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Generalized Berinde-Type contractions in partially ordered G_p-metric spaces

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Abstract: In this manuscript, we view generalized Berinde-type contractions, which is known as generalized almost contractions in the literature, in the framework of partially ordered G_p-metric spaces to get some common fixed point results for self-mappings f and g and some fixed point results for a single mapping f. Presented theorems generalize several previously obtained classical results. We also state some examples which show the validity of our results.

Keywords: Common fixed point, partially ordered set, G_p-metric space, weakly increasing maps, (c)-comparison function.

1 Introduction and preliminaries

Fixed point theory has been an important research field in solving deviational problems in nonlinear analysis. The prime goal of studies in fixed point theory is to obtain solutions for fixed point equation given by $Tx = x$, where T is a self mapping on X and $x \in X$. For this reason, numerous fixed point and common fixed point theorems have been proved for different generalizations of the term of metric space. One of this generalizations is G_p-metric space, which is defined by Zand and Nezhad [1] as a unification of the terms of partial metric space [2] and G-metric space [3]. Inspired by this remarkable study, Aydi et al. [4] obtained certain fixed point results which generalize the results of Ilić et al. [5] from partial metric space to G_p-metric space. In the light of these studies, many fixed point results for contraction type mappings on G_p-metric spaces have been considered. Some of this results are mentioned in [6–13].
Initially, we call to mind some essential definitions and results which shall be helpful for the rest of this study.

Definition 1. *[1] The pair (X, G_p) is called a G_p-metric space where X is a non empty set and $G_p : X \times X \times X \to [0, +\infty)$ is a function if the following axioms hold,*

G_{p_1}. $x = y = z$ *if* $G_p(x,y,z) = G_p(z,z,z) = G_p(y,y,y) = G_p(x,x,x)$,
G_{p_2}. $0 \le G_p(x,x,x) \le G_p(x,x,y) \le G_p(x,y,z)$ *for all* $x,y,z \in X$,
G_{p_3}. $G_p(x,y,z) = G_p(x,z,y) = G_p(y,z,x) = \ldots$, *symmetry in all three variables,*
G_{p_4}. $G_p(x,y,z) \le G_p(x,a,a) + G_p(a,y,z) - G_p(a,a,a)$ *for any* $x,y,z,a \in X$.

With G_{p_2} assumption, it is very easy to demonstrate that

$$G_p(x,x,y) = G_p(x,y,y)$$

holds for all $x,y \in X$. More precisely, the concerned space is symmetric. We give a fundamental example of G_p-metric space for a better comprehending of the topic, as the following.

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Example 1. [1] Let $X=[0,\infty)$ and $G_p : X\times X\times X\to X$ be a function identified by $G_p(x,y,z)=\max\{x,y,z\}$, for all $x,y,z\in X$. Clearly (X,G_p) is a symmetric G_p-metric space. However, it is not a G-metric space.

The next proposition gives some properties of a G_p-metric space.

Proposition 1. *[1] Let (X,G_p) be a G_p-metric space. In that case, for any x,y,z and $a\in X$, the following properties are true:*

i) $G_p(x,y,z)\le G_p(x,x,y)+G_p(x,x,z)-G_p(x,x,x)$,
ii) $G_p(x,y,y)\le 2G_p(x,x,y)-G_p(x,x,x)$,
iii) $G_p(x,y,z)\le G_p(x,a,a)+G_p(y,a,a)+G_p(z,a,a)-2G_p(a,a,a)$,
iv) $G_p(x,y,z)\le G_p(x,a,z)+G_p(a,y,z)-G_p(a,a,a)$.

The following proposition shows that to every G_p-metric space we can associate one metric.

Proposition 2. *[1] If (X,G_p) is a G_p-metric space, then (X,D_{G_p}) is a metric space where*

$$D_{G_p}(x,y)=G_p(x,y,y)+G_p(y,x,x)-G_p(x,x,x)-G_p(y,y,y)$$

for all $x,y\in X$.

Zand and Nezhad [1] also introduced the basic topological concepts like G_p-convergence, G_p-Cauchy sequence and G_p-completeness in G_p-metric spaces as follows.

Definition 2. *Let (X,G_p) be a G_p-metric space.*

i) A sequence $\{x_n\}$ is called G_p-convergent to $x\in X$ if $\lim_{m,n\to\infty}G_p(x,x_m,x_n)=G_p(x,x,x)$. A point $x\in X$ is said to be limit point of the sequence $\{x_n\}$ and denoted with $x_n\to x$,
ii) A sequence $\{x_n\}$ is said to be a G_p-Cauchy sequence if and only if $\lim_{m,n\to\infty}G_p(x_n,x_m,x_m)$ exits (and is finite),
iii) A G_p-metric space (X,G_p) is said to be G_p-complete if and only if every G_p-Cauchy sequence in X is G_p-convergent to $x\in X$ such that $G_p(x,x,x)=\lim_{m,n\to\infty}G_p(x_n,x_m,x_m)$.

The following proposition will be frequently used proving our results.

Proposition 3. *[1] Let (X,G_p) be a G_p-metric space. Then, for any sequence $\{x_n\}$ in X and a point $x\in X$ the following statements are equivalent,*

i) $\{x_n\}$ *is G_p-convergent to x,*
ii) $G_p(x_n,x_n,x)\to G_p(x,x,x)$ *as* $n\to\infty$,
iii) $G_p(x_n,x,x)\to G_p(x,x,x)$ *as* $n\to\infty$.

The following lemma, which given by Parvaneh et al. in [9], provides the characterizations of concepts of Cauchy and completeness for G_p-metric spaces.

Lemma 1. *i) A sequence $\{x_n\}$ is a G_p-Cauchy sequence in a G_p-metric space (X,G_p) if and only if it is a Cauchy sequence in the metric space (X,D_{G_p}).*
ii) A G_p-metric space (X,G_p) is G_p-complete if and only if the metric space (X,D_{G_p}) is complete. Moreover, $\lim_{n\to\infty}D_{G_p}(x,x_n)=0$ if and only if

$$\begin{aligned}\lim_{n\to\infty}G_p(x,x_n,x_n)&=\lim_{n\to\infty}G_p(x_n,x,x)=\lim_{n,m\to\infty}G_p(x_n,x_n,x_m)\\&=\lim_{n,m\to\infty}G_p(x_n,x_m,x_m)=G_p(x,x,x).\end{aligned}$$

The following useful lemmas have a crucial role in the proof of our main results.

Lemma 2. *[4] Let (X,G_p) be a G_p-metric space. Then*

i) If $G_p(x,y,z)=0$, then $x=y=z$,
ii) If $x\neq y$, then $G_p(x,y,y)>0$.

Lemma 3. *[9] Assume that $\{x_n\}\to x$ as $n\to\infty$ in a G_p-metric space (X,G_p) such that $G_p(x,x,x)=0$. Then, for every $y\in X$,*

i) $\lim_{n\to\infty}G_p(x_n,y,y)=G_p(x,y,y)$,
ii) $\lim_{n\to\infty}G_p(x_n,x_n,y)=G_p(x,x,y)$.

The following proposition of Zand and Nezhad [1] will be required in the sequel.

Proposition 4. *[1] Let (X_1,G_1) and (X_2,G_2) be G_p-metric spaces. Then a function $f:X_1\to X_2$ is G_p-continuous at a point $x\in X_1$ if and only if it is G_p-sequentially continuous at x; that is, whenever $\{x_n\}$ is G_p-convergent to x one has $\{f(x_n)\}$ is G_p-convergent to $f(x)$.*

Kaya et al. [12] given an important remark, which shows the relationship between G_p-continuity and D_{G_p}-continuity, as follows.

Remark. It is worth noting that the notions G_p-continuous and D_{G_p}-continuous of any function in the contex of G_p-metric space are incomparable, in general. Indeed, if $X=[0,+\infty)$, $G_p(x,y,z)=\max\{x,y,z\}$, $D_{G_p}(x,y)=|x-y|$, $f0=1$ and $fx=x^2$ for all $x>0$, $gx=|\sin x|$, then f is a G_p-continuous and D_{G_p}-discontinuous at point $x=0$; while g is a G_p-discontinuous and D_{G_p}-continuous at point $x=\pi$. Therefore, in this paper, we take that $T:X\to X$ continuous if both $T:(X,G_p)\to(X,G_p)$ and $T:(X,D_{G_p})\to(X,D_{G_p})$ are continuous.

