$.

Proof. (1) It is enough to prove that $g\alpha_\gamma Cl(A \cup B) \subseteq g\alpha_\gamma Cl(A) \cup g\alpha_\gamma Cl(B)$ since the other part follows directly from Lemma 2 (7). Let $x \notin g\alpha_\gamma Cl(A) \cup g\alpha_\gamma Cl(B)$. Then there exist two $g\alpha\gamma$-open sets U and V containing x such that $A \cap U = \phi$ and $B \cap V = \phi$. Since γ is a $g\alpha$-regular operation on $\tau_{g\alpha}$, then by Lemma 1 (1), $U \cap V$ is $g\alpha\gamma$-open in X such that $(U \cap V) \cap (A \cup B) = \phi$. Therefore, we have $x \notin g\alpha_\gamma Cl(A \cup B)$ and hence $g\alpha_\gamma Cl(A \cup B) \subseteq g\alpha_\gamma Cl(A) \cup g\alpha_\gamma Cl(B)$.

(2) Let $x \notin g\alpha Cl_\gamma(A) \cup g\alpha Cl_\gamma(B)$. Then there exist $g\alpha$-open sets V_1 and V_2 such that $x \in V_1$, $x \in V_2$, $A \cap \gamma(V_1) = \phi$ and $A \cap \gamma(V_2) = \phi$. Since γ is a $g\alpha$-regular operation on $\tau_{g\alpha}$, then there exists a $g\alpha$-open set U containing x such that $\gamma(U) \subseteq \gamma(V_1) \cap \gamma(V_2)$. Thus, we have $(A \cup B) \cap \gamma(U) \subseteq (A \cup B) \cap (\gamma(V_1) \cap \gamma(V_2))$. This implies that $(A \cup B) \cap \gamma(U) = \phi$ since $(A \cup B) \cap (\gamma(V_1) \cap \gamma(V_2)) = \phi$. This means that $x \notin g\alpha Cl_\gamma(A \cup B)$ and hence $g\alpha Cl_\gamma(A \cup B) \subseteq g\alpha Cl_\gamma(A) \cup g\alpha Cl_\gamma(B)$. Using Lemma 2 (7), we have the equality.

Definition 6. *An operation γ on $\tau_{g\alpha}$ is said to be $g\alpha$-open if for each $x \in X$ and for every $g\alpha$-open set V containing x, there exists a $g\alpha\gamma$-open set U containing x such that $U \subseteq \gamma(V)$.*

Theorem 5. *Let A be any subset of a topological space (X,τ). If γ is a $g\alpha$-open operation on $\tau_{g\alpha}$, then $g\alpha Cl_\gamma(A) = g\alpha_\gamma Cl(A)$, $g\alpha Cl_\gamma(g\alpha Cl_\gamma(A)) = g\alpha Cl_\gamma(A)$ and $g\alpha Cl_\gamma(A)$ is $g\alpha\gamma$-closed in X.*

Proof. First we need to show that $g\alpha_\gamma Cl(A) \subseteq g\alpha Cl_\gamma(A)$ since by Lemma 2 (2), we have $g\alpha Cl_\gamma(A) \subseteq g\alpha_\gamma Cl(A)$. Now let $x \notin g\alpha Cl_\gamma(A)$, then there exists a $g\alpha$-open set V containing x such that $A \cap \gamma(V) = \phi$. Since γ is a $g\alpha$-open on $\tau_{g\alpha}$, then there exists a $g\alpha\gamma$-open set U containing x such that $U \subseteq \gamma(V)$. So $A \cap U = \phi$ and hence by Theorem 3, $x \notin g\alpha_\gamma Cl(A)$. Therefore, $g\alpha_\gamma Cl(A) \subseteq g\alpha Cl_\gamma(A)$. Hence $g\alpha Cl_\gamma(A) = g\alpha_\gamma Cl(A)$. Moreover, using the above result and by Lemma 2 (8), we get $g\alpha Cl_\gamma(g\alpha Cl_\gamma(A)) = g\alpha Cl_\gamma(A)$ and by Lemma 2 (4b), we obtain $g\alpha Cl_\gamma(A)$ is $g\alpha\gamma$-closed in X.

