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Evaluation of generalized Mittag-Leffler function method on endemic disease model

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Abstract: The endemic disease is a world health problem and we suffer from them since the old years. There exist different models that express an endemic disease such as the model we will address during this paper (Susceptible, Exposed, Infections, Recovered) (SEIR) that caused by a wild type virus. We use the Generalized Mittag-Leffler Function Method (GMLFM) to obtain the analytical and numerical solution of the SEIR model. We comparing the results that obtained by using this method with the results that obtained by Runge-Kutta (RK4) method for taking classical order derivative of the governing equations.

Keywords: Fractional derivatives, non-linear system, generalized Mittag-Leffler function method, endemic model.

1 Introduction

One of the most exciting concepts of mathematical application in the biological fields is an endemic disease; many previous studies discussed and defined the endemic disease as a disease that is constantly present to a greater or lesser degree in people of a definite class or in people living in a particular location [1,2,3]. Such type of disease related to pathological conditions established and eternized within a population group, a country or continent without any external effects. It usually describes as an infectious disease that is transmitted directly or indirectly between humans and it is occurring at the usual predictable rate. Texas State Historical Association (TSHA) notes that endemic diseases occur in a small percentage of the population, while a majority of individuals show no signs of the illness.
In mathematical terms, we want to determine the non-linear stability of the endemic disease equilibrium to the introduction of a mutant viral strain. We assume that the original wild type virus has infection rate β, removal rate γ, α be the birth rate and d the disease-unrelated death rate, and disease-related death rate c, and that the mutant virus has corresponding rates β', γ', c'. We further assume that an individual infected with either a wild type or mutant virus gains immunity to subsequent infection from both wild type and mutant viral forms. Our model thus has a single susceptible class S, two distinct infective classes E and I depending on which virus causes the infection, and a single recovered class R. The corresponding differential equations shows this disease [4]

$$\begin{aligned}
\frac{dS(t)}{dt} &= \alpha N - dS(t) - S(t)(\beta E(t) + \beta' I(t)),\\
\frac{dE(t)}{dt} &= \beta S(t)E(t) - (d + c + \gamma)E(t),\\
\frac{dI(t)}{dt} &= \beta S(t)I(t) - (d + c' + \gamma')I(t),\\
\frac{dR(t)}{dt} &= \gamma E(t) + \gamma' I(t) - dR(t),
\end{aligned} \tag{1}$$

with initial conditions $S(0) = N_1,\ E(0) = N_2,\ I(0) = N_3,\ R(0) = N_4$ and $N = N_1 + N_2 + N_3 + N_4$, where N is total number of the individuals in the population. In recent years, the fractional order models were given much attention, because the biological models that involved fractional order derivative are more realistic and accurate as compared to the classical

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order models, for more detail [5,6,7,8]. In this manuscript, we considered the given modified form of model (1) by taking the derivative of the governing equations in fractional order. The arbitrary order shows the realistic biphasic decline behavior of infection of disease with a slower rate. Therefore, the modified model of arbitrary order α as the following

$$\begin{aligned} D^{\alpha}x(t) &= bb_1 - dx(t) - b_2x(t)y(t) - b_3x(t)z(t), \\ D^{\alpha}y(t) &= b_2x(t)y(t) - (b_4+d+k)y(t), \\ D^{\alpha}z(t) &= b_2x(t)z(t) - (b_5+d+k_1)z(t), \\ D^{\alpha}w(t) &= b_4y(t) + b_5z(t) - dw(t). \end{aligned} \tag{2}$$

In this model, we replace the variables and coefficients in system (1) as follow
$S(t)=x(t),\ E(t)=y(t),\ I(t)=z(t),\ R(t)=w(t)$ and $\alpha=b,\ N=b_1,\ \beta=b_2,\ \beta'=b_3,\ \gamma=b_4,\ \gamma'=b_5,\ c=k,\ c'=k_1.$
This paper organized as the follows. In Section 2, we show some necessary preliminaries of fractional calculus and Mittag-Leffler function. After that, we present how to use GMLF method to solve SEIR model in Section 3. In Section 4, we present the numerical results obtained by GMLFM and comparing these results with RK-4 method. Finally, the conclusion is given in Section 5.

2 Some necessary preliminaries in fractional calculus

In this part, we clarify the basic definitions of fractional calculus and important notations which are used further in this paper [10,11,12].

Definition 1. *The fractional integral of order $\alpha > 0$ in the Riemann-Liouville sense is defined as*

$$\begin{aligned} {}_aI_x^{\alpha}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-\xi)^{\alpha-1}f(\xi)d\xi,\ \ a\geq 0,\ \ x>a, \\ {}_aI_x^{0}f(x) &= f(x), \end{aligned}$$

where $\Gamma(\alpha)$ is Eular Gamma function defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}dt, \quad x\in\Re^+$$

Definition 2. *The Caputo fractional derivative of order α is defined by the following*

$${}_a^CD_x^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-\xi)^{n-\alpha-1}\, f^{(n)}(\xi)d\xi,\ \ n-1<\alpha<n,\ \ x>a.$$

At $\alpha=1,\ \ D=\frac{d}{dt}.$

Theorem 1. Let $f(x)$ be a differentiable function on the interval $[a,x]$. Then, for $\alpha>0$ we have

$$\begin{aligned} {}_a^CD_x^{\alpha}\ {}_aI_x^{\alpha}f(x) &= f(x), \\ {}_aI_x^{\alpha}\ {}_a^CD_x^{\alpha}f(x) &= f(x) - \sum_{k=0}^{n-1} f^k(a)\frac{(x-a)^k}{\Gamma(k+1)}. \end{aligned}$$