Definition 3. *[14] Let $(X,\preceq)$ be a partially ordered set. A pair (f,g) of self-maps of X is called weakly increasing if $fx\preceq gfx$ and $gx\preceq fgx$ for all $x\in X$.*

Berinde [15] introduced the term of a weak contraction mapping which is more general than a contraction mapping in 2004. But, in [16] Berinde redefine it as an almost contraction mapping that is more suitable. Berinde [15] established certain fixed point theorems for almost contractions in complete metric spaces. Moreover, Berinde [15] demonstrated that any strict contraction, the Kannan [17] and Zamfirescu [18] mappings and a large class of quasi-contractions are all almost contractions. Also, Berinde [19] introduced the notion of weak φ-contraction (or (φ,L)-weak contraction) using a comparison function. It is obvious that any almost contraction is a weak φ-contraction, but the opposite may not be true. On the other hand, Shaddad et al. [20] viewed the existence and uniqueness of a common fixed point for mappings providing some generalized Berinde type contractions in metric spaces. Furthermore, Altun and Acar [21] introduced the concepts of a (δ,L)-weak contraction and (φ,L)-weak contraction in partial metric spaces. In recent years, Türkoğlu and Öztürk [22] proved a fixed point theorem for mappings ensuring an almost generalized contractive condition in partial metric spaces. Quite recently, Aydi et al. [23] generalize the results obtained in [21,22]. In the literature, there are a lot of studies on common fixed points obtained by using Berinde-type contractions, see [24–29].
The prime purpose of this study is to establish fixed point and common fixed point theorems for generalized Berinde-type contractions in the context of partially ordered G_p-metric spaces and also generalize and extent the results of Barakat and Zidan [6], Aydi et al. [23], Shaddad et al. [20] and many other known corresponding theorems.

2 Main results

In this section, we state and prove our main results for self-mappings satisfying some generalized Berinde-type contractions in a partially ordered G_p-metric space which is complete.
Let us consider two sets $\Psi = \{\psi : [0,\infty) \to [0,\infty) : \psi$ is continuous, nondecreasing and $\psi(t) = 0 \Leftrightarrow t = 0\}$ and $\Phi = \{\phi : [0,\infty) \to [0,\infty) : \phi$ is lower semi-continuous, and $\phi(t) = 0 \Leftrightarrow t = 0\}$. Now, we give our initial result.

Theorem 1. *Let $(X,\preceq)$ be a partially ordered set and f and g be weakly increasing self-maps on a G_p-complete G_p-metric space X. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(G_p(fx,gy,gy)) \le \psi(\lambda u(x,y,y)) - \phi(\lambda u(x,y,y)) + LN(x,y), \tag{1}$$

for all comparable $x,y \in X$ where

$$u(x,y,y) \in \left\{ G_p(x,y,y), G_p(x,fx,fx), G_p(y,gy,gy), \frac{1}{2}[G_p(x,gy,gy)+G_p(y,fx,fx)] \right\},$$

and

$$N(x,y) = \min\{D_{G_p}(x,y), D_{G_p}(x,fx), D_{G_p}(y,gy), D_{G_p}(x,gy), D_{G_p}(y,fx)\},$$

with $L \ge 0$ and $0 \le \lambda \le 1$. If one of the following two cases is satisfied,

i) f or g is continuous,
ii) if a nondecreasing sequence $\{x_n\}$ converges to $z \in X$ implies $x_n \preceq z$ for all $n \in \mathbb{N}$,

then f and g have a common fixed point. Furthermore, the set of common fixed points of f and g is well ordered if and only if f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Then, we can construct a sequence $\{x_n\}$ defined by

$$x_{2n+1} = fx_{2n} \quad \text{and} \quad x_{2n+2} = gx_{2n+1} \quad \text{for} \quad n = 0,1,2,\dots$$

Since f and g are weakly increasing maps with respect to "$\preceq$", we get the following,

$$x_1 = fx_0 \preceq gfx_0 = gx_1 = x_2 \preceq fgx_1 = fx_2 = x_3,$$
$$x_3 = fx_2 \preceq gfx_2 = gx_3 = x_4 \preceq fgx_3 = fx_4 = x_5,$$

and proceeding this process we get

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \dots$$

Now, we suppose that $G_p(x_n,x_{n+1},x_{n+1}) = 0$ for some $n \in \mathbb{N}$. Without loss of generality, we can assume that $n = 2k$ for some $k \in \mathbb{N}$. Thus $G_p(x_{2k},x_{2k+1},x_{2k+1}) = 0$. Hence, we consider that $G_p(x_{2k+1},x_{2k+2},x_{2k+2}) > 0$. Since x_{2k} and x_{2k+1} are comparable, using (1), we have

$$\begin{aligned}\psi(G_p(x_{2k+1},x_{2k+2},x_{2k+2})) &= \psi(G_p(fx_{2k},gx_{2k+1},gx_{2k+1})) \\ &\le \psi(\lambda u(x_{2k},x_{2k+1},x_{2k+1})) - \phi(\lambda u(x_{2k},x_{2k+1},x_{2k+1})) + LN(x_{2k},x_{2k+1})\end{aligned}$$

where

$$u(x_{2k},x_{2k+1},x_{2k+1}) \in \left\{ \begin{array}{c} G_p(x_{2k},x_{2k+1},x_{2k+1}), G_p(x_{2k},fx_{2k},fx_{2k}), G_p(x_{2k+1},gx_{2k+1},gx_{2k+1}), \\ \frac{1}{2}[G_p(x_{2k},gx_{2k+1},gx_{2k+1}) + G_p(x_{2k+1},fx_{2k},fx_{2k})] \end{array} \right\}$$

and

$$N(x_{2k},x_{2k+1}) = \min \left\{ \begin{array}{c} D_{G_p}(x_{2k},x_{2k+1}), D_{G_p}(x_{2k},fx_{2k}), D_{G_p}(x_{2k+1},gx_{2k+1}), D_{G_p}(x_{2k},gx_{2k+1}), \\ D_{G_p}(x_{2k+1},fx_{2k}) \end{array} \right\},$$

i.e., $N(x_{2k},x_{2k+1}) = 0$. Thus, we have

$$\psi(G_p(x_{2k+1},x_{2k+2},x_{2k+2})) \le \psi(\lambda u(x_{2k},x_{2k+1},x_{2k+1})) - \phi(\lambda u(x_{2k},x_{2k+1},x_{2k+1})),$$

where

$$u(x_{2k},x_{2k+1},x_{2k+1}) \in \left\{ 0, G_p(x_{2k+1},x_{2k+2},x_{2k+2}), \frac{1}{2}[G_p(x_{2k},x_{2k+2},x_{2k+2}) + G_p(x_{2k+1},x_{2k+1},x_{2k+1})] \right\}.$$

Hence, we have three cases.

Case 1. $u(x_{2k},x_{2k+1},x_{2k+1}) = 0$. Then

$$\psi(G_p(x_{2k+1},x_{2k+2},x_{2k+2})) \le 0,$$

implies that $G_p(x_{2k+1},x_{2k+2},x_{2k+2}) = 0$ and so $x_{2k+1} = x_{2k+2}$, which is a contradiction.
Case 2. $u(x_{2k},x_{2k+1},x_{2k+1}) = G_p(x_{2k+1},x_{2k+2},x_{2k+2})$. Then

$$\begin{aligned} \psi(G_p(x_{2k+1},x_{2k+2},x_{2k+2})) &\le \psi(\lambda G_p(x_{2k+1},x_{2k+2},x_{2k+2})) - \phi(\lambda G_p(x_{2k+1},x_{2k+2},x_{2k+2})) \\ &< \psi(\lambda G_p(x_{2k+1},x_{2k+2},x_{2k+2})). \end{aligned}$$

Since ψ is nondecreasing, we have $G_p(x_{2k+1},x_{2k+2},x_{2k+2}) < \lambda G_p(x_{2k+1},x_{2k+2},x_{2k+2})$, which is impossible.
Case 3. $u(x_{2k},x_{2k+1},x_{2k+1}) = \frac{1}{2}[G_p(x_{2k},x_{2k+2},x_{2k+2}) + G_p(x_{2k+1},x_{2k+1},x_{2k+1})]$. Then,

$$\begin{aligned} \psi(G_p(x_{2k+1},x_{2k+2},x_{2k+2})) &\le \psi\left(\frac{\lambda}{2}[G_p(x_{2k},x_{2k+2},x_{2k+2}) + G_p(x_{2k+1},x_{2k+1},x_{2k+1})]\right) \\ &\quad - \phi\left(\frac{\lambda}{2}[G_p(x_{2k},x_{2k+2},x_{2k+2}) + G_p(x_{2k+1},x_{2k+1},x_{2k+1})]\right) \\ &< \psi\left(\frac{\lambda}{2}[G_p(x_{2k},x_{2k+2},x_{2k+2}) + G_p(x_{2k+1},x_{2k+1},x_{2k+1})]\right). \end{aligned}$$

Since ψ is nondecreasing, we have

$$\begin{aligned} G_p(x_{2k+1},x_{2k+2},x_{2k+2}) &< \frac{\lambda}{2}[G_p(x_{2k},x_{2k+2},x_{2k+2}) + G_p(x_{2k+1},x_{2k+1},x_{2k+1})] \\ &= \frac{\lambda}{2}G_p(x_{2k+1},x_{2k+2},x_{2k+2}), \end{aligned}$$

which is a contradiction since $\lambda \in [0,1]$.

