New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2018.325

Uniform $\mathscr{I}$-Lacunary statistical convergence on time scales

Omer Kisi and Erhan Guler

Deparment of Mathematics, Bartın University, Bartın, Turkey

Received: 3 June 2018, Accepted: 1 August 2018
Published online: 24 December 2018.

Abstract: In this paper, we define m-uniform $\mathscr{I}$-statistical convergence, (θ,m)-uniform $\mathscr{I}$-lacunary statistical convergence and $(\mathscr{I}_\theta,m)$-uniform strongly p-lacunary summability of functions on an arbitrary time scale. Also, by using m-uniform and (λ,m)-uniform density of the subset of the time scale, we will focus on constructing concepts of $(\mathscr{I}_\lambda,m)$-uniform statistically convergence and $(\mathscr{I}_\lambda,m)$-uniformly strongly p-summability of functions on time scale. Some inclusion relations about these new concepts are also presented.

Keywords: Statistical convergence, time scale, lacunary sequence, ideal convergence.

1 Introduction and backround

The idea of statistical convergence goes back to the study of Zygmund [39] which was published in 1935. Statistical convergence of number sequences was formally introduced by Fast [11] and Steinhaus [38] independently in the same year. Over the years and under different names, statistical convergence has been discussed in Fourier analysis, ergodic theory, number theory, approximation theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Connor [4], Fridy [13], Mohiuddine et al. [28], Rath and Tripathy [29], Tripathy [30], Belen and Mohiuddine [31], Maddox [34] and references therein.

The concept of lacunary statistical convergence was defined by Fridy and Orhan [14]. Also, Fridy and Orhan [15] gave the relationships between the lacunary statistical convergence and the Cesàro summability.

Mursaleen [5] defined λ-statistical convergence by using the λ sequence. In [8], Borwein introduced and studied strongly summable functions. Strongly summable number sequences and statistically convergent number seqeunces were studied by Maddox [34], Nuray and Aydın [17], and Et et al. [36]. Nuray [18] studied on λ-statistically convergent functions, λ-strong summable and λ-statistically convergent functions. Furthermore, Nuray and Aydın [17] introduced and studied strongly lacunary summable functions.

Kostyrko et al. [20] introduced the concept of $\mathscr{I}$-convergence of sequences in a metric space and studied some properties of this convergence.

Recently, the idea of statistical convergence and lacunary convergence was further extended by Das et al. [6] to $\mathscr{I}$-statistical convergence and $\mathscr{I}$-lacunary statistical convergence, respectively.

* Corresponding author e-mail: okisi@bartin.edu.tr

A time scale T is an arbitrary, nonempty, closed subset of real numbers. The calculus of time scale was introduced by Hilger in his Ph.D. thesis supervised by Auldbach in 1988 [10]. It allows to unify the usual differential and integral calculus for one variable. One can replace the range of definition $\mathbb{R}$ of the functions under consideration by an arbitrary time scale T. This method of calculus is effective in modeling some practical life problems for example one needs both discrete and continuous time variables to modeling prey and predator populations. Recently, time scale theory has been applied to different areas by many authors (see [9], [22], [23], [37]).

Also a large numbers of very important functions on time scales have been applied to solve various dynamic equations, the expression of Green's functions of some boundary value problem [26] or oscillation properties of first and second order nonlinear equations [27].

So in view of recent applications of time scales in real life problems, it seems very natural to extend the interesting concept of convergence on time scales by using ideals which we mainly do here.

Statistical convergence is applied to time scales for different purposes by various authors in the literature. Seyyidoglu and Tan [21] gave some important notions such as Δ-convergence and Δ-Cauchy sequences by using Δ-density and investigate their relations on T.

Turan and Duman [22] introduced density and statistical convergence of Δ-measurable real-valued functions defined on T.

Uniform density was studied by Balaz and Salat [2], Brown and Freedman [7], Raimi [19] and Maddox [35]. The notion of m-uniform statistical convergence was first introduced by Nuray [16]. Furthermore, Altin et al. [1] expressed m- and (λ,m)-uniform density of a set and m- and (λ,m)-uniform statistical convergence on T. Also, Yilmaz et al. [24] defined λ-statistical convergence on T. Turan and Duman [23] gave the definitions of lacunary sequence and lacunary statistical convergence on T. Yilmaz et al. [25] introduced uniform lacunary statistical convergence on time scale.

The definition of p-Cesáro summability on time scales was given by Turan and Duman [22].

Measure theory on time scales was first constructed by Guseinov [37] and Lebesque Δ-integral on time scales introduced by Cabada and Vivero [3].

Here, our aim is to move some notions and properties about lacunary statistical convergence to time scale calculus by using ideal. Before our new concepts, we recall the main features of the time scale theory.

First we recall some basic concepts related to time scales and summability theory. We should note that throughout the paper, we consider that T is a time scales satisfying $\inf T = t_o > 0$ and $\sup T = \infty$.

Let $\mathscr{F}_1$ denote the family of all left closed and right open intervals of T of the form $[a,b)_T$. Let $m_1 : \mathscr{F}_1 \to [0,\infty)$ be the set function on $\mathscr{F}_1$ such that $m_1([a,b)_T) = b-a$. Then, it is known that m_1 is a countably additive measure on $\mathscr{F}_1$. Now, the Caratheodory extension of the set function m_1 associated with family $\mathscr{F}_1$ is said to be the Lebesgue Δ-measure on T and is denoted by μ_Δ. (see [32], [33]).

