Constancy of $\varphi$-holomorphic sectional curvature of an indefinite Sasaki-like almost contact manifold with $B$-metric

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Received: 1 January 2017, Accepted: 22 July 2018
Published online: 10 December 2018.

Abstract: The aim of the present paper is to establish a criterion for an indefinite Sasaki-like almost contact manifold with $B$-metric to reduce to a space of $\varphi$-holomorphic sectional curvature.

Keywords: Sasaki-like almost contact manifolds, $B$-metric, $\varphi$-holomorphic sectional curvature.

1 Introduction

Ganchev et al. [3] defined the odd-dimensional version of almost complex manifolds with Norden metric [8,4,2] known as the almost contact manifolds with $B$-metric (Norden metric). Later, Ivanov et al. [5] introduced a new class of almost contact manifolds with $B$-metric namely Sasaki-like almost contact Complex Riemannian manifolds with $B$-metric, which is analogue to indefinite Sasakian manifold.

Tanno [9] proved the following result for an almost Hermitian manifold $(M^{2n}, g, J)$ to reduce to a space of constant holomorphic sectional curvature.

**Theorem 1.** [9] Let dimension $(2n \geq 4)$, assume that almost Hermitian manifold $(M^{2n}, g, J)$ satisfies

$$R(JX, JY, JZ, JX) = R(X, Y, Z, X),$$

(1)

for every tangent vectors $X, Y$ and $Z$. Then $(M^{2n}, g, J)$ is of constant holomorphic sectional curvature at $x$, if and only if,

$$R(X, JX)X \text{ is proportional to } JX,$$

(2)

for every tangent vector $X$ at $x$ in $M$.

Tanno [9] also extended the above Theorem 1 for the Sasakian manifolds as follows.

**Theorem 2.** [9] A Sasakian manifold $(M^{2n+1}, \phi, \eta, \xi, g)$ of dimension $\geq 5$, is of constant $\phi$-sectional curvature if and only if

$$R(X, \phi X)X \text{ is proportional to } \phi X$$

(3)

for every vector field $X$ such that $g(X, \xi) = 0$, where $\xi$ is a characteristic vector field of $M$.

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Further, Nagaich [7] generalized the Theorem 1 for an *indefinite almost Hermitian manifold* and provided the following characterization.

**Theorem 3.** Let \((M^{2n}, g, J)\) of dimension \(2n\), where \(n \geq 2\) be an indefinite almost Hermitian manifold satisfying (1). Then \(M\) is of constant holomorphic sectional curvature at \(p\), if and only if,

\[
R(X, JX)X \text{ is proportional to } JX,
\]

for every tangent vector \(X\) at \(p \in M\).

And later, Kumar et al.[6] proved the generalized version of the Theorem 2 for an *indefinite Sasakian manifold* as follows.

**Theorem 4.** Let \((M^{2n+1}, \varphi, \xi, \eta)\) \((2n \geq 4)\) be an indefinite Sasakian manifold. Then \(M\) is of constant \(\varphi\)-sectional curvature if and only if

\[
R(X, \varphi X)X \text{ is proportional to } \varphi X
\]

for every tangent vector field \(X\) such that \(g(X, \xi) = 0\), where \(\xi\) is a characteristic vector field of \(M\).

Recently, we have generalized the Theorem 3 to the setting of an *almost complex manifold with Norden metric* as

**Theorem 5.** [1] Let \((M^{2n}, g, J)\) \((2n \geq 4)\) be an indefinite almost complex manifold with Norden metric satisfying (1). Then \((M^{2n}, g, J)\) is of constant holomorphic sectional curvature at \(p\) if and only if

\[
R(X, JX)X \text{ is proportional to } \alpha X + \beta JX,
\]

where \(\alpha\) and \(\beta\) are the functions of holomorphic sectional curvature \(H(X)\), for every tangent vector \(X\) at \(p \in M\).

In this paper, we have extended the Theorem 5 to the setting of a Sasaki-like almost contact manifold with \(B\)-metric to reduce to a space of constant \(\varphi\)-holomorphic sectional curvature.