Thus our supposition that $G_p(x_{2k+1},x_{2k+2},x_{2k+2}) > 0$ is not true. Therefore, we conclude that $G_p(x_{2k+1},x_{2k+2},x_{2k+2}) = 0$ and so $x_{2k+1} = x_{2k+2}$. Then x_{2k} becomes a common fixed point of f and g since $x_{2k} = fx_{2k} = gx_{2k}$. Thus, we may presume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Now, we shall show that $G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leq G_p(x_{2n}, x_{2n+1}, x_{2n+1})$. Arguing by contradiction, we suppose $G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}) > G_p(x_{2n}, x_{2n+1}, x_{2n+1})$. Since x_{2n} and x_{2n+1} are comparable, by (1) we get

$$\begin{aligned}\psi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) &= \psi(G_p(fx_{2n}, gx_{2n+1}, gx_{2n+1})) \\ &\leq \psi(\lambda u(x_{2n}, x_{2n+1}, x_{2n+1})) - \phi(\lambda u(x_{2n}, x_{2n+1}, x_{2n+1})) \\ &+ LN(x_{2n}, x_{2n+1}), \end{aligned} \tag{2}$$

where

$$u(x_{2n}, x_{2n+1}, x_{2n+1}) \in \left\{ \begin{array}{c} G_p(x_{2n}, x_{2n+1}, x_{2n+1}), G_p(x_{2n}, fx_{2n}, fx_{2n}), G_p(x_{2n+1}, gx_{2n+1}, gx_{2n+1}), \\ \frac{1}{2}[G_p(x_{2n}, gx_{2n+1}, gx_{2n+1}) + G_p(x_{2n+1}, fx_{2n}, fx_{2n})], \end{array} \right\}$$

and

$$N(x_{2n}, x_{2n+1}) = \min \left\{ \begin{array}{c} D_{G_p}(x_{2n}, x_{2n+1}), D_{G_p}(x_{2n}, fx_{2n}), D_{G_p}(x_{2n+1}, gx_{2n+1}), \\ D_{G_p}(x_{2n}, gx_{2n+1}), D_{G_p}(x_{2n+1}, fx_{2n}). \end{array} \right\} = 0.$$

Hence, (2) becomes

$$\psi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) \leq \psi(\lambda u(x_{2n}, x_{2n+1}, x_{2n+1})) - \phi(\lambda u(x_{2n}, x_{2n+1}, x_{2n+1})),$$

where

$$u(x_{2n}, x_{2n+1}, x_{2n+1}) \in \left\{ \begin{array}{c} G_p(x_{2n}, x_{2n+1}, x_{2n+1}), G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ \frac{1}{2}[G_p(x_{2n}, x_{2n+2}, x_{2n+2}) + G_p(x_{2n+1}, x_{2n+1}, x_{2n+1})]. \end{array} \right\}.$$

Hence, we have three cases.

Case 1. $u(x_{2n}, x_{2n+1}, x_{2n+1}) = G_p(x_{2n}, x_{2n+1}, x_{2n+1})$. Then

$$\begin{aligned}\psi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) &\leq \psi(\lambda G_p(x_{2n}, x_{2n+1}, x_{2n+1})) - \phi(\lambda G_p(x_{2n}, x_{2n+1}, x_{2n+1})) \\ &< \psi(\lambda G_p(x_{2n}, x_{2n+1}, x_{2n+1})).\end{aligned}$$

Since ψ is nondecreasing, we have $G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}) < \lambda G_p(x_{2n}, x_{2n+1}, x_{2n+1})$, which is a contradiction.
Case 2. $u(x_{2n}, x_{2n+1}, x_{2n+1}) = G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})$. Then

$$\begin{aligned}\psi(G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) &\leq \psi(\lambda G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})) - \phi(\lambda G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})), \\ &< \psi(\lambda G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})).\end{aligned}$$

Since ψ is nondecreasing, we have $G_p(x_{2n+1}, x_{2n+2}, x_{2n+2}) < \lambda G_p(x_{2n+1}, x_{2n+2}, x_{2n+2})$, which is impossible.

Case 3. $u(x_{2n},x_{2n+1},x_{2n+1}) = \frac{1}{2}[G_p(x_{2n},x_{2n+2},x_{2n+2}) + G_p(x_{2n+1},x_{2n+1},x_{2n+1})]$. Then

$$\begin{aligned}\psi(G_p(x_{2n+1},x_{2n+2},x_{2n+2})) &\le \psi\left(\frac{\lambda}{2}[G_p(x_{2n},x_{2n+2},x_{2n+2}) + G_p(x_{2n+1},x_{2n+1},x_{2n+1})]\right)\\ &\quad - \phi\left(\frac{\lambda}{2}[G_p(x_{2n},x_{2n+2},x_{2n+2}) + G_p(x_{2n+1},x_{2n+1},x_{2n+1})]\right),\\ &\le \psi\left(\frac{\lambda}{2}[G_p(x_{2n},x_{2n+2},x_{2n+2}) + G_p(x_{2n+1},x_{2n+1},x_{2n+1})]\right).\end{aligned}$$

Since ψ is nondecreasing, we have

$$\begin{aligned}G_p(x_{2n+1},x_{2n+2},x_{2n+2}) &\le \frac{\lambda}{2}[G_p(x_{2n},x_{2n+2},x_{2n+2}) + G_p(x_{2n+1},x_{2n+1},x_{2n+1})],\\ &\le \frac{\lambda}{2}[G_p(x_{2n},x_{2n+1},x_{2n+1}) + G_p(x_{2n+1},x_{2n+2},x_{2n+2})],\end{aligned}$$

which leads to

$$G_p(x_{2n+1},x_{2n+2},x_{2n+2}) \le \frac{\lambda}{2-\lambda} G_p(x_{2n},x_{2n+1},x_{2n+1}),$$

but $G_p(x_{2n+1},x_{2n+2},x_{2n+2}) > G_p(x_{2n},x_{2n+1},x_{2n+1})$, hence

$$G_p(x_{2n+1},x_{2n+2},x_{2n+2}) < \frac{\lambda}{2-\lambda} G_p(x_{2n+1},x_{2n+2},x_{2n+2}),$$

which is unfeasible as $\lambda/(2-\lambda) \le 1$.

Therefore, we obtain

$$G_p(x_{2n+1},x_{2n+2},x_{2n+2}) \le G_p(x_{2n},x_{2n+1},x_{2n+1}). \tag{3}$$

By similar arguments as above, we can show that

$$G_p(x_{2n},x_{2n+1},x_{2n+1}) \le G_p(x_{2n-1},x_{2n},x_{2n}). \tag{4}$$

By (3) and (4), we have

$$G_p(x_{n+1},x_{n+2},x_{n+2}) \le G_p(x_n,x_{n+1},x_{n+1}), \tag{5}$$

for all $n \in \mathbb{N}$. Therefore, the sequence $\{G_p(x_n,x_{n+1},x_{n+1})\}$ is a decreasing sequence and bounded below. Hence, $\{G_p(x_n,x_{n+1},x_{n+1})\}$ is convergent and so there exists $r \ge 0$ such that

$$\lim_{n\to\infty} G_p(x_n,x_{n+1},x_{n+1}) = r. \tag{6}$$

Next, we want to show that $r = 0$. We have two cases.

Case 1. When $u(x_n,x_{n+1},x_{n+1}) \in \{G_p(x_n,x_{n+1},x_{n+1}), G_p(x_{n+1},x_{n+2},x_{n+2})\}$, as ψ is continuous and ϕ is lower semi-continuous and from (6) we obtain

$$\psi(r) \le \psi(\lambda r) - \phi(\lambda r).$$

If $\lambda = 0$, then we have $\psi(r) = 0$, that is, $r = 0$. If $\lambda \ne 0$, then we get $\phi(\lambda r) \le \psi(\lambda r) - \psi(r) \le 0$. Thus $\phi(\lambda r) = 0$, which implies $r = 0$.