In this case, it is known that if $a \in T \setminus \{\max T\}$, then the single point set $\{a\}$ is Δ-measurable function and $\mu_\Delta(a) = \sigma(a) - a$. If $a,b \in T$ and $a \leq b$, then $\mu_\Delta([a,b)_T) = \sigma(b) - \sigma(a)$ and $\mu_\Delta([a,b)_T) = \sigma(b) - a$. (see [37]).

Now, we give a generalization of the study [23] in a different form where $\theta = \{k_{t-t_0+1}\}$ is a lacunary sequence on T.

Definition 1. [25] *Let Ω be a Δ-measurable subset of T and θ be a lacunary sequence. Then, we define the set $\Omega(t,\theta)$ by*

$$\Omega(t,\theta) = \left\{s \in \left(k_{t-2t_0+1}, k_{t-t_0+1}\right]_T : s \in \Omega\right\},$$

for $t \in T$. In this case, the θ-density of Ω on T is introduced by

$$\delta_T^{\theta}(\Omega) = \lim_{t\to\infty} \frac{\mu_\Delta(\Omega(t,\theta))}{\mu_\Delta\left(\left(k_{t-2t_0}, k_{t-t_0}\right]_T\right)},$$

provided that the above limit exists.

Definition 2. [25] *Let $f : T \to \mathbb{R}$ be a Δ-measurable subset of T and θ be a lacunary sequence. Then, f is lacunary statistically convergent to a real number L on T if*

$$\lim_{t\to\infty} \frac{\mu_\Delta\left(s \in \left(k_{t-2t_0+1}, k_{t-t_0+1}\right]_T : |f(s) - L| \geq \varepsilon\right)}{\mu_\Delta\left(\left(k_{t-2t_0}, k_{t-t_0}\right]_T\right)} = 0,$$

for each $\varepsilon > 0$ and $t \in T$. In this case, s_T^{θ}-$\lim_{t\to\infty}(f(t)) = L$. The set of all lacunary functions on T will be denoted by s_T^{θ}. $\left(k_{t-2t_0+1}, k_{t-t_0+1}\right]$ turns to $(k_{r-1}, k_r]$ for $t = r$, $t_0 = 1$ and $T = \mathbb{N}$. In this case, lacunary statistical convergence on time scales is reduced to classical lacunary statistical convergence which is defined by Fridy and Orhan [14].

Uniform density of an arbitrary set was introduced by Raimi [19] as follows.

Definition 3. [19] *A subset $E \subset \mathbb{N}$ is uniform density if*

$$u(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{\infty} \chi_E(j+m) = a,$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n} |E \cap \{m+1, \ldots, m+n\}| = a,$$

uniformly in m, where $m = 0, 1, 2, \ldots$ and χ_E is characteristic function of E.

Subsequently, uniformly density was studied by Balaz and Salat [2]. Later, m-uniform statistical convergence is introduced by Nuray [16] in the following manner.

Definition 4. [16] *Let $x = (x_k)$ be a real or complex valued sequence. If*

$$\lim_{n\to\infty} \frac{1}{n} |\{m \leq k < n+m : |x_k - L| \geq \varepsilon\}| = 0,$$

uniformly in m, then $x = (x_k)$ is said to be m-uniform statistically convergent to L for all $\varepsilon > 0$.

Based on Definition 4, we can generalize m-uniform statistical convergence to lacunary type sequences as follows:

Definition 5. *Let $K \subset \mathbb{N}$ and θ be a lacunary sequence. Then, we define the (θ,m)-uniform density of K by*

$$\delta_T^{\theta}(K) = \lim_{r\to\infty} \frac{1}{h_{r,m}} |\{k_{r-1+m} < k \leq k_{r+m} : k \in K\}| = 0,$$

uniformly in $m \geq 0$, where $h_{r,m} = k_{r+m} - k_{r+m-1}$.

Definition 6. *A sequence $x=(x_k)$ is said to be (θ,m)-uniform lacunary statistical convergent to a real number L if*

$$\lim_{r\to\infty}\frac{1}{h_{r,m}}\left|\{k_{r-1+m}<k\le k_{r+m}:|x_k-L|\ge\varepsilon\}\right|=0,$$

for all $\varepsilon>0$, uniformly in m.

Now, we define above notions on time scale T and we give the definition of m-uniform density, m-uniform statistical convergence, (θ,m)-density, (θ,m)-uniform lacunary statistical convergence on T, where $\theta=\left(k_{t-t_0+m+1}\right]$ is a lacunary sequence for $t\in T$.

Definition 7. [1] *Let Ω be a Δ_m-measurable subset of T . Then, one defines the set $\Omega(t,m)$ by*

$$\Omega(t,m)=\{s\in[m+t_0-1,t+m)_T : s\in\Omega\},$$

for $t\in T$. In this case, m-uniform density of Ω on T , denoted by $\delta_T^m(\Omega)$ is defined as follows:

$$\delta_T^m(\Omega)=\lim_{t\to\infty}\frac{\mu_{\Delta_m}(\Omega(t,m))}{\mu_{\Delta_m}([m+t_0-1,t+m)_T)},$$

provided that the above limit exists.

Definition 8. [1] *Let $f:T\to\mathbb{R}$ be a Δ-measurable function. Then, one says that f is m-uniform statistically convergent to real number L on T if*

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(s\in([m+t_0-1,t+m)_T):|f(s)-L|\ge\varepsilon\right)}{\mu_\Delta([m+t_0-1,t+m)_T)}=0,$$

uniformly in m for every $\varepsilon>0$. In this case we write $s_T^m\text{-}\lim_{t\to\infty}(f(t))=L$.