**Theorem 6.** Let \((M^{2n+1}, \varphi, \zeta, \eta)\) be an indefinite Sasaki-like almost contact manifold with \(B\)-metric. Then \(M\) is of constant \(\varphi\)-holomorphic sectional curvature if and only if

\[
R(X, \varphi X)X \text{ is proportional to } \gamma X + \delta \varphi X,
\]

where \(\gamma\) and \(\delta\) are the functions of \(\varphi\)-holomorphic sectional curvature \(H(X)\), for every tangent vector \(X\) such that \(g(X, \zeta) = 0\), where \(\zeta\) is a characteristic vector field of \(M\).

**2 Preliminaries**

**2.1 Almost contact manifold with \(B\)-metric**

Let \((M^{2n+1}, \varphi, \xi, \eta)\) be an almost contact manifold with \(B\)-metric \(\bar{g}\), i.e., \(M\) is a \((2n+1)\)-dimensional smooth manifold endowed with an almost contact structure \((\varphi, \xi, \eta)\) and equipped with a pseudo-Riemannian metric \(\bar{g}\), such that the following relations are satisfied [3],

\[
\varphi \xi = 0, \quad \eta(\xi) = 1,
\]

\[
\eta(X) = \bar{g}(X, \xi), \quad \eta(\varphi X) = 0,
\]
the tensor field $F$ of type $(0,3)$ is defined on $\tilde{M}$ as follows
\[ F(X,Y,Z) = \tilde{g}((\nabla_X \phi)Y, Z) \]
and the following properties hold in general [3]:
\[ F(X,Y,Z) = F(X,Y) = F(X,\phi Y, \phi Z) + \eta(Y)F(X,\zeta, Z) + \eta(Z)F(X,Y, \zeta), \]
for any $X,Y,Z \in T\tilde{M}$. The relations of $F$ with $\nabla_X \phi$ and $\nabla \eta$ are given by:
\[ (\nabla_X \phi)Y = g(\nabla_X \zeta, Y) = F(X,\phi Y, \phi \zeta), \quad \eta(\nabla_X \phi) = 0, \quad \phi(\nabla_X \phi) = \nabla_X \zeta \]
In [3], Ganchev et al. defined eleven basic classes $\tilde{F}_i (i = 1,2, ..., 11)$ of almost contact manifolds with $B$-metric and classified the almost contact manifolds with $B$-metric in terms of the tensor $F$. The intersection of these basic classes is the class $F_0$, which is analogue to Kaehler manifold with Norden metric and is determined by the condition
\[ F(X,Y,Z) = 0(\nabla \phi = \nabla \eta = \nabla \zeta = 0). \]

**Definition 1. [5]** An almost contact manifold $(\tilde{M}, \phi, \zeta, \eta, \tilde{g})$ with $B$-metric is called Sasaki-like if the structure tensors $(\phi, \zeta, \eta, \tilde{g})$ satisfy the following equalities
\[ F(X,Y,Z) = F(\zeta,Y,Z) = F(\zeta,\zeta, Z) = 0, \]
\[ F(X,Y, \zeta) = -\tilde{g}(X,Y). \]

Also, the covariant derivative $\nabla \phi$ satisfies the following equality
\[ (\nabla_X \phi)Y = -\tilde{g}(X,Y)\zeta - \eta(Y)X + 2\eta(X)\eta(Y)\zeta. \]
A non-zero tangent vector field $U$ is classified in the following types
(i) spacelike if $\tilde{g}(U,U) > 0$,
(ii) timelike if $\tilde{g}(U,U) < 0$,
(iii) null (lightlike) if $\tilde{g}(U,U) = 0$, $U \neq 0$. 

\[ \tilde{g}(\phi X, \phi Y) = -\tilde{g}(X,Y) + \eta(X)\eta(Y), \]
\[ \tilde{g}(\phi X, Y) = \tilde{g}(X, \phi Y), \]
\[ \tilde{g}(X,Y) = \tilde{g}(\phi X, Y) + \eta(X)\eta(Y), \]
2.2 Curvature properties