Case 2. When $u(x_n,x_{n+1},x_{n+1}) = \frac{1}{2}[G_p(x_n,x_{n+2},x_{n+2}) + G_p(x_{n+1},x_{n+1},x_{n+1})]$, we suppose that $r \neq 0$, then

$$\begin{aligned}\psi(G_p(x_{n+1},x_{n+2},x_{n+2})) &\leq \psi\left(\frac{\lambda}{2}[G_p(x_n,x_{n+2},x_{n+2}) + G_p(x_{n+1},x_{n+1},x_{n+1})]\right), \\ &- \phi\left(\frac{\lambda}{2}[G_p(x_n,x_{n+2},x_{n+2}) + G_p(x_{n+1},x_{n+1},x_{n+1})]\right), \\ &\leq \psi\left(\frac{\lambda}{2}[G_p(x_n,x_{n+2},x_{n+2}) + G_p(x_{n+1},x_{n+1},x_{n+1})]\right), \\ &\leq \psi\left(\frac{\lambda}{2}[G_p(x_n,x_{n+1},x_{n+1}) + G_p(x_{n+1},x_{n+2},x_{n+2})]\right).\end{aligned}$$

Now, we get two subcases.

Subcase 1.$\lambda < 1$. Then as $n \to \infty$ we get $\psi(r) \leq \psi(\lambda r)$, which causes a contradiction if $r \neq 0$.

Subcase 2.$\lambda = 1$. Then

$$\begin{aligned}\psi(G_p(x_{n+1},x_{n+2},x_{n+2})) &\leq \psi\left(\frac{1}{2}[G_p(x_n,x_{n+2},x_{n+2}) + G_p(x_{n+1},x_{n+1},x_{n+1})]\right), \\ &\leq \psi\left(\frac{1}{2}[G_p(x_n,x_{n+1},x_{n+1}) + G_p(x_{n+1},x_{n+2},x_{n+2})]\right).\end{aligned}$$

As $n \to \infty$, we have

$$\begin{aligned}\psi(r) &\leq \lim_{n\to\infty} \psi\left(\frac{1}{2}[G_p(x_n,x_{n+2},x_{n+2}) + G_p(x_{n+1},x_{n+1},x_{n+1})]\right), \\ &\leq \psi(r),\end{aligned}$$

i.e.,

$$\lim_{n\to\infty} \psi\left(\frac{1}{2}[G_p(x_n,x_{n+2},x_{n+2}) + G_p(x_{n+1},x_{n+1},x_{n+1})]\right) = \psi(r).$$

Since ψ is a continuous function, we obtain

$$\lim_{n\to\infty}[G_p(x_n,x_{n+2},x_{n+2}) + G_p(x_{n+1},x_{n+1},x_{n+1})] = 2r. \tag{7}$$

By taking the lower limit as $n \to \infty$ in

$$\begin{aligned}\psi(G_p(x_{n+1},x_{n+2},x_{n+2})) &\leq \psi\left(\frac{1}{2}[G_p(x_n,x_{n+2},x_{n+2}) + G_p(x_{n+1},x_{n+1},x_{n+1})]\right) \\ &- \phi\left(\frac{1}{2}[G_p(x_n,x_{n+2},x_{n+2}) + G_p(x_{n+1},x_{n+1},x_{n+1})]\right),\end{aligned}$$

and using (7), we have

$$\psi(r) \leq \psi(r) - \liminf_{n\to\infty} -\phi\left(\frac{1}{2}[G_p(x_n,x_{n+2},x_{n+2}) + G_p(x_{n+1},x_{n+1},x_{n+1})]\right) \leq \psi(r) - \phi(r),$$

which implies that $\phi(r) \leq 0$. Hence $\phi(r) = 0$ and then $r = 0$. This is a contradiction. In that case, from the above we obtain $r = 0$, i.e.,

$$\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = \lim_{n\to\infty} G_p(x_{n+1}, x_n, x_n) = 0. \tag{8}$$

Since $G_p(x_n, x_n, x_n) \leq G_p(x_n, x_{n+1}, x_{n+1})$, we get by (8)

$$\lim_{n\to\infty} G_p(x_n, x_n, x_n) = 0 \tag{9}$$

for all $n \in \mathbb{N}$. On the other hand, we have

$$D_{G_p}(x_n, x_{n+1}) = G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_n, x_n) - G_p(x_n, x_n, x_n) - G_p(x_{n+1}, x_{n+1}, x_{n+1}).$$

Letting $n \to \infty$ in the previous equality and using (8) and (9), we get

$$\lim_{n\to\infty} D_{G_p}(x_n, x_{n+1}) = 0.$$

Next, we denote that $\{x_n\}$ is a G_p-Cauchy sequence in X. That is, we show that for every $\varepsilon > 0$, there exists an integer k such that for all $m > n \geq k$,

$$G_p(x_n, x_m, x_m) < \varepsilon,$$

i.e., we prove that $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = 0$. For this, it is sufficient to prove that $\{x_{2n}\}$ is a G_p-Cauchy sequence in X. We argue by contradiction. Hypothesize that $\{x_{2n}\}$ is not a G_p-Cauchy sequence in X. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{2n(k)}\}$ and $\{x_{2m(k)}\}$ of $\{x_{2n}\}$ such that $m(k) > n(k) \geq k$ and

$$G_p(x_{2n(k)}, x_{2m(k)}, x_{2m(k)}) \geq \varepsilon, \tag{10}$$

where $m(k)$ is the smallest positive integer with $m(k) > n(k)$ such that (10) holds, i.e.,

$$G_p(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}) < \varepsilon. \tag{11}$$

So by using rectangle inequality and (10), (11) we get

$$\begin{aligned}
\varepsilon &\leq G_p(x_{2n(k)}, x_{2m(k)}, x_{2m(k)}) \\
&\leq G_p(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}) + G_p(x_{2m(k)-1}, x_{2m(k)}, x_{2m(k)}), \\
&\leq G_p(x_{2n(k)}, x_{2n(k)+1}, x_{2n(k)+1}) + G_p(x_{2n(k)+1}, x_{2m(k)-1}, x_{2m(k)-1}) \\
&+ G_p(x_{2m(k)-1}, x_{2m(k)}, x_{2m(k)}), \\
&\leq G_p(x_{2n(k)}, x_{2n(k)+1}, x_{2n(k)+1}) + G_p(x_{2n(k)+1}, x_{2m(k)}, x_{2m(k)}) \\
&+ G_p(x_{2m(k)}, x_{2m(k)-1}, x_{2m(k)-1}) + G_p(x_{2m(k)-1}, x_{2m(k)}, x_{2m(k)}), \\
&\leq G_p(x_{2n(k)}, x_{2n(k)+1}, x_{2n(k)+1}) + G_p(x_{2n(k)+1}, x_{2n(k)}, x_{2n(k)}) \\
&+ G_p(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}) + G_p(x_{2m(k)-1}, x_{2m(k)}, x_{2m(k)}) \\
&+ G_p(x_{2m(k)}, x_{2m(k)-1}, x_{2m(k)-1}) + G_p(x_{2m(k)-1}, x_{2m(k)}, x_{2m(k)}), \\
&< 2G_p(x_{2n(k)}, x_{2n(k)+1}, x_{2n(k)+1}) + \varepsilon + 3G_p(x_{2m(k)-1}, x_{2m(k)}, x_{2m(k)}).
\end{aligned}$$

Letting $k \to \infty$, in the above inequality and using (8) we have

$$\begin{aligned}\lim_{k\to\infty} G_p(x_{2n(k)}, x_{2m(k)}, x_{2m(k)}) &= \lim_{k\to\infty} G_p(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}),\\ &= \lim_{k\to\infty} G_p(x_{2n(k)+1}, x_{2m(k)-1}, x_{2m(k)-1}),\\ &= \lim_{k\to\infty} G_p(x_{2n(k)+1}, x_{2m(k)}, x_{2m(k)})\\ &= \varepsilon.\end{aligned}$$

By the definition of $u(x,y,y)$ and $N(x,y)$ and using previous limits we get that

$$\lim_{k\to\infty} u(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}) \in \{\varepsilon, 0\} \quad \text{and} \quad \lim_{k\to\infty} N(x_{2n(k)}, x_{2m(k)-1}) = 0.$$

Indeed,

$$u(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}) \in \left\{ \begin{array}{c} G_p(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}), G_p(x_{2n(k)}, fx_{2n(k)}, fx_{2n(k)}),\\ G_p(x_{2m(k)-1}, gx_{2m(k)-1}, gx_{2m(k)-1}),\\ \frac{1}{2}[G_p(x_{2n(k)}, gx_{2m(k)-1}, gx_{2m(k)-1}) + G_p(x_{2m(k)-1}, fx_{2n(k)}, fx_{2n(k)})], \end{array} \right\}$$

and

$$N(x_{2n(k)}, x_{2m(k)-1}) = \min \left\{ \begin{array}{c} D_{G_p}(x_{2n(k)}, x_{2m(k)-1}), D_{G_p}(x_{2n(k)}, fx_{2n(k)}), D_{G_p}(x_{2m(k)-1}, gx_{2m(k)-1}),\\ D_{G_p}(x_{2n(k)}, gx_{2m(k)-1}), D_{G_p}(x_{2m(k)-1}, fx_{2n(k)}). \end{array} \right\}.$$