Example 2. Let $X = \{1,2,3\}$ and let $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$. Then $\tau_\alpha = \tau_{g\alpha} = \tau$. Define an operation γ: $\tau_{g\alpha} \to P(X)$ by $\gamma(A) = \alpha Cl(A)$ for every $A \in \tau_{g\alpha}$. Clearly, $\tau_{g\alpha\gamma} = \{\phi, X\}$. So γ is not $g\alpha$-open on $\tau_{g\alpha}$. If $A = \{1\}$, then $g\alpha_\gamma Cl(A) = X$ and $g\alpha Cl_\gamma(A) = \{1,3\}$. Therefore, $g\alpha Cl_\gamma(A) \neq g\alpha_\gamma Cl(A)$, $g\alpha Cl_\gamma(g\alpha Cl_\gamma(A)) \neq g\alpha Cl_\gamma(A)$ and $g\alpha Cl_\gamma(A)$ is not $g\alpha\gamma$-closed in X.

Theorem 6. *Let A be any subset of a topological space (X,τ) and let γ be an operation on $\tau_{g\alpha}$. Then the following statements are equivalent:*

(1) *A is $g\alpha\gamma$-open.*

(2) $g\alpha Cl_\gamma(X \backslash A) = X \backslash A$.

(3) $g\alpha_\gamma Cl(X \backslash A) = X \backslash A$.

(4) *$X \backslash A$ is $g\alpha\gamma$-closed.*

Proof. Clear.

Lemma 3. *Let (X,τ) be a topological space and let γ be a $g\alpha$-regular operation on $\tau_{g\alpha}$. Then $g\alpha_\gamma Cl(A) \cap U \subseteq g\alpha_\gamma Cl(A \cap U)$ holds for every $g\alpha\gamma$-open set U and every subset A of X.*

Proof. Suppose that $x \in g\alpha_\gamma Cl(A) \cap U$ for every $g\alpha\gamma$-open set U, then $x \in g\alpha_\gamma Cl(A)$ and $x \in U$. Let V be any $g\alpha\gamma$-open set of X containing x. Since γ is $g\alpha$-regular on $\tau_{g\alpha}$. So by Lemma 1 (1), $U \cap V$ is $g\alpha\gamma$-open containing x. Since $x \in g\alpha_\gamma Cl(A)$, then by Theorem 3, we have $A \cap (U \cap V) \neq \phi$. This means that $(A \cap U) \cap V \neq \phi$. Therefore, again by Theorem 3, we obtain that $x \in g\alpha_\gamma Cl(A \cap U)$. Thus, $g\alpha_\gamma Cl(A) \cap U \subseteq g\alpha_\gamma Cl(A \cap U)$.

3 $g\alpha\gamma$-separation axioms

This section studies properties of some types of separation axioms called $g\alpha\gamma$-T_i for $i \in \{0, \frac{1}{2}, 1, 2\}$.

Definition 7. *A space (X,τ) is called $g\alpha\gamma$-T_0 if for any two distinct points x, y in X, there exists a $g\alpha$-open set V such that $x \in V$ and $y \notin \gamma(V)$ or $y \in V$ and $x \notin \gamma(V)$.*

Definition 8. *A space (X,τ) is called $g\alpha\gamma$-T_1 if for any two distinct points x, y in X, there exist two $g\alpha$-open sets U and V containing x and y respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$.*

Definition 9. *A space (X,τ) is called $g\alpha\gamma$-T_2 if for any two distinct points x, y in X, there exist two $g\alpha$-open sets U and V containing x and y respectively such that $\gamma(U) \cap \gamma(V) = \phi$.*

Theorem 7. *The space (X,τ) is $g\alpha\gamma$-T_1 if and only if for every point $x \in X$, $\{x\}$ is a $g\alpha\gamma$-closed set in X.*