Let the curvature tensor $R$ of $\bar{\nabla}$ on $\bar{M}$ is given by

$$R(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z.$$ 

The corresponding curvature $(0,4)$-tensor with respect to $\bar{g}$ is given by

$$R(X,Y,Z,W) = \bar{g}(R(X,Y)Z,W)$$

and satisfies the following properties

$$RX,Y,Z,W = -R(Y,X,Z,W) = -R(X,Y,W,Z),$$

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0,$$

$$R(X,Y,Z,W) = -R(X,Y,\varphi Z,\varphi W),$$

for all tangent vector fields $X,Y,Z,W$ on $\bar{M}$.

The associated curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ on $\tilde{M}$ is defined as

$$\tilde{R}(X,Y,Z,W) = R(X,Y,\varphi Z,\varphi W).$$

Thus, for the curvature tensor $R$, we have

$$R(X,Y,Z,\varphi W) = R(X,Y,\varphi Z,\varphi W). \quad (18)$$

Let $\alpha$ denote a non-degenerate 2-plane in the tangent space $T_p\bar{M}$. Then the sectional curvature for $\alpha$ with respect to $\bar{g}$ and $R$ is given by

$$K(\alpha, p) = \frac{R(U,V,U,V)}{\bar{g}(U,U)\bar{g}(V,V) - \bar{g}(U,V)^2}. \quad (19)$$

where $\{U,V\}$ is an orthogonal basis of $\alpha$ and $p \in \bar{M}$.

**Definition 2.** A 2-plane $\alpha = \{U, \varphi U\}$, where $U$ is orthonormal to $\zeta$ is known as $\varphi$-holomorphic section (respectively, a $\zeta$-section) if $\alpha = \varphi \alpha$ (respectively, $\zeta \in \alpha$) and the curvature associated with this is said to be $\varphi$-holomorphic sectional curvature, denoted by $H(U)$ and given as

$$H(U) = \frac{R(U,\varphi U,U,\varphi U)}{\bar{g}(U,U)\bar{g}(\varphi U,\varphi U) - \bar{g}(U,\varphi U)^2}. \quad (20)$$

Moreover, if $H(U)$ is always constant with respect to every unit tangent vector $U \in T\bar{M}$, then $\bar{M}$ is said to be of constant $\varphi$-holomorphic sectional curvature or a Sasakian space form.

2.3 Sasaki-like almost contact manifold with B-metric

In [5], Ivanov defined the odd dimensional version of an indefinite Kaehler manifold known as Sasaki-like almost contact manifold with $B$-metric and proved the following result.
Lemma 1. [5] For a Sasaki-like almost contact manifold $(\tilde{M}, \varphi, \zeta, \eta, \bar{g})$ with $B$-metric the next formula holds

$$R(X, Y, \varphi Z, \varphi U) - R(X, Y, Z, \varphi U) = \{\bar{g}(Y, Z) - 2\eta(Y)\eta(Z)\} \bar{g}(X, \varphi U) + \{\bar{g}(Y, U) - 2\eta(Y)\eta(U)\} \bar{g}(X, \varphi Z)$$

$$- \{\bar{g}(X, Z) - 2\eta(X)\eta(Z)\} \bar{g}(Y, \varphi U) - \{\bar{g}(X, U) - 2\eta(X)\eta(U)\} \bar{g}(Y, \varphi Z).$$ (21)

In particular, we have

$$R(X, Y)\zeta = \eta(Y)X - \eta(X)Y$$ (22)

and

$$R(\zeta, X)\zeta = -X$$

The equation (21) further implies

$$R(X, Y)\varphi Z = \varphi R(X, Y)Z - 2\varphi \eta(Z)R(X, Y)\zeta - \bar{g}(X, Z)\varphi Y + \bar{g}(X, \varphi Z)Y$$

$$- \bar{g}(Y, \varphi Z)X + 2\{\bar{g}(Y, \varphi Z)\eta(X) - \bar{g}(X, \varphi Z)\eta(Y)\}\zeta.$$ (23)