Let $k \to \infty$, we get

$$\lim_{k\to\infty} u(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}) \in \{\varepsilon, 0\} \quad \text{and} \quad \lim_{k\to\infty} N(x_{2n(k)}, x_{2m(k)-1}) = 0.$$

As $x_{2n(k)}$ and $x_{2m(k)-1}$ are comparable, we can apply condition (1) to obtain

$$\begin{aligned}\psi(G_p(x_{2n(k)+1}, x_{2m(k)}, x_{2m(k)})) &= \psi(G_p(fx_{2n(k)}, gx_{2m(k)-1}, gx_{2m(k)-1})),\\ &\leq \psi(\lambda u(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1})) - \phi(\lambda u(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1})),\\ &+ LN(x_{2n(k)}, x_{2m(k)-1}).\end{aligned}$$

Passing to the limit when $k \to \infty$ we obtain that

$$\begin{aligned}\psi(\varepsilon) &= \liminf_{k\to\infty} \psi(G_p(x_{2n(k)+1}, x_{2m(k)}, x_{2m(k)})),\\ &\leq \liminf_{k\to\infty} \psi(\lambda u(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1})), - \liminf_{k\to\infty} \phi(\lambda u(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1})),\\ &\leq \psi(\lambda\varepsilon) - \phi(\lambda\varepsilon).\end{aligned}$$

If $\lambda = 0$, then we have $\psi(\varepsilon) = 0$, that is, $\varepsilon = 0$. If $\lambda \neq 0$, then we get $\phi(\lambda\varepsilon) \leq \psi(\lambda\varepsilon) - \psi(\varepsilon) \leq 0$. Thus $\phi(\lambda\varepsilon) = 0$, which implies $\varepsilon = 0$, which is impossible. Consequently, $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = 0$ and thus $\{x_n\}$ is a G_p-Cauchy sequence in the G_p-complete G_p-metric space (X, G_p). Then, from Lemma 1 $\{x_n\}$ is a Cauchy sequence in the metric space (X, D_{G_p}).

Completeness of (X,G_p) yields that (X,D_{G_p}) is also complete. Then there exists $z\in X$ such that

$$\lim_{n\to\infty} D_{G_p}(x_n,z)=0. \tag{12}$$

Since $\lim_{n,m\to\infty} G_p(x_n,x_m,x_m)=0$, (17) and part (ii) of Lemma 1 yield that

$$\begin{aligned}\lim_{n\to\infty} G_p(x_n,z,z) &= \lim_{n\to\infty} G_p(x_n,x_n,z),\\ &= \lim_{n,m\to\infty} G_p(x_n,x_m,x_m),\\ &= G_p(z,z,z)\\ &= 0.\end{aligned}$$

Let us now denote that z is a common fixed point of f and g.

i)If f is a continuous self map on X, (12) implies that $fx_{2n}\to fz$ as $n\to\infty$. Since $x_{2n+1}\to z$, by the uniqueness of the limit in metric space (X,D_{G_p}), we obtain that $fz=z$. Assume that $gz\neq z$. Also, because $z\preceq z$, from (1) we get

$$\psi(G_p(z,gz,gz))=\psi(G_p(fz,gz,gz))\le\psi(\lambda u(z,z,z))-\phi(\lambda u(z,z,z))+LN(z,z)$$

where

$$u(z,z,z)\in\left\{0,G_p(z,gz,gz),\frac{G_p(z,gz,gz)}{2}\right\} \quad \text{and} \quad N(z,z)=0.$$

If $u(z,z,z)=0$, we get $\psi(G_p(z,gz,gz))=0$, which means that $G_p(z,gz,gz)=0$, namely $z=gz$. This is a contradiction. If $u(z,z,z)=G_p(z,gz,gz)$ or $u(z,z,z)=\frac{G_p(z,gz,gz)}{2}$, we obtain

$$\psi(G_p(z,gz,gz))<\psi(\lambda G_p(z,gz,gz)) \quad \text{or} \quad \psi(G_p(z,gz,gz))<\psi\left(\frac{\lambda}{2}G_p(z,gz,gz)\right),$$

which is a impossible.

Hence, we have $z=gz$. The proof is similar if g is continuous.

ii) Further, if f and g are not continuous then by given assumption we have $x_n\preceq z$ for all $n\in\mathbb{N}$. Thus for the subsequences $\{x_{2n(k)}\}$ and $\{x_{2n(k)+1}\}$ of x_n we have $x_{2n(k)}\preceq z$ and $x_{2n(k)+1}\preceq z$. Therefore, we get

$$\begin{aligned}\psi(G_p(fz,x_{2n(k)+2},x_{2n(k)+2})) &= \psi(G_p(fz,gx_{2n(k)+1},gx_{2n(k)+1})),\\ &\le \psi(\lambda u(z,x_{2n(k)+1},x_{2n(k)+1}))-\phi(\lambda u(z,x_{2n(k)+1},x_{2n(k)+1}))\\ &\quad +LN(z,x_{2n(k)+1})\end{aligned}$$

where

$$u(z,x_{2n(k)+1},x_{2n(k)+1})\in\left\{\begin{array}{c} G_p(z,x_{2n(k)+1},x_{2n(k)+1}),G_p(z,fz,fz),\\ G_p(x_{2n(k)+1},gx_{2n(k)+1},gx_{2n(k)+1}),\\ \frac{1}{2}[G_p(z,gx_{2n(k)+1},gx_{2n(k)+1})+G_p(x_{2n(k)+1},fz,fz)]\end{array}\right\}$$

and

$$N(z,x_{2n(k)+1})=\min\left\{\begin{array}{c} D_{G_p}(z,x_{2n(k)+1}),D_{G_p}(z,fz),D_{G_p}(x_{2n(k)+1},gx_{2n(k)+1}),\\ D_{G_p}(z,gx_{2n(k)+1}),D_{G_p}(x_{2n(k)+1},fz)\end{array}\right\}.$$

Let $k \to \infty$, we get

$$\begin{aligned}\psi(G_p(fz,z,z)) &= \liminf_{k\to\infty} \psi(G_p(fz,x_{2n(k)+2},x_{2n(k)+2})) \\ &\leq \liminf_{k\to\infty} \psi(\lambda u(z,x_{2n(k)+1},x_{2n(k)+1})) \\ &- \liminf_{k\to\infty} \phi(\lambda u(z,x_{2n(k)+1},x_{2n(k)+1}))\end{aligned}$$

where

$$\lim_{k\to\infty} u(z,x_{2n(k)+1},x_{2n(k)+1}) \in \left\{0, G_p(z,fz,fz), \frac{G_p(z,fz,fz)}{2}\right\}.$$

If, $G_p(z,fz,fz) \neq 0$, then

$$\psi(G_p(fz,z,z)) < \psi(\lambda G_p(fz,z,z)) \quad \text{or} \quad \psi(G_p(fz,z,z)) < \psi\left(\frac{\lambda}{2}G_p(fz,z,z)\right),$$

which is a contradiction. Hence, we obtain $G_p(z,fz,fz) = 0$, that is $z = fz$.

In a similar manner, when we take $x = x_{2n(k)}$ and $y = z$ in (1) for all n we attain $z = gz$. Then, z is a common fixed point of f and g.

Now, suppose that the set of common fixed points of f and g is well ordered. Then common fixed of f and g is unique. Assume on contrary that, let w be another common fixed point of f and g. As z ad w are comparable, from (1) we have

$$\psi(G_p(z,w,w)) = \psi(G_p(fz,gw,gw)) \leq \psi(\lambda u(z,w,w)) - \phi(\lambda u(z,w,w)) + LN(z,w),$$

where

$$u(z,w,w) \in \{0, G_p(z,w,w)\} \quad \text{and} \quad N(z,w) = 0.$$

Then we obtain $z = w$. Conversely, if f and g have only one common fixed point then the set of common fixed point of f and g being singleton is well ordered.