Proof. Let x be a point of a $g\alpha\gamma$-T_1 space (X,τ). Then for any point $y \in X$ such that $x \neq y$, there exists a $g\alpha$-open set V_y such that $y \in V_y$ but $x \notin \gamma(V_y)$. Thus, $y \in \gamma(V_y) \subseteq X \backslash \{x\}$. This implies that $X \backslash \{x\} = \cup\{\gamma(V_y) : y \in X \backslash \{x\}\}$. It is shown that $X \backslash \{x\}$ is $g\alpha\gamma$-open in (X,τ). Hence $\{x\}$ is $g\alpha\gamma$-closed in (X,τ).

Conversely, let $x, y \in X$ such that $x \neq y$. By hypothesis, we get $X \backslash \{y\}$ and $X \backslash \{x\}$ are $g\alpha\gamma$-open sets such that $x \in X \backslash \{y\}$ and $y \in X \backslash \{x\}$. Therefore, there exist $g\alpha$-open sets U and V such that $x \in U$, $y \in V$, $\gamma(U) \subseteq X \backslash \{y\}$ and $\gamma(V) \subseteq X \backslash \{x\}$. So, $y \notin \gamma(U)$ and $x \notin \gamma(V)$. This implies that (X,τ) is $g\alpha\gamma$-T_1.

Theorem 8. *Let γ be a $g\alpha$-open operation on $\tau_{g\alpha}$. Then (X,τ) is a $g\alpha\gamma$-T_0 space if and only if $g\alpha Cl_\gamma(\{x\}) \neq g\alpha Cl_\gamma(\{y\})$ for every distinct points x, y of X.*

Proof. Let x, y be any two distinct points of a $g\alpha\gamma$-T_0 space (X,τ). Then by definition, we assume that there exists a $g\alpha\gamma$-open set V such that $x \in V$ and $y \notin \gamma(V)$. Since γ is a $g\alpha$-open operation on $\tau_{g\alpha}$, then there exists a $g\alpha\gamma$-open set U such that $x \in U$ and $U \subseteq \gamma(V)$. Hence $y \in X \backslash \gamma(V) \subseteq X \backslash U$. Since $X \backslash U$ is a $g\alpha\gamma$-closed set in (X,τ). Then we obtain that $g\alpha Cl_\gamma(\{y\}) \subseteq X \backslash U$ and therefore $g\alpha Cl_\gamma(\{x\}) \neq g\alpha Cl_\gamma(\{y\})$.

Conversely, suppose for any $x, y \in X$ with $x \neq y$, we have $g\alpha Cl_\gamma(\{x\}) \neq g\alpha Cl_\gamma(\{y\})$. Now, we assume that there exists $z \in X$ such that $z \in g\alpha Cl_\gamma(\{x\})$, but $z \notin g\alpha Cl_\gamma(\{y\})$. If $x \in g\alpha Cl_\gamma(\{y\})$, then $\{x\} \subseteq g\alpha Cl_\gamma(\{y\})$, which implies that $g\alpha Cl_\gamma(\{x\}) \subseteq g\alpha Cl_\gamma(\{y\})$ (by Lemma 2 (5)). This implies that $z \in g\alpha Cl_\gamma(\{y\})$. This contradiction shows that $x \notin g\alpha Cl_\gamma(\{y\})$. This means that by Definition 4, there exists a $g\alpha$-open set V such that $x \in V$ and $\gamma(V) \cap \{y\} = \phi$. Thus, we have that $x \in V$ and $y \notin \gamma(V)$. It gives that the space (X,τ) is $g\alpha\gamma$-T_0.