Definition 9. [25] *Let Ω be a Δ-measurable subset of T and θ be a lacunary sequence. Then, we define the set $\Omega(t,\theta,m)$ by*

$$\Omega(t,\theta,m)=\left\{s\in\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_T : s\in\Omega\right\},$$

for $t\in T$. In this case, (θ,m)-uniform density of Ω on T is defined by

$$\lim_{t\to\infty}\frac{\mu_\Delta(\Omega(t,\theta))}{\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_T\right)},$$

provided that the above limit exists.

Definition 10. [25] *Let $f:T\to\mathbb{R}$ be a Δ-measurable function and θ be a lacunary sequence. Then, f is (θ,m)-uniform lacunary statistically convergent to real number L on T if*

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(s\in\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_T:|f(s)-L|\ge\varepsilon\right)}{\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_T\right)}=0,$$

uniformly in m, for all $\varepsilon>0$ and $t\in T$. In this case, $s_T^{\theta,m}\text{-}\lim_{t\to\infty}(f(t))=L$. The set of all (θ,m)-uniform lacunary statistically convergent functions on T will be denoted by $s_T^{\theta,m}$.

2 Main results

In this study, we define m-uniform $\mathscr{I}$-statistical convergence, (θ,m)-uniform $\mathscr{I}$-lacunary statistical convergence and $(\mathscr{I}_\theta,m)$-uniform strongly p-lacunary summability on an arbitrary time scale. Also, we define $(\mathscr{I}_\lambda,m)$-uniform statistically convergence and $(\mathscr{I}_\lambda, m)$-uniformly strongly p-summability on time scale. Some inclusion relations about these new concepts are also presented.

In the main part of our study, we introduce the notion of $\mathscr{I}$-convergence on time scales. For this purpose, we consider the measurable space $(T, M(m_1^*))$ equipped with Lebesgue Δ-measure μ_Δ. We can give some examples of ideals of time scales. $\mathscr{I} = \{A \in M(m_1^*) : \mu_\Delta(A) = 0\}$ and $\mathscr{I}_\Delta = \{A \in M(m_1^*) : \delta_T(A) = 0\}$ are ideals on a time scale. An ideal $\mathscr{I}$ of a time scale T is said to be B-admissible if it contains all bounded subsets of T.

Definition 11. *Let $f : T \to \mathbb{R}$ be a Δ-measurable function. We say that f is $\mathscr{I}$-convergent to a number L on T if for every $\varepsilon > 0$,*

$$A(\varepsilon) : \{t \in T : |f(t) - L| \geq \varepsilon\} \in \mathscr{I}.$$

If we take $\mathscr{I} = \mathscr{I}_\Delta$, $\mathscr{I}$-convergence reduced to statistical convergence on time scale T.

Definition 12. *Let $f : T \to \mathbb{R}$ be a Δ-measurable function and $\mathscr{I}$ be a B-admissible ideal. We say that f is said to be m-uniform $\mathscr{I}$-statistically convergent to L or $s_T^{\mathscr{I},m}$-convergent to L on T to a number L uniformly in m if every $\varepsilon > 0$, $\delta > 0$,*

$$\left\{t \in T : \frac{|\{\mu_{\Delta_m}(s \in ([m + t_0 - 1, t + m)_T) : |f(s) - L| \geq \varepsilon)\}|}{\mu_{\Delta_m}([m + t_0 - 1, t + m)_T)} \geq \delta\right\} \in \mathscr{I}.$$

In this case, one writes $f \to L\left(s_T^{\mathscr{I},m}\right)$. The set of all m-uniform $\mathscr{I}$-statistically convergent functions on T will be denoted by $s_T^{\mathscr{I},m}$.

Definition 13. *Let $f : T \to \mathbb{R}$ be a Δ-measurable function. f is a m-uniform $\mathscr{I}$-statistical Cauchy function on T if there exists a number $t_1 > t_0 \in T$ such that*

$$\left\{t \in T : \frac{|\{\mu_{\Delta_m}(s \in ([m + t_0 - 1, t + m)_T) : |f(s) - f(t_1)| \geq \varepsilon)\}|}{\mu_{\Delta_m}([m + t_0 - 1, t + m)_T)} \geq \delta\right\} \in \mathscr{I},$$

for each $\varepsilon > 0$, $\delta > 0$ uniformly in m.

Theorem 1. *Let $f : T \to \mathbb{R}$ be a Δ-measurable function, then f is m-uniform $\mathscr{I}$-statistically convergent on T if and only if f is m-uniform $\mathscr{I}$-statistical Cauchy function on T.*

Proof. We can prove this by using techniques similar to Theorem 3 of [16].

Definition 14. *Let $f : T \to \mathbb{R}$ be a Δ-measurable function, θ be lacunary sequence and $\mathscr{I}$ be a B-admissible ideal. Then, f is $(\mathscr{I}_\theta, m)$-uniform lacunary statistically convergent to L on T or $s_T^{\mathscr{I}_\theta,m}$-convergent to L on T if*

$$\left\{t \in T : \frac{|\{\mu_{\Delta_m}\left(s \in \left(k_{t-2t_0+m+1}, k_{t-t_0+m+1}\right]_T : |f(s) - L| \geq \varepsilon\right)\}|}{\mu_{\Delta_m}\left(\left(k_{t-2t_0+m}, k_{t-t_0+m}\right]_T\right)} \geq \delta\right\} \in \mathscr{I},$$

uniformly in m, for all $\varepsilon > 0$, $\delta > 0$. In this case, one writes $f \to L\left(s_T^{\mathscr{I}_\theta,m}\right)$. The set of all (θ, m)-uniform $\mathscr{I}$-statistically convergent functions on T will be denoted by $s_T^{\mathscr{I}_\theta,m}$.