Corollary 1. *Let $(X,\preceq)$ be a partially ordered set and f and g be weakly increasing self-maps on a G_p-complete G_p-metric space X. Assume that $\psi \in \Psi$ and $\phi \in \Phi$ such that*

$$\psi(G_p(fx,gy,gy)) \leq \psi(\lambda M(x,y,y)) - \phi(\lambda M(x,y,y)) + LN(x,y)$$

for all comparable $x,y \in X$ where

$$M(x,y,y) = \max\left\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,gy,gy), \frac{1}{2}(G_p(x,gy,gy) + G_p(y,fx,fx))\right\}$$

and

$$N(x,y) = \min\{D_{G_p}(x,y), D_{G_p}(x,fx), D_{G_p}(y,gy), D_{G_p}(x,gy), D_{G_p}(y,fx)\}$$

with $L \geq 0$ and $0 \leq \lambda \leq 1$. If one of the following two cases is satisfied

i) f or g is continuous,

ii) if a nondecreasing sequence $\{x_n\}$ converges to $z \in X$ implies $x_n \preceq z$ for all $n \in \mathbb{N}$,

then f and g have a common fixed point. Furthermore, the set of common fixed points of f and g is well ordered if and only if f and g have a unique common fixed point.

Proof. Since $M(x,y,y) \in \left\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,gy,gy), \frac{1}{2}(G_p(x,gy,gy)+G_p(y,fx,fx))\right\}$, the result follows from Theorem 1.

Remark. In Corollary 1,

i) If $L=0$ and $\lambda=1$, we get Theorem 2.1 of Barakat and Zidan [6].
ii) If $\psi(t)=t$ for all $t\in[0,\infty)$, $L=0$ and $\lambda=1$, we get Corollary 2.1 of Barakat and Zidan [6].
iii) If $\psi(t)=t$, $\phi(t)=(1-k)t$ for all $t\in[0,\infty)$ where $k\in[0,1)$, $L=0$ and $\lambda=1$, we get Corollary 2.4 of Barakat and Zidan [6].

Corollary 2. *Let $(X,\preceq)$ be a partially ordered set and f and g be weakly increasing self-maps on a G_p-complete G_p-metric space X satisfying*

$$G_p(fx,gy,gy) \le \alpha u(x,y,y) + LN(x,y)$$

for all comparable $x,y\in X$ where

$$u(x,y,y) \in \left\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,gy,gy), \frac{1}{2}(G_p(x,gy,gy)+G_p(y,fx,fx))\right\}$$

and

$$N(x,y) = \min\{D_{G_p}(x,y), D_{G_p}(x,fx), D_{G_p}(y,gy), D_{G_p}(x,gy), D_{G_p}(y,fx)\}$$

with $L\ge 0$ and $0\le\alpha<1$. If one of the following two cases is satisfied

i) f or g is continuous;
ii) if a nondecreasing sequence $\{x_n\}$ converges to $z\in X$ implies $x_n\preceq z$ for all $n\in\mathbb{N}$;

then f and g have a common fixed point. Furthermore, the set of common fixed points of f and g is well ordered if and only if f and g have a unique common fixed point.

Proof. It suffices to get $\psi(t)=t$ and $\phi(t)=(1-k)t$ with $k<1$ in Theorem 1.

Corollary 3. *Let $(X,\preceq)$ be a partially ordered set and f be a nondecreasing self-map on a G_p-complete G_p-metric space X satisfying*

$$G_p(fx,fy,fy) \le \alpha u(x,y,y) + LN(x,y)$$

for all comparable $x,y\in X$ where

$$u(x,y,y) \in \left\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy), \frac{1}{2}(G_p(x,fy,fy)+G_p(y,fx,fx))\right\}$$

and

$$N(x,y) = \min\{D_{G_p}(x,y), D_{G_p}(x,fx), D_{G_p}(y,fy), D_{G_p}(x,fy), D_{G_p}(y,fx)\}$$

with $L\ge 0$ and $0\le\alpha<1$. If there exists $x_0\in X$ with $x_0\preceq fx_0$ and one of the following two cases is satisfied

i) f is continuous,
ii) if a nondecreasing sequence $\{x_n\}$ converges to $z\in X$ implies $x_n\preceq z$ for all $n\in\mathbb{N}$;

then f has a fixed point. Furthermore, the set of fixed points of f is well ordered if and only if f has a unique fixed point.

Proof. If follows by taking $f=g$ in Corollary 2.

Now, let $\mathscr{F}$ be the set of functions $\varphi:[0,\infty)\to[0,\infty)$ satisfying the following hypotheses:

i) φ is monotone increasing,
ii) $\sum_{n=0}^{\infty}\varphi^n(t)$ converges for all $t>0$.

Take in consideration that if $\varphi\in\mathscr{F}$, φ is called a (c)-comparison function. It can be proved easily that if φ is a (c)-comparison function, then $\varphi(t)<t$ for any $t>0$. Our second main result is as follows.

Theorem 2. *Let $(X,\preceq)$ be a partially ordered set and f and g be weakly increasing self-maps on a G_p-complete G_p-metric space X. There exist $\varphi\in\mathscr{F}$ and $L\geq 0$ such that for all comparable $x,y\in X$*

$$G_p(fx,gy,gy)\leq\varphi(M(x,y,y))+L\min\{D_{G_p}(x,y),D_{G_p}(x,fx),D_{G_p}(y,gy),D_{G_p}(x,gy),D_{G_p}(y,fx)\}\tag{13}$$

where

$$M(x,y,y)=\max\left\{G_p(x,y,y),G_p(x,fx,fx),G_p(y,gy,gy),\frac{1}{2}(G_p(x,gy,gy)+G_p(y,fx,fx))\right\}.$$

If one of the following two cases is satisfied

i)f or g is continuous,
ii)if a nondecreasing sequence $\{x_n\}$ converges to $z\in X$ implies $x_n\preceq z$ for all $n\in\mathbb{N}$,

then f and g have a common fixed point. Furthermore, the set of common fixed points of f and g is well ordered if and only if f and g have a unique common fixed point.

Proof. Choose $x_0\in X$. Then, we can construct a sequence $\{x_n\}$ defined by

$$x_{2n+1}=fx_{2n}\quad\text{and}\quad x_{2n+2}=gx_{2n+1}\quad\text{for}\quad n=0,1,2,\dots$$

As f and g are weakly increasing maps with respect to "$\preceq$", we get the following:

$$x_1\preceq x_2\preceq\cdots\preceq x_n\preceq x_{n+1}\preceq\dots$$

Suppose first that $G_p(x_n,x_{n+1},x_{n+1})=0$ for some $n\in\mathbb{N}$. Then the sequence $\{x_n\}$ is constant for n. Indeed, let $n=2k$ for some $k\in\mathbb{N}$. Then $G_p(x_{2k},x_{2k+1},x_{2k+1})=0$. Now, we assume $G_p(x_{2k+1},x_{2k+2},x_{2k+2})>0$. Since x_{2k} and x_{2k+1} are comparable, using (13), we get

$$\begin{aligned}G_p(x_{2k+1},x_{2k+2},x_{2k+2})&=G_p(fx_{2k},gx_{2k+1},gx_{2k+1}),\\&\leq\varphi(M(x_{2k},x_{2k+1},x_{2k+1}))+L\min\{D_{G_p}(x_{2k},x_{2k+1}),D_{G_p}(x_{2k},fx_{2k}),\\&D_{G_p}(x_{2k+1},gx_{2k+1}),D_{G_p}(x_{2k},gx_{2k+1}),D_{G_p}(x_{2k+1},fx_{2k})\},\end{aligned}\tag{14}$$

$$\begin{aligned}M(x_{2k},x_{2k+1},x_{2k+1})&=\max\left\{\begin{array}{c}G_p(x_{2k},x_{2k+1},x_{2k+1}),G_p(x_{2k},fx_{2k},fx_{2k}),G_p(x_{2k+1},gx_{2k+1},gx_{2k+1}),\\\frac{1}{2}[G_p(x_{2k},gx_{2k+1},gx_{2k+1})+G_p(x_{2k+1},fx_{2k},fx_{2k})]\end{array}\right\}\\&=\max\left\{G_p(x_{2k+1},x_{2k+2},x_{2k+2}),\frac{G_p(x_{2k},x_{2k+2},x_{2k+2})+G_p(x_{2k+1},x_{2k+1},x_{2k+1})}{2}\right\}\\&=G_p(x_{2k+1},x_{2k+2},x_{2k+2}).\end{aligned}$$