Definition 10. *A subset A of a space (X,τ) is said to be $g\alpha\gamma$-generalized closed (in short $g\alpha\gamma g$.closed) if $g\alpha Cl_\gamma(A) \subseteq V$ whenever $A \subseteq V$ and V is a $g\alpha\gamma$-open set in X.*

Lemma 4. *Let (X,τ) be a topological space and let γ be an operation on $\tau_{g\alpha}$. A set A in (X,τ) is $g\alpha\gamma g$.closed if and only if $A \cap g\alpha_\gamma Cl(\{x\}) \neq \phi$ for every $x \in g\alpha Cl_\gamma(A)$.*

Proof. Suppose that A is $g\alpha\gamma g$.closed in X and suppose (if possible) that there exists an element $x \in g\alpha Cl_\gamma(A)$ such that $A \cap g\alpha_\gamma Cl(\{x\}) = \phi$. This follows that $A \subseteq X \backslash g\alpha_\gamma Cl(\{x\})$. Since $g\alpha_\gamma Cl(\{x\})$ is $g\alpha\gamma$-closed and A is $g\alpha\gamma g$.closed in X, then $X \backslash g\alpha_\gamma Cl(\{x\})$ is $g\alpha\gamma$-open and so $g\alpha Cl_\gamma(A) \subseteq X \backslash g\alpha_\gamma Cl(\{x\})$. This means that $x \notin g\alpha Cl_\gamma(A)$, which is a contradiction. Hence $A \cap g\alpha_\gamma Cl(\{x\}) \neq \phi$.

Conversely, let $V \in \tau_{g\alpha\gamma}$ such that $A \subseteq V$. To show that $g\alpha Cl_\gamma(A) \subseteq V$, let $x \in g\alpha Cl_\gamma(A)$. By hypothesis, $A \cap g\alpha_\gamma Cl(\{x\}) \neq \phi$. So there exists an element $y \in A \cap g\alpha_\gamma Cl(\{x\})$. Therefore $y \in A \subseteq V$ and $y \in g\alpha_\gamma Cl(\{x\})$. By Theorem 3, $\{x\} \cap V \neq \phi$. Hence $x \in V$ and so $g\alpha Cl_\gamma(A) \subseteq V$. Thus, A is $g\alpha\gamma g$.closed in (X,τ).

Theorem 9. *Let A be a subset of topological space (X,τ) and let γ be an operation on $\tau_{g\alpha}$. If A is $g\alpha\gamma g$.closed, then $g\alpha Cl_\gamma(A)\backslash A$ does not contain any non-empty $g\alpha\gamma$-closed set.*

Proof. Let E be a non-empty $g\alpha\gamma$-closed set in X such that $E\subseteq g\alpha Cl_\gamma(A)\backslash A$. Then $E\subseteq X\backslash A$ implies that $A\subseteq X\backslash E$. Since $X\backslash E$ is $g\alpha\gamma$-open and A is $g\alpha\gamma g$.closed, then $g\alpha Cl_\gamma(A)\subseteq X\backslash E$. That is $E\subseteq X\backslash g\alpha Cl_\gamma(A)$. Hence $E\subseteq X\backslash g\alpha Cl_\gamma(A)\cap g\alpha Cl_\gamma(A)\backslash A\subseteq X\backslash g\alpha Cl_\gamma(A)\cap g\alpha Cl_\gamma(A)=\phi$. This shows that $E=\phi$, which is a contradiction. Therefore, $E\nsubseteq g\alpha Cl_\gamma(A)\backslash A$.

Theorem 10. *If $\gamma\colon\ \tau_{g\alpha}\to P(X)$ is a $g\alpha$-open operation, then the converse of the Theorem 9 is true.*

Proof. Let V be a $g\alpha\gamma$-open set in (X,τ) such that $A\subseteq V$. Since $\gamma\colon\ \tau_{g\alpha}\to P(X)$ is a $g\alpha$-open operation, by Theorem 5, $g\alpha Cl_\gamma(A)$ is $g\alpha\gamma$-closed in X. Thus, using Theorem 1, we have $g\alpha Cl_\gamma(A)\cap X\backslash V$ is a $g\alpha\gamma$-closed set in (X,τ). Since $X\backslash V\subseteq X\backslash A$, then $g\alpha Cl_\gamma(A)\cap X\backslash V\subseteq g\alpha Cl_\gamma(A)\backslash A$. Using the assumption of the converse of Theorem 9, $g\alpha Cl_\gamma(A)\subseteq V$. Therefore, A is $g\alpha\gamma g$.closed in (X,τ).