Definition 15. *Let $f : T \to \mathbb{R}$ be a Δ-measurable function and $0 < p < \infty$. Then f is strongly $\mathscr{I}$-Cesáro p-summable on T if there exists some L such that*

$$\left\{t \in T : \left|\frac{1}{\mu_\Delta([t_0, t]_T)} \cdot \int_{[t_0, t]_T} |f(s) - L|^p \Delta s\right| \geq \varepsilon\right\} \in \mathscr{I},$$

if for each $\varepsilon > 0$. *In this case, one writes* $f \to L\left[W_p^{\mathscr{I}}\right]_T$. *The set of all strongly* $\mathscr{I}$*-Cesáro p-summable functions on* T *will be denoted by* $\left[W_p^{\mathscr{I}}\right]_T$.

Definition 16. *Let* $f : T \to \mathbb{R}$ *be a* Δ*-measurable function and* θ *be a lacunary sequence. Assume also that* $0 < p < \infty$. *Then,* f *is* $(\mathscr{I}_\theta, m)$*-uniform strongly p-lacunary summable on* T *if there exists some* L *such that*

$$\left\{ t \in T : \left| \frac{1}{\mu_{\Delta_m}\left(\left(k_{t-2t_0+m}, k_{t-t_0+m}\right]_T\right)} \cdot \int\limits_{\left(k_{t-2t_0+m+1}, k_{t-t_0+m+1}\right]_T} |f(s) - L|^p \Delta s \right| \geq \varepsilon \right\} \in \mathscr{I},$$

uniformly in m if for each $\varepsilon > 0$. *In this case, one writes* $f \to L\left(\left[W_p^{\mathscr{I}_\theta, m}\right]_T\right)$. *The set of all* $(\mathscr{I}_\theta, m)$*-uniform strongly p-lacunary summable functions on* T *will be denoted by* $\left[W_p^{\mathscr{I}_\theta, m}\right]_T$.

Now, we recall the definition of (λ, m)-uniform density on T .

Definition 17. [1] *Let* $\Omega(t, m, \lambda)$ *be a* $\Delta_{(\lambda, m)}$*-measurable subset of* T. *Then, one defines the set* $\Omega(t, m, \lambda)$ *by*

$$\Omega(t, m, \lambda) = \{ s \in [t + m - \lambda_t + t_0 - 1, t + m) : s \in \Omega \}$$

for $t \in T$. *In this case, one defines the* (λ, m)*-uniformly density of* Ω *on* T *denoted by* $\delta_T^{(\lambda, m)}(\Omega)$, *as follows:*

$$\delta_T^{(\lambda, m)}(\Omega) : \lim_{t \to \infty} \frac{\mu_{\Delta_{(\lambda, m)}}(\Omega(t, m, \lambda))}{\mu_{\Delta_{(\lambda, m)}}([t + m - \lambda_t + t_0 - 1, t + m)_T)}.$$

Definition 18. *Let* $f : T \to \mathbb{R}$ *be a* $\Delta_{(\lambda, m)}$*-measurable function. We say that* f *is said to be* $(\mathscr{I}_\lambda, m)$*-uniform statistically convergent to a real number* L *on* T *uniformly in m if every* $\varepsilon > 0$, $\delta > 0$,

$$\left\{ t \in T : \frac{\left| \left\{ \mu_{\Delta_{(\lambda, m)}}(s \in [t + m - \lambda_t + t_0 - 1, t + m)_T : |f(s) - L| \geq \varepsilon) \right\} \right|}{\mu_{\Delta_{(\lambda, m)}}([t + m - \lambda_t + t_0 - 1, t + m)_T)} \geq \delta \right\} \in \mathscr{I}.$$

In this case, one writes $f \to L\left(\widetilde{s}_T^{(\mathscr{I}_\lambda, m)}\right)$. *The set of all* $(\mathscr{I}_\lambda, m)$*-uniform statistically convergent functions on* T *will be denoted by* $\widetilde{s}_T^{(\mathscr{I}_\lambda, m)}$.

Definition 19. *Let* $f : T \to \mathbb{R}$ *be a* Δ*-measurable function and* $0 < p < \infty$. *Then* f *is* $(\mathscr{I}_\lambda, m)$*-uniformly strongly p-summable on* T *if there exists some* L *such that*

$$\left\{ t \in T : \left| \frac{1}{\mu_{\Delta_{(\lambda, m)}}([t + m - \lambda_t + t_0 - 1, t + m)_T)} \cdot \int\limits_{[t+m-\lambda_t+t_0-1, t+m)_T} |f(s) - L|^p \Delta s \right| \geq \varepsilon \right\} \in \mathscr{I},$$

uniformly in m if for each $\varepsilon > 0$. *In this case, one writes* $f \to L\left(\left[\widetilde{W}_p^{\mathscr{I}(\lambda, m)}\right]_T\right)$.

Theorem 2. $s_T^{\mathscr{I}, m} \subset \widetilde{s}_T^{(\mathscr{I}_\lambda, m)}$ *if and only if*

$$\liminf_{t \to \infty} \frac{\mu_{\Delta_{(\lambda, m)}}([t + m - \lambda_t + t_0 - 1, t + m)_T)}{\mu_{\Delta_m}([m + t_0 - 1, t + m)_T)} > 0.$$

Proof. Assume that

$$\liminf \frac{\mu_{\Delta_{(\lambda,m)}}\left([t+m-\lambda_t+t_0-1,t+m)_T\right)}{\mu_{\Delta_m}\left([m+t_0-1,t+m)_T\right)} > 0.$$