Replacing $Y$ by $\varphi X$ and $Z$ by $\varphi X$ in above equation (23) and use of (22) yields,

$$R(X, \varphi X)X = -\{R(X, \varphi X)\varphi X + (\eta(X))^2 \varphi X + 2\bar{g}(X, \varphi X)X + 2\bar{g}(\varphi X, \varphi X)\varphi X - 3\bar{g}(X, \varphi X)\eta(X)\}\zeta.$$ (24)

### 3 Constancy of $\varphi$-holomorphic sectional curvature

Now we will prove the main result.

**Proof.** Initially assume that $\tilde{M}$ be an indefinite Sasaki-like almost contact manifold with $B$-metric, then using formula (20), we obtain

$$R(X, \varphi X)X = -H(X)\varphi X + H(X)\varphi X.$$ (25)

where $X$ denotes a unit tangent vector such that $\bar{g}(X, \varphi X) = \rho(\neq 0)$. By using the fact that $\tilde{M}$ is having constant $\varphi$-holomorphic sectional curvature and the equation (25), the necessity of the assertion follows. To prove the converse part, the following two cases have been considered.

**Case I.** For the space-like, or in other words, $\bar{g}(X, X) = \bar{g}(Y, Y)$. Let $\{X, Y\}$ denote an orthonormal pair of vectors in $\tilde{M}$ such that

$$\bar{g}(X, X) = -\bar{g}(\varphi X, \varphi X) = 1,$$

$$\bar{g}(Y, Y) = -\bar{g}(\varphi Y, \varphi Y) = 1,$$

$$\bar{g}(X, \varphi X) = \bar{g}(Y, \varphi Y) = \rho(\neq 0)$$

and

$$\bar{g}(X, \varphi Y) = \bar{g}(\varphi X, Y) = 0.$$

In this case, $X^{**}$ and $Z^{**}$ be defined by

$$X^{**} = \cos \theta X + \sin \theta Y$$

and

$$Z^{**} = -\sin \theta X + \cos \theta Y.$$
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Clearly, \( \{X^*, Z^*\} \) also form an orthonormal pair of vectors in \( \bar{M} \) and using the above relation (7), we have

\[
R(X^*, \phi X^*) X^* \sim \gamma X^* + \delta \phi X^*.
\]

Taking inner product of above equation with \( \phi Z^* \), we have

\[
R(X^*, \phi X^*, X^*, \phi Z^*) = 0.
\]

Also, by using the linear properties of Riemannian curvature tensor \( R \), we obtain

\[
cos \theta \sin \theta \{ -\cos^2 \theta R(X, \phi X, X, \phi X) + \sin^2 \theta R(Y, \phi Y, Y, \phi Y) + (\cos^2 \theta - \sin^2 \theta) R(Y, \phi Y, X, \phi X) \} = 0.
\]  \( \text{(26)} \)

Considering \( \theta = \frac{\pi}{4} \) yields,

\[
H(X) = H(Y).
\]

If \( \{Z, W\} \) is a \( \phi \)-holomorphic section then \( \phi Z = pZ + qZ \), for any scalars \( p \) and \( q \). Thus, \( \{Z, \phi Z\} = \{Z, pZ + qZ\} = \{Z, W\} \) and similarly \( \{W, \phi W\} = \{Z, W\} \). Therefore, \( \{Z, \phi Z\} = \{W, \phi W\} \) and hence \( H(Z) = H(W) \).

On the contrary if \( \{Z, W\} \) is not a \( \phi \)-holomorphic section then there must exist unit vectors \( X \in \{Z, \phi Z\}^\perp \) and \( Y \in \{W, \phi W\}^\perp \) that determine a \( \phi \)-holomorphic section \( \{X, Y\} \) and thus, we have

\[
H(Z) = H(X) = H(Y) = H(W),
\]

which proves that any \( \phi \)-holomorphic section has the same \( \phi \)-holomorphic sectional curvature.