Corollary 1. *Let A be a $g\alpha\gamma g$.closed subset of topological space (X,τ) and let γ be an operation on $\tau_{g\alpha}$. Then A is $g\alpha\gamma$-closed if and only if $g\alpha Cl_\gamma(A)\backslash A$ is $g\alpha\gamma$-closed.*

Proof. Let A be a $g\alpha\gamma$-closed set in (X,τ). By Lemma 2 (4b), $g\alpha Cl_\gamma(A)=A$ and so $g\alpha Cl_\gamma(A)\backslash A=\phi$ which is $g\alpha\gamma$-closed.

Conversely, suppose that $g\alpha Cl_\gamma(A)\backslash A$ is $g\alpha\gamma$-closed and A is $g\alpha\gamma g$.closed. By Theorem 9, $g\alpha Cl_\gamma(A)\backslash A$ does not contain any non-empty $g\alpha\gamma$-closed set and since $g\alpha Cl_\gamma(A)\backslash A$ is $g\alpha\gamma$-closed subset of itself, then $g\alpha Cl_\gamma(A)\backslash A=\phi$ implies that $g\alpha Cl_\gamma(A)\cap X\backslash A=\phi$. So $g\alpha Cl_\gamma(A)=A$. Hence A is $g\alpha\gamma$-closed in (X,τ).

Theorem 11. *Let (X,τ) be a topological space and let γ be an operation on $\tau_{g\alpha}$. If a subset A of X is $g\alpha\gamma g$.closed and $g\alpha\gamma$-open, then A is $g\alpha\gamma$-closed.*

Proof. Let A be $g\alpha\gamma g$.closed and $g\alpha\gamma$-open in X, then $g\alpha Cl_\gamma(A)\subseteq A$ and so, by Lemma 2 (4b), A is $g\alpha\gamma$-closed.

Theorem 12. *Let (X,τ) be a topological space with an operation γ on $\tau_{g\alpha}$. For each point $x\in X$, $X\backslash\{x\}$ is either $g\alpha\gamma g$.closed or $g\alpha\gamma$-open.*

Proof. Suppose that $X\backslash\{x\}$ is not $g\alpha\gamma$-open. Then X is the only $g\alpha\gamma$-open set containing $X\backslash\{x\}$. This implies that $g\alpha Cl_\gamma(X\backslash\{x\})\subseteq X$. Thus $X\backslash\{x\}$ is a $g\alpha\gamma g$.closed set in X.

Corollary 2. *Let (X,τ) be a topological space with an operation γ on $\tau_{g\alpha}$. For each point $x\in X$, either $\{x\}$ is $g\alpha\gamma$-closed or $X\backslash\{x\}$ is $g\alpha\gamma g$.closed.*

Proof. Suppose that $\{x\}$ is not $g\alpha\gamma$-closed, then $X\backslash\{x\}$ is not $g\alpha\gamma$-open. By Theorem 12, $X\backslash\{x\}$ is $g\alpha\gamma g$.closed in X.

Definition 11. *The $\tau_{g\alpha\gamma}$-kernel of a subset A of a space (X,τ), denoted by $\tau_{g\alpha\gamma}$-ker(A), is defined as is the intersection of all $g\alpha\gamma$-open sets of (X,τ) containing A.*

Theorem 13. *Let $A\subseteq(X,\tau)$ and let γ be an operation on $\tau_{g\alpha}$. Then A is $g\alpha\gamma g$.closed if and only if $g\alpha Cl_\gamma(A)\subseteq\tau_{g\alpha\gamma}$-ker(A).*