Therefore, the expression (14) turns into,

$$G_p(x_{2k+1},x_{2k+2},x_{2k+2})\leq\varphi(G_p(x_{2k+1},x_{2k+2},x_{2k+2}))<G_p(x_{2k+1},x_{2k+2},x_{2k+2}),$$

which is a contradiction. So $G_p(x_{2k+1},x_{2k+2},x_{2k+2}) = 0$ and $x_{2k+1} = x_{2k+2}$. Hence, the sequence $\{x_n\}$ is constant and x_{2k} is a common fixed point of f and g. Thus, we may suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. From (13), we obtain

$$\begin{aligned} G_p(x_{2n+1},x_{2n+2},x_{2n+2}) &= G_p(fx_{2n},gx_{2n+1},gx_{2n+1}) \\ &\leq \varphi(M(x_{2n},x_{2n+1},x_{2n+1})) \\ &+ L\min\left\{ \begin{array}{c} D_{G_p}(x_{2n},x_{2n+1}), D_{G_p}(x_{2n},fx_{2n}), D_{G_p}(x_{2n+1},gx_{2n+1}), \\ D_{G_p}(x_{2n},gx_{2n+1}), D_{G_p}(x_{2n+1},fx_{2n}) \end{array} \right\} \\ &= \varphi(M(x_{2n},x_{2n+1},x_{2n+1})). \end{aligned} \tag{15}$$

As explained in the proof of Theorem 1, we may get

$$M(x_{2n},x_{2n+1},x_{2n+1}) = \max\{G_p(x_{2n},x_{2n+1},x_{2n+1}), G_p(x_{2n+1},x_{2n+2},x_{2n+2})\}.$$

If for some $n \in \mathbb{N}$, $M(x_{2n},x_{2n+1},x_{2n+1}) = G_p(x_{2n+1},x_{2n+2},x_{2n+2})$, then by (**??**), we obtain that

$$G_p(x_{2n+1},x_{2n+2},x_{2n+2}) \leq \varphi(G_p(x_{2n+1},x_{2n+2},x_{2n+2})) < G_p(x_{2n+1},x_{2n+2},x_{2n+2}),$$

which is a contradiction. Thus, for all $n \in \mathbb{N}$, we get $M(x_{2n},x_{2n+1},x_{2n+1}) = G_p(x_{2n},x_{2n+1},x_{2n+1})$. Using (15), we get that

$$G_p(x_{2n+1},x_{2n+2},x_{2n+2}) \leq \varphi(G_p(x_{2n},x_{2n+1},x_{2n+1})). \tag{16}$$

By similar arguments as above, we can show that

$$G_p(x_{2n},x_{2n+1},x_{2n+1}) \leq \varphi(G_p(x_{2n-1},x_{2n},x_{2n})). \tag{17}$$

By (16) and (17), we have

$$G_p(x_n,x_{n+1},x_{n+1}) \leq \varphi(G_p(x_{n-1},x_n,x_n)).$$

By using mathematical induction, we obtain

$$G_p(x_n,x_{n+1},x_{n+1}) \leq \varphi^n(G_p(x_0,x_1,x_1)).$$

So, we can conclude that

$$\lim_{n\to\infty} G_p(x_n,x_{n+1},x_{n+1}) = 0. \tag{18}$$

For $n,m \in \mathbb{N}$ with $m > n$, we get

$$\begin{aligned} G_p(x_n,x_m,x_m) &\leq \sum_{k=n}^{m-1} G_p(x_k,x_{k+1},x_{k+1}) - \sum_{k=n+1}^{m-1} G_p(x_k,x_k,x_k) \\ &\leq \sum_{k=n}^{m-1} G_p(x_k,x_{k+1},x_{k+1}) \\ &\leq \sum_{k=n}^{\infty} G_p(x_k,x_{k+1},x_{k+1}) \\ &\leq \sum_{k=n}^{\infty} \varphi^k(G_p(x_0,x_1,x_1)) \end{aligned}$$

Since φ is (c)-comparison function, we have that $\sum_{k=0}^{\infty}\varphi^k(G_p(x_0,x_1,x_1))$ converges and hence $\lim_{n\to\infty}\sum_{k=n}^{\infty}\varphi^k(G_p(x_0,x_1,x_1))=0$. So, $\lim_{n,m\to\infty}G_p(x_n,x_m,x_m)=0$. This implies that $\{x_n\}$ is a G_p-Cauchy sequence in the G_p-metric space (X,G_p). Then, from Lemma 1 $\{x_n\}$ is a Cauchy sequence in the metric space (X,D_{G_p}). By G_p-completeness of X, (X,D_{G_p}) is also complete. Then there exists $z\in X$ such that

$$\lim_{n\to\infty}D_{G_p}(x_n,z)=0. \tag{19}$$

Since $\lim_{n,m\to\infty}G_p(x_n,x_m,x_m)=0$, (19) and part (ii) of Lemma 1 yield that

$$\lim_{n\to\infty}G_p(x_n,z,z)=\lim_{n\to\infty}G_p(x_n,x_n,z)=G_p(z,z,z)=0. \tag{20}$$

Now we will distinguish the cases (i) and (ii) of Theorem 2.

i) If f is a continuous self map on X, (19) implies that $fx_{2n}\to fz$ as $n\to\infty$. Since $x_{2n+1}\to z$, by the uniqueness of the limit in metric space (X,D_{G_p}), we obtain that $fz=z$. Assume that $gz\neq z$. Also, because $z\preceq z$, from (**??**) we get

$$\begin{aligned}
G_p(z,gz,gz)&=G_p(fz,gz,gz)\\
&\leq\varphi(M(z,z,z))+L\min\{D_{G_p}(z,z),D_{G_p}(z,fz),D_{G_p}(z,gz)\}\\
&=\varphi(M(z,z,z))\\
&=\varphi(G_p(z,gz,gz))\\
&<G_p(z,gz,gz)
\end{aligned}$$

because of the properties of φ. This is a contradiction and hence $z=gz$. The proof is similar if g is continuous.

ii) If f and g are not continuous then by given assumption we have $x_n\preceq z$ for all $n\in\mathbb{N}$. Thus for the subsequences $\{x_{2n(k)}\}$ and $\{x_{2n(k)+1}\}$ of x_n we have $x_{2n(k)}\preceq z$ and $x_{2n(k)+1}\preceq z$. Therefore, we get

$$\begin{aligned}
G_p(fz,x_{2n(k)+2},x_{2n(k)+2})&=G_p(fz,gx_{2n(k)+1},gx_{2n(k)+1})\\
&\leq\varphi(M(z,x_{2n(k)+1},x_{2n(k)+1}))\\
&+L\min\left\{\begin{array}{c}D_{G_p}(z,x_{2n(k)+1}),D_{G_p}(z,fz),D_{G_p}(x_{2n(k)+1},gx_{2n(k)+1}),\\ D_{G_p}(z,gx_{2n(k)+1}),D_{G_p}(x_{2n(k)+1},fz)\end{array}\right\}
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
M(z,x_{2n(k)+1},x_{2n(k)+1})&=\max\left\{\begin{array}{c}G_p(z,x_{2n(k)+1},x_{2n(k)+1}),G_p(z,fz,fz),\\ G_p(x_{2n(k)+1},gx_{2n(k)+1},gx_{2n(k)+1}),\\ \frac{1}{2}[G_p(z,gx_{2n(k)+1},gx_{2n(k)+1})+G_p(x_{2n(k)+1},fz,fz)]\end{array}\right\}\\
&=\max\left\{\begin{array}{c}G_p(z,x_{2n(k)+1},x_{2n(k)+1}),G_p(z,fz,fz),\\ G_p(x_{2n(k)+1},x_{2n(k)+2},x_{2n(k)+2}),\\ \frac{1}{2}[G_p(z,x_{2n(k)+2},x_{2n(k)+2})+G_p(x_{2n(k)+1},fz,fz)]\end{array}\right\}.
\end{aligned}$$

Suppose that $G_p(z,fz,fz)>0$. From (18) and (20), there exists $n_0\in\mathbb{N}$ such that for all $n>n_0$, we get

$$G_p(x_n,x_{n+1},x_{n+1})<\frac{1}{3}G_p(z,fz,fz). \tag{22}$$

Similarly, there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$, we can write

$$G_p(x_n, z, z) < \frac{1}{3} G_p(z, fz, fz). \tag{23}$$

Then for all $n > \max\{n_0, n_1\}$, by using (22), (23) and rectangle inequality we have

$$\begin{aligned}
&\frac{1}{2}[G_p(z, x_{2n(k)+2}, x_{2n(k)+2}) + G_p(x_{2n(k)+1}, fz, fz)] \\
&\leq \frac{1}{2}[G_p(z, x_{2n(k)+2}, x_{2n(k)+2}) + G_p(x_{2n(k)+1}, z, z) + G_p(z, fz, fz)] \\
&\leq \frac{1}{2}\left[\frac{1}{3} G_p(z, fz, fz) + \frac{1}{3} G_p(z, fz, fz) + G_p(z, fz, fz)\right] \\
&= \frac{5}{6} G_p(z, fz, fz).
\end{aligned} \tag{24}$$