Now, let the \( \text{dim}(\bar{M}) = 5 \) and using the properties of curvature tensor \( R \), the following relations hold.

\[
R(X, \phi X) X = H(X) \{ -\rho X + \phi X \} \tag{27}
\]

\[
R(X, \phi X) Y = \frac{1}{1 + \rho^2} \{ R(X, \phi X, Y, \phi Y) (\rho Y - \phi Y) \}
\]

\[
R(X, \phi Y) X = \frac{1}{1 + \rho^2} \{ R(X, \phi Y, X, \phi Y) (\rho X + \phi Y) \}
\]

\[
R(Y, \phi X) X = \frac{1}{1 + \rho^2} \{ R(Y, \phi X, X, \phi Y) (\rho Y - \phi Y) \}
\]

\[
R(X, \phi Y) Y = \frac{1}{1 + \rho^2} \{ R(X, \phi Y, Y, \phi X) (\rho X - \phi X) \}
\]

\[
R(Y, \phi X) Y = \frac{1}{1 + \rho^2} \{ R(Y, \phi X, Y, \phi X) (\rho X + \phi X) \}
\]

\[
R(Y, \phi Y) X = \frac{1}{1 + \rho^2} \{ R(Y, \phi Y, X, \phi X) (\rho X - \phi X) \}
\]

\[
R(Y, \phi Y) Y = H(Y) \{ -\rho Y + \phi Y \}. \tag{28}
\]

Now, define \( X^* = dX + eY \) where \( d^2 + e^2 = 1 \), then making use of the above algebraic relations (28), we have

\[
R(X^*, \phi X^*) X^* = E_1 X + E_2 Y + E_3 \phi X + E_4 \phi Y,
\]  \( \text{(29)} \)

where

\[
E_3 = d^3 H(X) - \frac{de^2}{1 + \rho^2} E_5, \quad E_4 = e^3 H(X) - \frac{d^2 e}{1 + \rho^2} E_5,
\]
and
\[ E_5 = R(X, \varphi Y, Y, \varphi X) + R(Y, \varphi X, Y, \varphi X) + R(Y, \varphi Y, X, \varphi X). \]

On the other hand, equation (27) yields,
\[ R(X^{**}, \varphi X^{**})X^{**} = H(X^{**})\{ -\rho X^{**} + \varphi X^{**}\} = H(X^{**})\{ \rho dX + \rho eY - d\varphi X - e\varphi Y\}. \]  

Comparing the equations (29) and (30), we obtain
\[ d^2H(X) - \frac{e^2}{(1 + \rho^2)}E_5 = H(X^{**}), \quad e^2H(X) - \frac{d^2}{(1 + \rho^2)}E_5 = H(X^{**}), \]

upon solving the above equations, we have
\[ E_5 = -(1 + \rho^2)H(X) \]

and hence consequently
\[ H(X^{**}) = (d^2 + e^2)H(X) = H(X). \]

Similarly, on the parallel lines, we prove that
\[ H(Y^{**}) = H(Y). \]

Thus, we have proved that the manifold \( \bar{M} \) is of constant \( \varphi \)-holomorphic sectional curvature.

**Case II:** When the metric is timelike, or in other words, \( \bar{g}(X, X) = -\bar{g}(Y, Y) \), where either \( X \) and \( Y \) are spacelike and timelike vectors, respectively or vice versa. Let \( \{X, Y\} \) denote a pair of orthonormal vectors in \( \bar{M} \) such that
\[
\begin{align*}
\bar{g}(X, X) &= -\bar{g}(\varphi X, \varphi X) = 1, \\
\bar{g}(Y, Y) &= -\bar{g}(\varphi Y, \varphi Y) = -1, \\
\bar{g}(X, \varphi X) &= -\bar{g}(Y, \varphi Y) = \rho(\neq 0) \\
\end{align*}
\]

and
\[ \bar{g}(X, \varphi Y) = \bar{g}(\varphi X, Y) = 0. \]