Then, there exists a $\delta > 0$ such that

$$\frac{\mu_{\Delta_{(\lambda,m)}}\left([t+m-\lambda_t+t_0-1,t+m)_T\right)}{\mu_{\Delta_m}\left([m+t_0-1,t+m)_T\right)} \geq \delta$$

for sufficiently large t. For given $\varepsilon > 0$ we have,

$$\mu_{\Delta_m}\left(s \in [m+t_0-1,t+m)_T : |f(s)-L| \geq \varepsilon\right) \supseteq \mu_{\Delta_{(\lambda,m)}}\left(s \in [t+m-\lambda_t+t_0-1,t+m)_T : |f(s)-L| \geq \varepsilon\right.$$

Therefore,

$$\begin{aligned}
\frac{\mu_{\Delta_m}\left(s \in [m+t_0-1,t+m)_T : |f(s)-L| \geq \varepsilon\right)}{\mu_{\Delta_m}\left([m+t_0-1,t+m)_T\right)} &\geq \frac{\mu_{\Delta_{(\lambda,m)}}\left(s \in [t+m-\lambda_t+t_0-1,t+m)_T : |f(s)-L| \geq \varepsilon\right)}{\mu_{\Delta_m}\left([m+t_0-1,t+m)_T\right)} \\
&= \frac{\mu_{\Delta_{(\lambda,m)}}\left([t+m-\lambda_t+t_0-1,t+m)_T\right)}{\mu_{\Delta_m}\left([m+t_0-1,t+m)_T\right)} . \frac{\mu_{\Delta_{(\lambda,m)}}\left(s \in [t+m-\lambda_t+t_0-1,t+m)_T : |f(s)-L| \geq \varepsilon\right)}{\mu_{\Delta_{(\lambda,m)}}\left([t+m-\lambda_t+t_0-1,t+m)_T\right)} \\
&\geq \delta . \frac{\mu_{\Delta_{(\lambda,m)}}\left(s \in [t+m-\lambda_t+t_0-1,t+m)_T : |f(s)-L| \geq \varepsilon\right)}{\mu_{\Delta_{(\lambda,m)}}\left([t+m-\lambda_t+t_0-1,t+m)_T\right)} .
\end{aligned}$$

Then, for any $\eta > 0$ we get

$$\begin{aligned}
&\left\{ t \in T : \frac{\left|\left\{\mu_{\Delta_{(\lambda,m)}}\left(s \in [t+m-\lambda_t+t_0-1,t+m)_T : |f(s)-L| \geq \varepsilon\right)\right\}\right|}{\mu_{\Delta_{(\lambda,m)}}\left([t+m-\lambda_t+t_0-1,t+m)_T\right)} \geq \eta \right\} \\
&\qquad \subseteq \left\{ t \in T : \frac{\left|\left\{\mu_{\Delta_m}\left(s \in ([m+t_0-1,t+m)_T) : |f(s)-L| \geq \varepsilon\right)\right\}\right|}{\mu_{\Delta_m}\left([m+t_0-1,t+m)_T\right)} \geq \eta.\delta \right\} \in \mathscr{I},
\end{aligned}$$

and this completes the proof.

We will now investigate the relationship between $(\mathscr{I}_\lambda, m)$-uniform statistically convergence and $(\mathscr{I}_\lambda, m)$-uniformly strongly p-summability of functions on time scale.

Now, for the proof of Theorem 3 we give the following lemma.

Lemma 1. [1] *Let $f : T \to \mathbb{R}$ be a Δ-measurable function and*

$$\Omega(t,m,\lambda) = \{s \in [t+m-\lambda_t+t_0-1,t+m) : s \in \Omega\}$$

for $\varepsilon > 0$. In this case, we have

$$\mu_{\Delta_m}(\Omega(t,m,\lambda)) \leq \frac{1}{\varepsilon} \int_{\Omega(t,m,\lambda)} |f(s)-L|\,\Delta s \leq \frac{1}{\varepsilon} \int_{[t+m-\lambda_t+t_0-1,t+m)_T} |f(s)-L|\,\Delta s.$$

Proof. This can be proved by using a method similar to the approach in ([22]).

Theorem 3. *Let $f : T \to \mathbb{R}$ be a $\Delta_{(\lambda,m)}$-measurable function, $L \in \mathbb{R}$ and $0 < p < \infty$. Then, one gets the following.*

(i) $\left[\widetilde{W}_p^{\mathscr{I}(\lambda,m)}\right]_T \subset \widetilde{s}_T^{(\mathscr{I}_\lambda,m)}$.

(ii) *If f is $(\mathscr{I}_\lambda, m)$-uniformly strongly p-summable to L, then $f \to L\left(\widetilde{s}_T^{(\mathscr{I}_\lambda,m)}\right)$.*
(iii) *If $f \to L\left(\widetilde{s}_T^{(\mathscr{I}_\lambda,m)}\right)$ and f is a bounded function, then f is $(\mathscr{I}_\lambda, m)$-uniformly strongly p-summable to L.*

Proof. (i) Let $\varepsilon > 0$ and $f \to L\left(\left[\widetilde{W}_p^{\mathscr{I}(\lambda,m)}\right]_T\right)$. We can write

$$\int_{[t+m-\lambda_t+t_0-1,t+m)_T} |f(s)-L|^p \Delta s \geq \int_{\Omega(t,m,\lambda)} |f(s)-L|^p \Delta s \geq \varepsilon^p.\mu_{\Delta_{(\lambda,m)}}(\Omega(t,m,\lambda)),$$

and so,

$$\frac{1}{\varepsilon^p.\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)} \int_{[t+m-\lambda_t+t_0-1,t+m)_T} |f(s)-L|^p \Delta s$$
$$\geq \frac{\left|\left\{\mu_{\Delta_{(\lambda,m)}}(s \in [t+m-\lambda_t+t_0-1,t+m)_T : |f(s)-L| \geq \varepsilon)\right\}\right|}{\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)}.$$