Proof. Suppose that A is $g\alpha\gamma g$.closed. Then $g\alpha Cl_\gamma(A)\subseteq V$, whenever $A\subseteq V$ and V is $g\alpha\gamma$-open. Let $x\in g\alpha Cl_\gamma(A)$. By Lemma 4, $A\cap g\alpha_\gamma Cl(\{x\})\neq\phi$. So there exists a point z in X such that $z\in A\cap g\alpha_\gamma Cl(\{x\})$ which implies that $z\in A\subseteq V$ and $z\in g\alpha_\gamma Cl(\{x\})$. By Theorem 3, $\{x\}\cap V\neq\phi$. This concludes that $x\in\tau_{g\alpha\gamma}$-ker(A). Therefore, $g\alpha Cl_\gamma(A)\subseteq\tau_{g\alpha\gamma}$-ker$(A)$.

Conversely, let $g\alpha Cl_\gamma(A)\subseteq\tau_{g\alpha\gamma}$-ker$(A)$. Let V be a $g\alpha\gamma$-open set containing A. Let x be a point in X such that $x\in g\alpha Cl_\gamma(A)$. Then $x\in\tau_{g\alpha\gamma}$-ker(A). Now, we have $x\in V$, because $A\subseteq V$ and $V\in\tau_{g\alpha\gamma}$. Therefore $g\alpha Cl_\gamma(A)\subseteq\tau_{g\alpha\gamma}$-ker$(A)\subseteq V$. Thus A is $g\alpha\gamma g$.closed in X.

Definition 12. *A space (X,τ) is called $g\alpha\gamma$-$T_{\frac{1}{2}}$ if every $g\alpha\gamma g$.closed set in X is $g\alpha\gamma$-closed.*

Theorem 14. *A space (X,τ) is $g\alpha\gamma$-$T_{\frac{1}{2}}$ if and only if for each $x\in X$, $\{x\}$ is either $g\alpha\gamma$-closed or $g\alpha\gamma$-open.*

Proof. Let X be a $g\alpha\gamma$-$T_{\frac{1}{2}}$ space and let $\{x\}$ be not a $g\alpha\gamma$-closed set in (X,τ). By Corollary 2, $X\backslash\{x\}$ is $g\alpha\gamma g$.closed. Since (X,τ) is $g\alpha\gamma$-$T_{\frac{1}{2}}$, then $X\backslash\{x\}$ is $g\alpha\gamma$-closed which means that $\{x\}$ is $g\alpha\gamma$-open in X.

Conversely, let E be a $g\alpha\gamma g$.closed set in (X,τ). We have to show that E is $g\alpha\gamma$-closed (that is $g\alpha Cl_\gamma(E)=E$ (by Lemma 2 (4b))). It is sufficient to show that $g\alpha Cl_\gamma(E)\subseteq E$. Let $x\in g\alpha Cl_\gamma(E)$. By hypothesis $\{x\}$ is $g\alpha\gamma$-closed or $g\alpha\gamma$-open for each $x\in X$. We consider two cases:

Case (1): Let $\{x\}$ be a $g\alpha\gamma$-closed set. Suppose that $x\notin E$, then $x\in g\alpha Cl_\gamma(E)\backslash E$ contains a non-empty $g\alpha\gamma$-closed set $\{x\}$. Since E is $g\alpha\gamma g$.closed set, so this leads us to contradiction according to Theorem 9. Thus $x\in E$. Therefore $g\alpha Cl_\gamma(E)\subseteq E$ and so $g\alpha Cl_\gamma(E)=E$. This means that E is $g\alpha\gamma$-closed in (X,τ). Hence (X,τ) is $g\alpha\gamma$-$T_{\frac{1}{2}}$ space.

Case (2): let $\{x\}$ be a $g\alpha\gamma$-open set. By Theorem 3, $E\cap\{x\}\neq\phi$ which implies that $x\in E$. So $g\alpha Cl_\gamma(E)\subseteq E$. By Lemma 2 (4b), E is $g\alpha\gamma$-closed. Therefore, (X,τ) is $g\alpha\gamma$-$T_{\frac{1}{2}}$ space.