Further, we define \( X^{**} \) and \( Z^{**} \) by
\[ X^{**} = \cosh \theta X + \sinh \theta Y \]

and
\[ Z^{**} = -\sinh \theta \varphi X + \cosh \theta \varphi Y. \]

then \( X^{**}, Z^{**} \) form an orthonormal pair of vectors in \( \bar{M} \) and therefore making use of the relation (7), we have
\[ R(X^{**}, \varphi X^{**})X^{**} \sim \gamma X^{**} + \delta \varphi X^{**}. \]

Taking inner product of above equation with \( Z^{**} \), we obtain,
\[ R(X^{**}, \varphi X^{**}, X^{**}, Z^{**}) = 0, \]

further, using the linearity properties of curvature tensor, we have
\[ \cosh \theta \sinh \theta \{ \cos^2 h \theta H(X) - \sin^2 h \theta H(Y) - (\cos^2 h \theta - \sin^2 h \theta)R(X, \varphi X, Y, \varphi Y) \} = 0. \]
Considering $\theta = \frac{d}{4}$, we get

$$H(X) = H(Y).$$

Further, using the same argument given in Case I, we obtain that any holomorphic section has same sectional curvature.

Now, assuming the $\text{dim}(\bar{M}) = 5$ and using the curvature properties of curvature tensor $R$, we have the following relations

$$R(X, \varphi X)X = -H(X)(\rho X - \varphi X)$$

$$R(X, \varphi X)Y = \frac{1}{1 + \rho^2}[R(X, \varphi X, Y)(-\rho Y + \varphi Y)]$$

$$R(X, \varphi Y)X = \frac{1}{1 + \rho^2}[R(X, \varphi X, Y)(-\rho Y + \varphi Y)]$$

$$R(Y, \varphi X)X = \frac{1}{1 + \rho^2}[R(X, \varphi X, Y)(-\rho Y + \varphi Y)]$$

$$R(Y, \varphi Y)X = \frac{1}{1 + \rho^2}[R(X, \varphi X, Y)(X + \rho \varphi X) + R(X, \varphi Y, X)(\rho X - \varphi X)]$$

$$R(Y, \varphi Y)Y = -H(Y)(-\rho Y + \varphi Y).$$

Now, define $X'' = dX + eY$ with $d^2 - e^2 = 1$, then using the above relations, we have

$$R(X'', \varphi X'')X'' = E_3X + E_2Y + E_3\varphi X + E_4\varphi Y,$$

where

$$E_3 = d^2H(X)(1 + \rho^2) - \frac{de^2}{(1 + \rho^2)}E_5, \quad E_4 = -e^3H(X)(1 + \rho^2) + \frac{d^2e}{(1 + \rho^2)}E_5,$$

and $E_5 = R(X, \varphi Y, Y, \varphi X) + R(Y, \varphi X, Y, \varphi Y) + R(Y, \varphi X, X, \varphi X)$. On the other hand, using (32), we have

$$R(X'', \varphi X'')X'' = -H(X'')(\rho dX + \rho eY - d\varphi X - e\varphi Y).$$

Comparing (34) and (35), we obtain

$$d^2H(X) - \frac{e^2}{(1 + \rho^2)}E_5 = H(X''), \quad -e^2H(X) + \frac{d^2}{(1 + \rho^2)}E_5 = H(X''),$$

on solving these equations, we obtain

$$E_5 = (1 + \rho^2)H(X)$$

and consequently

$$H(X'') = (d^2 - e^2)H(X) = H(X).$$

Similarly, we can prove

$$H(Y'') = H(Y).$$

Thus, the manifold $\bar{M}$ is of constant $\varphi$-holomorphic sectional curvature.
Hence, we conclude that Theorem 2 can be derived by considering \( g(X, \phi X) = \rho = 0 \), in Theorem 6.

Similarly, by taking \( g(X, \phi X) = \rho = 0 \), in Theorem 6, the constancy of \( \phi \)-holomorphic sectional curvature can be derived for an indefinite almost Sasakian manifold with some minor changes and thus, Theorem 6 provides a generalization of Theorem 4.

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

**References**