Then for any $\delta > 0$,

$$\left\{t \in T : \frac{\left|\left\{\mu_{\Delta_{(\lambda,m)}}(s\in[t+m-\lambda_t+t_0-1,t+m)_T : |f(s)-L| \geq \varepsilon^p)\right\}\right|}{\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)} \geq \delta\right\}$$
$$\subseteq \left\{t \in T : \left|\frac{1}{\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)}.\int_{[t+m-\lambda_t+t_0-1,t+m)_T} |f(s)-L|^p \Delta s\right| \geq \varepsilon^p.\delta\right\}.$$

Since right hand belogs to $\mathscr{I}$ then left hand also belongs to $\mathscr{I}$ and this completes the proof.
(ii) Let f be $(\mathscr{I}_\lambda, m)$-uniformly strongly p-summable to L. For given $\varepsilon > 0$, let

$$\Omega(t,m,\lambda) = \{s \in [t+m-\lambda_t+t_0-1,t+m) : s \in \Omega\}$$

then it follows from Lemma 1 [1] that

$$\varepsilon^p.\mu_{\Delta_{(\lambda,m)}}(\Omega(t,m,\lambda)) \leq \int_{[t+m-\lambda_t+t_0-1,t+m)_T} |f(s)-L|^p \Delta s.$$

Dividing both sides of the last inequality by $\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)$, we obtain

$$\frac{\mu_{\Delta_{(\lambda,m)}}(\Omega(t,m,\lambda))}{\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)} \leq \frac{1}{\varepsilon^p \mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)}.\int_{[t+m-\lambda_t+t_0-1,t+m)_T} |f(s)-L|^p \Delta s.$$

Therefore for all $\delta > 0$ we have,

$$\left\{t \in T : \frac{\left|\left\{\mu_{\Delta_{(\lambda,m)}} s\in[t+m-\lambda_t+t_0-1,t+m)_T : |f(s)-L| \geq \varepsilon^p\right\}\right|}{\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)} \geq \delta\right\}$$
$$\subseteq \left\{t \in T : \left|\frac{1}{\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)}.\int_{[t+m-\lambda_t+t_0-1,t+m)_T} |f(s)-L|^p \Delta s\right| \geq \varepsilon^p.\delta\right\},$$

which yields $f \to L\left(\widetilde{s}_T^{(\mathscr{I}_\lambda,m)}\right)$.

(iii) Let f is a bounded function and $f \to L\left(\widetilde{s}_T^{(\mathscr{I}_\lambda,m)}\right)$. Then there exists a positive number M such that $|f(s)| \le M$ for all $s \in T$, and also

$$\left\{t \in T : \frac{\mu_{\Delta_m}(\Omega(t,m,\lambda))}{\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)} \ge \delta\right\} \in \mathscr{I}.$$

where $\Omega(t,m,\lambda)$ is as before. Since

$$\begin{aligned}\int\limits_{[t+m-\lambda_t+t_0-1,t+m)_T} |f(s)-L|^p \Delta s &= \int\limits_{\Omega(t,m,\lambda)} |f(s)-L|^p \Delta s + \int\limits_{[t+m-\lambda_t+t_0-1,t+m)_T \setminus \Omega(t,m,\lambda)} |f(s)-L|^p \Delta s \\ &\le (M+|L|^p).\int\limits_{\Omega(t,m,\lambda)} \Delta s + \varepsilon^p.\int\limits_{[t+m-\lambda_t+t_0-1,t+m)_T} \Delta s \\ &= (M+|L|^p).\mu_{\Delta_m}(\Omega(t,m,\lambda)) + \varepsilon^p.\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)\end{aligned}$$

we obtain

$$\frac{1}{\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)} \int\limits_{[t+m-\lambda_t+t_0-1,t+m)_T} |f(s)-L|^p \Delta s \le (M+|L|^p)\frac{\mu_{\Delta_m}(\Omega(t,m,\lambda))}{\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)} + \varepsilon^p.$$

Since ε is arbitrary,

$$\left\{t \in T : \left|\frac{1}{\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)}.\int\limits_{[t+m-\lambda_t+t_0-1,t+m)_T} |f(s)-L|^p \Delta s\right| \ge \varepsilon\right\}$$

$$\subseteq \left\{t \in T : \frac{\mu_{\Delta_m}(\Omega(t,m,\lambda))}{\mu_{\Delta_{(\lambda,m)}}([t+m-\lambda_t+t_0-1,t+m)_T)} \ge \frac{\varepsilon}{(M+|L|^p)}\right\} \in \mathscr{I}.$$

Therefore, we have $f \to L\left(\left[\widetilde{W}_p^{\mathscr{I}(\lambda,m)}\right]_T\right)$.

Definition 20. *Let $f : T \to \mathbb{R}$ be a Δ-measurable function and θ be a lacunary sequence. Assume also that $0 < p < \infty$. Then, f is $(\mathscr{I}_\theta, m)$-uniform strongly p-lacunary summable on T if there exists some $L \in \mathbb{R}$ such that*

$$\left\{t \in T : \frac{1}{\mu_\Delta\left(\left(k_{t-2t_0+m}, k_{t-t_0+m}\right]_T\right)}.\int\limits_{\left(k_{t-2t_0+m}, k_{t-t_0+m}\right]_T} |f(s)-L|^p \Delta s \ge \varepsilon\right\} \in \mathscr{I}.$$

In that case, $f \to L\left(\left[W^m_{\mathscr{I}_{\theta p}}\right]_T\right)$. The set of all $(\mathscr{I}_\theta, m)$-uniform strongly p-lacunary summable functions on T will be denoted by $\left[W^m_{\mathscr{I}_{\theta p}}\right]_T$. We will now investigate the relationship between (θ,m)-uniform $\mathscr{I}$-statistically convergence and $(\mathscr{I}_\theta, m)$-uniform strongly p-lacunary summability on time scale.