Theorem 15. *For any topological space (X,τ) and any operation γ on $\tau_{g\alpha}$, the following properties hold.*

(1) *Every $g\alpha\gamma$-T_2 space is $g\alpha\gamma$-T_1.*
(2) *Every $g\alpha\gamma$-T_1 space is $g\alpha\gamma$-$T_{\frac{1}{2}}$.*
(3) *Every $g\alpha\gamma$-$T_{\frac{1}{2}}$ space is $g\alpha\gamma$-T_0.*

Proof. The proofs can be followed from their definitions.

The converse of each statement in Theorem 15 is not true in general as shown by the following examples.

Example 3. Let $X=\{1,2,3\}$ and let τ be the discrete topology on X. If γ: $\tau_{g\alpha}\to P(X)$ is an operation on $\tau_{g\alpha}$ defined by For every $A\in\tau_{g\alpha}$

$$\gamma(A)=\begin{cases}A \text{ if } A=\{1,2\} \text{ or } \{1,3\} \text{ or } \{2,3\}\\ X \text{ otherwise,}\end{cases}$$

then the space (X,τ) is $g\alpha\gamma$-T_1, but it is not $g\alpha\gamma$-T_2.

Example 4. Let $X=\{1,2,3\}$ and let τ be all subsets of X. Define an operation γ on $\tau_{g\alpha}$ as follows: For every set $A\in\tau_{g\alpha}$

$$\gamma(A)=\begin{cases}A \text{ if } A=\{1\} \text{ or } \{3\} \text{ or } \{1,3\} \text{ or } \{2,3\}\\ X \text{ otherwise}\end{cases}$$

Clearly, $\tau_{g\alpha\gamma}=\{\phi,X,\{1\},\{3\},\{1,3\},\{2,3\}\}$. Thus (X,τ) is $g\alpha\gamma$-$T_{\frac{1}{2}}$ but it is not $g\alpha\gamma$-T_1.

Example 5. Let $X=\{1,2,3\}$ and $\tau=\{\phi,X,\{1\},\{2\},\{1,2\}\}$. Then $\tau_{g\alpha}=\tau_\alpha=\tau$. Define an operation γ on $\tau_{g\alpha}$ as follows. For every set $A\in\tau_{g\alpha}$

$$\gamma(A)=\begin{cases}A & \text{if } 2\in A\\ Cl(A) & \text{if } 2\notin A\end{cases}$$

Thus, $\tau_{g\alpha\gamma}=\{\phi,X,\{2\},\{1,2\}\}$. Hence the space (X,τ) is $g\alpha\gamma$-T_0, but it is not $g\alpha\gamma$-$T_{\frac{1}{2}}$. Since $\{1\}$ is neither $g\alpha\gamma$-closed nor $g\alpha\gamma$-open in X by Theorem 14.

Remark. In 2018, Ameen [5], examined, respectively, that the set of preopen (b-open and β-open) subsets coincides with set of pg-open (bg-open and βg-open) subsets of all spaces (X,τ). Defining operations like γ on the later classes of sets turns to be identical to γ on some sets already exist in the literature. Namely, γ operation, respectively, on pg-open (bg-open and βg-open) sets will be the same as γ on preopen [10] (b-open [12] and β-open [20]) sets. This remark can be applied to all kind of open sets involving the operation γ. For undefined terms in the remark, we refer the reader to Definition 1.6 in [5].

4 Conclusions

In this paper, we introduced a γ operation on $\tau_{g\alpha}$. We analyzed $g\alpha\gamma$-open sets of (X,τ) via γ operation on $\tau_{g\alpha}$ operation. In addition, $g\alpha\gamma$-T_i spaces where $i = 0,1,2$, have been studied. Finally, we defined $g\alpha\gamma$-generalized closed sets and then the space $g\alpha\gamma$-$T_{\frac{1}{2}}$ has been investigated.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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