Hence, for all $n > \max\{n_0, n_1\}$, from (22), (23) and (24) we conclude that

$$\begin{aligned}
M(z, x_{2n(k)+1}, x_{2n(k)+1}) &= \max \left\{ \begin{array}{c} G_p(z, x_{2n(k)+1}, x_{2n(k)+1}), G_p(z, fz, fz), \\ G_p(x_{2n(k)+1}, x_{2n(k)+2}, x_{2n(k)+2}), \\ \frac{1}{2}[G_p(z, x_{2n(k)+2}, x_{2n(k)+2}) + G_p(x_{2n(k)+1}, fz, fz)] \end{array} \right\} \\
&\leq G_p(z, fz, fz).
\end{aligned}$$

So, by inequality (21), for all $n > \max\{n_0, n_1\}$ we obtain

$$\begin{aligned}
G_p(fz, x_{2n(k)+2}, x_{2n(k)+2}) &\leq \varphi(G_p(z, fz, fz)) \\
&\quad + L \min \left\{ \begin{array}{c} D_{G_p}(z, x_{2n(k)+1}), D_{G_p}(z, fz), \\ D_{G_p}(x_{2n(k)+1}, gx_{2n(k)+1}), D_{G_p}(z, gx_{2n(k)+1}), \\ D_{G_p}(x_{2n(k)+1}, fz) \end{array} \right\}.
\end{aligned}$$

Now, passing to the limit when $k \to \infty$ in last inequality, we get

$$G_p(fz, z, z) \leq \varphi(G_p(z, fz, fz)) < G_p(z, fz, fz) = G_p(fz, z, z),$$

which is a contradiction. Hence, we have $z = fz$.
In a similar way, when we take $x = x_{2n(k)}$ and $y = z$ in (13) for all n we get $z = gz$. Then, z is a common fixed point of f and g.

The rest of the Theorem 2 can be proved in similar way as Theorem 1.

Taking $f = g$ in Theorem 2, we have the following result.

Corollary 4. *Let $(X, \preceq)$ be a partially ordered set and f be a nondecreasing self-map on a G_p-complete G_p-metric space X. There exist $\varphi \in \mathscr{F}$ and $L \geq 0$ such that for all comparable $x, y \in X$*

$$G_p(fx, fy, fy) \leq \varphi(M(x, y, y)) + L \min\{D_{G_p}(x, y), D_{G_p}(x, fx), D_{G_p}(y, fy), D_{G_p}(x, fy), D_{G_p}(y, fx)\}$$

where

$$M(x, y, y) = \max\left\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, fy, fy), \frac{1}{2}(G_p(x, fy, fy) + G_p(y, fx, fx))\right\}.$$

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$ and one of the following two cases is satisfied

i) f is continuous;
ii) if a nondecreasing sequence $\{x_n\}$ converges to $z \in X$ implies $x_n \preceq z$ for all $n \in \mathbb{N}$;

then f has a fixed point. Furthermore, the set of fixed points of f is well ordered if and only if f has a unique fixed point.

Now we give some examples making effective our obtained results.

Example 2. Let $X = [0,1]$. Define a G_p-metric $G_p : X \times X \times X \to [0,\infty)$ by the formula $G_p(x,y,z) = \max\{x,y,z\}$. Therefore, for any $x,y \in X$

$$D_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y) = |x-y|.$$

Then (X,G_p) is a G_p-complete symmetric G_p-metric space. Let us define a partial order $\preceq$ on X by $x \preceq y$ if and only if $y \leq x$. Then, $(X,\preceq)$ is a partially ordered set. Also, consider the functions $\psi : [0,\infty) \to [0,\infty)$ and $\phi : [0,\infty) \to [0,\infty)$ defined by

$$\psi(t) = t \quad \text{and} \quad \phi(t) = \frac{t}{1+t},$$

respectively. Clearly the function $\psi \in \Psi$, that is, ψ is continuous, nondecreasing and $\psi(t) = 0 \Leftrightarrow t = 0$ and also $\phi \in \Phi$, that is, ϕ is lower semi-continuous, and $\phi(t) = 0 \Leftrightarrow t = 0$. Furthermore, define $f,g : X \to X$ as $fx = \frac{x^2}{1+x}$ and $gx = 0$. Since

$$f(gx) = f(0) = 0 \leq gx$$

for all $x \in X$, we have $gx \preceq fgx$. Similarly, we get $fx \preceq gfx$ since

$$g(fx) = g\left(\frac{x^2}{1+x}\right) = 0 \leq \frac{x^2}{1+x} = fx$$

for all $x \in X$. So f and g are weakly increasing mappings. Also, f is continuous in X with respect to the standard metric and G_p-metric. Indeed, let $\{x_n\}$ be a sequence converging to x in (X,G_p), then

$$\lim_{n\to\infty} \max\{x_n,x\} = \lim_{n\to\infty} G_p(x_n,x,x) = G_p(x,x,x) = x,$$

hence by definition of f, we have

$$\begin{aligned}
\lim_{n\to\infty} G_p(fx_n,fx,fx) &= \lim_{n\to\infty} \max\{fx_n,fx\} \\
&= \lim_{n\to\infty} \max\left\{\frac{x_n^2}{1+x_n},\frac{x^2}{1+x}\right\} \\
&= \frac{x^2}{1+x} \\
&= G_p(fx,fx,fx), \qquad (25)
\end{aligned}$$

that is, $\{fx_n\}$ converges to fx in (X,G_p). On the other hand, if $\{x_n\}$ converges to x in (X,D_{G_p}), hence

$$\lim_{n\to\infty} D_{G_p}(x_n,x) = \lim_{n\to\infty} |x_n - x| = 0.$$

Thus, by definition of D_{G_p} and f, one can find

$$\lim_{n\to\infty} D_{G_p}(fx_n, fx) = \lim_{n\to\infty}\left|\frac{x_n^2}{1+x_n} - \frac{x^2}{1+x}\right| = 0. \tag{26}$$

By convergences (25) and (26) yield that f is a continuous mapping.

Now, let us show that the contraction condition of Corollary 1 is satisfied. Then, for all $x,y\in X$ with $y\le x$, we get

$$\begin{aligned}\psi(G_p(fx,gy,gy)) &= \max\left\{\frac{x^2}{1+x},0\right\} = \frac{x^2}{1+x} = x - \frac{x}{1+x}\\ &= \psi(M(x,y,y)) - \phi(M(x,y,y))\\ &\le \psi(M(x,y,y)) - \phi(M(x,y,y)) + LN(x,y)\end{aligned}$$

for all $L\ge 0$ and $\lambda = 1$, since $M(x,y,y) = x$. Therefore, all hypothesis of Corollary 1 are satisfied and f and g have a unique common fixed point in X. It is seen that 0 is unique common fixed point of f and g.

Example 3. Let $X=[0,1]$ and $G_p : X\times X\times X \to [0,\infty)$ be defined by $G_p(x,y,z) = \max\{x,y,z\}$. We endow X with a partial order $\preceq$ given by $x\preceq y$ if and only if $y\le x$. Then, (X,G_p) is partially ordered G_p-complete symmetric G_p-metric space. Consider the mapping $\varphi : [0,\infty)\to[0,\infty)$ defined by $\varphi(t)=\frac{t}{2}$. By induction, we have $\varphi^n(t)=\frac{t}{2^n}$ for all $n\ge 1$, so it is clear that φ is a (c)-comparison function. Also, the mappings $f,g : X\to X$ are defined by

$$fx = \frac{x}{3} \quad\text{and}\quad gx = \frac{x}{4},$$

respectively. In that case, f and g are weakly increasing. Indeed, given $x\in X$. Since

$$f(gx) = f\left(\frac{x}{4}\right) = \frac{x}{12} \le \frac{x}{4} = gx,$$

we have $gx\preceq fgx$. Similarly, we can show that $fx\preceq gfx$. Moreover, f is a continuous mapping in X with respect to the standard metric and G_p-metric. Now, we show that f and g satisfy the contractive condition (13) for all $x,y\in X$ with $y\le x$. Then, by definition of f and g, we get

$$\begin{aligned}G_p(fx,gy,gy) &= \max\{\frac{x}{3},\frac{y}{4}\} = \frac{x}{3}\\ &\le \frac{x}{2} = \varphi(M(x,y,y)) \le \varphi(M(x,y,y)) + LN(x,y)\end{aligned}$$

for all $L\ge 0$, since $M(x,y,y) = x$. Then (13) is verified. Applying Theorem 2, f and g have a unique common fixed point, which is $z=0$.

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