Theorem 4. *Let $f : T \to \mathbb{R}$ be a Δ-measurable function, θ be a lacunary sequence. Assume also that $0 < p < \infty$ and $L \in \mathbb{R}$. Then,*

(i) *If f is $(\mathscr{I}_\theta, m)$-uniform strongly p-lacunary summable to L, then $f \to L\left(s_T^{\mathscr{I}_\theta,m}\right)$.*

(ii) *If $f \to L\left(s_T^{\mathscr{I}_\theta,m}\right)$ and f is a bounded function, then f is $(\mathscr{I}_\theta, m)$-uniform strongly p-lacunary summable to L.*

Proof. (i) Suppose f is (θ,m)-uniform strongly p $\mathscr{I}$-lacunary summable to L. For given $\varepsilon > 0$, let

$$\Omega(t,\theta,m) = \left\{s \in \left(k_{t-2t_0+m+1}, k_{t-t_0+m+1}\right]_T : |f(s)-L| \ge \varepsilon\right\}$$

on T. Then it follows

$$\varepsilon^p.\mu_\Delta\left(\Omega\left(t,\theta,m\right)\right) \leq \int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_T} |f(s)-L|^p \Delta s$$

from Lemma 1 in [1]. Dividing this inequality by $\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_T\right)$, we obtain

$$\frac{\mu_\Delta\left(\Omega\left(t,\theta,m\right)\right)}{\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_T\right)} \leq \frac{1}{\varepsilon^p}.\frac{\int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_T} |f(s)-L|^p \Delta s}{\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_T\right)}.$$

Then, for any $\delta > 0$

$$\left\{t\in T : \frac{\left|\left\{\mu_\Delta\left(s\in\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_T : |f(s)-L|\geq\varepsilon\right)\right\}\right|}{\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_T\right)} \geq \delta\right\}$$

$$\subseteq \left\{t\in T : \frac{1}{\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_T\right)}.\int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_T} |f(s)-L|^p \Delta s \geq \varepsilon^p.\delta\right\},$$

which yields that $f \to L\left(s_T^{\mathscr{I}_\theta,m}\right)$.

(ii) Let f is a bounded function and $f \to L\left(s_T^{\mathscr{I}_\theta,m}\right)$. Then there exists a positive number M such that $|f(s)| \leq M$ for all $s \in T$. Since

$$\begin{aligned}\int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_T} |f(s)-L|^p \Delta s &= \int\limits_{\Omega(t,\theta,m)} |f(s)-L|^p \Delta s + \int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_T \setminus \Omega(t,\theta,m)} |f(s)-L|^p \Delta s \\ &\leq \left(M+|L|^p\right)\int\limits_{\Omega(t,\theta,m)} \Delta s + \varepsilon^p.\int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_T} \Delta s \\ &= \left(M+|L|^p\right).\mu_\Delta\left(\Omega\left(t,\theta,m\right)\right) + \varepsilon^p.\mu_\Delta\left(\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_T\right),\end{aligned}$$

we obtain

$$\left\{t\in T : \left|\frac{1}{\mu_\Delta\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_T}.\int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_T} |f(s)-L|^p \Delta s\right| \geq \varepsilon\right\} \subseteq \left\{t\in T : \frac{\left|\mu_\Delta\left(\Omega\left(t,\theta,m\right)\right)\right|}{\mu_\Delta\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_T} \geq \frac{\varepsilon}{\left(M+|L|^p\right)}\right\} \in \mathscr{I}.$$

Therefore, we have $f \to L\left(\left[W_{\mathscr{I}_{\theta_p}}^m\right]_T\right)$.

Theorem 5. *Let* $\theta = \left\{k_{t-t_0+m+1}\right\}$ *be a lacunary sequence for* $t \in T$. *Then*

(i) $\left[W_{\mathscr{I}_{\theta_p}}^m\right]_T \subset \left[W_{\mathscr{I}_p}^m\right]_T$ *if* $\limsup_t\left(\frac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) < \infty$,

(ii) $\left[W_{\mathscr{I}_p}^m\right]_T \subset \left[W_{\mathscr{I}_{\theta_p}}^m\right]_T$ *if* $\liminf_t\left(\frac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) > 1$,

(iii) $\left[W_{\mathscr{I}_p}^m\right]_T = \left[W_{\mathscr{I}_{\theta_p}}^m\right]_T$ *if* $1 < \liminf_t\left(\frac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) < \limsup_t\left(\frac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) < \infty$.

Proof. We can prove by using similar techniques to Theorem 2.2, Theorem 2.3 and Theorem 2.4 of [17] in case of $p = 1$ by using B-admissible ideal.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Y. Altin, H. Koyunbakan and E. Yilmaz, Uniform statistical convergence on time scales, J. Appl. Math., (2014) Art. ID 471437.

[2] V. Báláz, T. Šalát, Uniform density u and corresponding $\mathscr{I}_u$-convergence, Math. Commun., 11(1) (2006) 1-7.

[3] A. Cabada and D. R. Vivero, Expression of the Lebesque Δ-integral on time scales as a usual Lebesque integral; application to the calculus of Δ-antiderivates, Math. Comp. Model., 43 (2006) 194-207.

[4] J. S. Connor, The statistical and strong p-Cesàro convergence of sequences, Analysis, 8 (1988) 47-63.

[5] M. Mursaleen, $\lambda-$Statistical Convergence, Math. Slovaca, 50(1) (2000) 111-115.

[6] P. Das, E. Savaş and S. Kr. Ghosal, On generalized of certain summability methods using ideals, Appl. Math. Letter, 36 (2011) 1509-1514.

[7] T. C. Brown, A. R. Freedman, The uniform density of sets of integers and Fermat's last theorem," La Soci´et´e Royale du Canada. L'Acad´emie des Sciences, 12(1) (1990) 1–6.

[8] D. Borwein, Linear functionals connected with strong Cesàro summability, Journal of the London Mathematical Society, 40 (1965) 628–634.

[9] M. Bohner and G. Sh. Guseinov, Multiple integration on time scales, Dynamic Systems and Applications, 14(3) (2005) 579–606.

[10] S. Hilger, Ein Makettenkalkl mit Anwendung auf Zentrumsmannigfaltigkeiten Ph.D. Thesis, Universtat Wurzburg, 1988.

[11] Fast H., Sur la convergence statistique, Coll. Math., 2 (1951) 241-244.

[12] Freedman A. R., Sember J. J. and Rapnael M., Some Cesàro type summability spaces, Proc. London. Math. Soc., 37(3) (1978) 508-520.

[13] J. A. Fridy, On statistical convergence, Analysis, 5 (1985) 301-313.

[14] J. A. Fridy and C. Orhan, Lacunary statistical convergence, Pac. J. Math., 160(1) (1993) 43-51.

[15] J. A. Fridy and C. Orhan, Lacunary statistical summability, J. Math. Anal. Appl., 173(2) (1993) 497-504.

[16] F. Nuray, Uniform statistical convergence, Sci. Engineer. J. Firat Univ., 11(3) (1999) 219-222.

[17] F. Nuray, B. Aydin, Strongly summable and statistically convergent functions, Inform. Tech. Valdymas, 30 (1) (2004) 74-76.

[18] F. Nuray, λ-strongly summable and λ-statistically convergent functions, Iranian Journal of Science and Technology, Transaction A: Science, 34(4) (2010) 335–338.

[19] R. A. Raimi, Convergence, density, and μ-density of bounded sequences, Proc. Amer. Math. Soc., 14 (1963) 708-712.

[20] P. Kostyrko, T. Šalát T. and W. Wilezyński,, $\mathscr{I}$-Convergence, Real Anal. Exchange, 26(2) (2000) 669-686.

[21] M. S. Seyyidoglu and N. O. Tan, A note on statistical convergence on time scale, J. Inequal. Appl., (2012) Art. ID 219.

[22] C. Turan and O. Duman, Statistical convergence on time scales and its characterizations, Advances in Applied Mathematics and Approximation Theory, Springer Proc. Math. Stat., 41 (2013) 57-71.

[23] C. Turan and O. Duman, Convergence Methods on Time Scales, 11th international conference of numerical analysis and Applied Mathematics, AIP Conference Proc., 1558 (2013) 1120-1123.

[24] E. Yilmaz, Y. Altin and H. Koyunbakan, λ-Statistical convergence on time scales, Dyn. Cont. Disc. Impul. Syst. Ser. A: Math. Anal., 23 (2016) 69-78.

[25] E. Yilmaz, S. A. Mohiuddine, Y. Altin and H. Koyunbakan, Uniform Lacunary Statistical convergence on time scales, International Journal of Analysis and Applications, 14(1) (2017) 99-106.

[26] Cabada, A., Extremal Solutions and green's functions of higher order periodic boundary value problems in time scales, J. Math. Anal. Appl., 290 (2004) 35-54.

[27] Erbe, L., Peterson, A. and Saker, S. H., Oscillation criteria for second-order non-linear dynamic equations in time scales, J. London Math. Soc., 67(3) (2003) 701-714.
[28] S. A. Mohiuddine, A. Alotaibi and M. Mursaleen, Statistical summability $(C;1)$ and a Korovkin type approximation theorem, J. Inequal. Appl., (2012), Art. ID 172.
[29] D. Rath, B. C. Tripathy, On statistically convergent and statistically Cauchy sequences, Indian J. Pure Appl. Math., 25(4) (1994) 381-386.
[30] B. C. Tripathy, On statistical convergence, Proc. Est. Aca. Sci. Phy. 47(4) (1998) 299-303.
[31] C. Belen and S. A. Mohiuddine, Generalized weighted statistical convergence and application, Appl. Math. Comput., 219 (2013), 9821-9826.
[32] Pal, S. K. and Malik, P., On a critria of weak Ideal convergence and some Further results on weak Ideal Summability, Southest Asian Bulletin on Mathematics, 39 (2015) 685-694.
[33] Rudin, W., Real and Complex Analysis, Third ed., McGraw-Hill Co., 1987.
[34] I. J. Maddox, Spaces of strongly summable sequences, Quart. J. Math., 18(1) (1967) 345-355.
[35] I. J. Maddox, A new type of convergence, Mathematical Proceedings of the Cambridge Philosophical Society, 83(1) (1978) 61–64.
[36] M. Et, S. A. Mohiuddine and A. Alotaibi, On λ-statistical convergence and strongly λ-summable functions of order α , J. Inequal. Appl., (2013) Art. ID 469.
[37] Guseinov G. S., Integration on time scales, J. Math. Anal. Appl., 285(1) (2003) 107-127.
[38] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math., 2 (1951) 73-74.
[39] A. Zygmund, Trigonometrical Series, Monogr. Mat., vol. 5. Warszawa-Lwow 1935.