Definition 3. *The two parameter Mittag-Leffler functions $E_{\alpha,\beta}$ is defined by*

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n\alpha+\beta)} \quad \alpha,\beta>0, \tag{3}$$

when $\beta=1$ this function called one parameter Mittag-Leffler functions E_{α} and defined as follows

$$E_{\alpha}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n\alpha+1)}, \tag{4}$$

moreover, when $\alpha = 1$, $\beta = 1$ *then*

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)} = e^x.$$

3 Main results

In this section, the GMLFM is applied to SEIR which considered as one of the most suitable example of nonlinear system where the GMLFM has a wide rang of applications whether linear or nonlinear systems. The analysis of the GMLFM is described as the following

$$y_i(t) = E_\alpha(a_i t^\alpha) = \sum_{n=0}^{\infty} a_i^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \quad i = 1,2,3..... \quad , \tag{5}$$

$$^cD^\alpha y_i(t) = \sum_{n=1}^{\infty} a_i^n \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}, \quad i = 1,2,3..... \quad , \tag{6}$$

where $y_i(t)$, $i = 1,2,3,...$ are decomposed by an infinite series of components [13,14,15]. The Caputo fractional derivative is played pivotal role in the GMLFM, for more details see [10]. To solve system (2) by using GMLFM , let

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} a^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ y(t) &= \sum_{n=0}^{\infty} d^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ z(t) &= \sum_{n=0}^{\infty} l^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ w(t) &= \sum_{n=0}^{\infty} f^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \end{aligned} \tag{7}$$

By applying Caputo fractional derivative as defined in definition(2) on equations (7) we have

$$\begin{aligned} ^CD^\alpha x(t) &= \sum_{n=1}^{\infty} a^n \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} = \sum_{n=0}^{\infty} a^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ ^CD^\alpha y(t) &= \sum_{n=1}^{\infty} d^n \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} = \sum_{n=0}^{\infty} d^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ ^CD^\alpha z(t) &= \sum_{n=1}^{\infty} l^n \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} = \sum_{n=0}^{\infty} l^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ ^CD^\alpha w(t) &= \sum_{n=1}^{\infty} f^n \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} = \sum_{n=0}^{\infty} f^{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \end{aligned} \tag{8}$$

Substituting from equations (7) and (8) in system (2) we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{a^{n+1}}{\Gamma(n\alpha+1)} t^{n\alpha} &= bb_1 - d \sum_{n=0}^{\infty} \frac{a^n}{\Gamma(n\alpha+1)} t^{n\alpha} - b_2 \sum_{n=0}^{\infty} c^n t^{n\alpha} - b_3 \sum_{n=0}^{\infty} c_1^n t^{n\alpha}, \\ \sum_{n=0}^{\infty} \frac{d^{n+1}}{\Gamma(n\alpha+1)} t^{n\alpha} &= b_2 \sum_{n=0}^{\infty} c^n t^{n\alpha} - (b_4 + d + k) \sum_{n=0}^{\infty} \frac{d^n}{\Gamma(n\alpha+1)} t^{n\alpha}, \\ \sum_{n=0}^{\infty} \frac{l^{n+1}}{\Gamma(n\alpha+1)} t^{n\alpha} &= b_2 \sum_{n=0}^{\infty} c_1^n t^{n\alpha} - (b_5 + d + k_1) \sum_{n=0}^{\infty} \frac{l^n}{\Gamma(n\alpha+1)} t^{n\alpha}, \\ \sum_{n=0}^{\infty} \frac{f^{n+1}}{\Gamma(n\alpha+1)} t^{n\alpha} &= b_4 \sum_{n=0}^{\infty} \frac{d^n}{\Gamma(n\alpha+1)} t^{n\alpha} + b_5 \sum_{n=0}^{\infty} \frac{l^n}{\Gamma(n\alpha+1)} t^{n\alpha} - d \sum_{n=0}^{\infty} \frac{f^n}{\Gamma(n\alpha+1)} t^{n\alpha}, \end{aligned}$$

where

$$c^n = \sum_{k=0}^{n} \frac{a^k d^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)},$$
$$c_1^n = \sum_{k=0}^{n} \frac{a^k l^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)}.$$

Then after compilation the summations we find

$$\begin{aligned} bb_1 &= \sum_{n=0}^{\infty} \Big(\frac{a^{n+1}}{\Gamma(n\alpha+1)} + d\frac{a^n}{\Gamma(n\alpha+1)} + b_2 c^n + b_3 c_1^n \Big) t^{n\alpha}, \\ 0 &= \sum_{n=0}^{\infty} \Big(\frac{d^{n+1}}{\Gamma(n\alpha+1)} - b_2 c^n + (b_4+d+k)\frac{d^n}{\Gamma(n\alpha+1)} \Big) t^{n\alpha}, \\ 0 &= \sum_{n=0}^{\infty} \Big(\frac{l^{n+1}}{\Gamma(n\alpha+1)} - b_2 c_1^n + (b_5+d+k_1)\frac{l^n}{\Gamma(n\alpha+1)} \Big) t^{n\alpha}, \\ 0 &= \sum_{n=0}^{\infty} \Big(\frac{f^{n+1}}{\Gamma(n\alpha+1)} - b_4\frac{d^n}{\Gamma(n\alpha+1)} - b_5\frac{l^n}{\Gamma(n\alpha+1)} + d\frac{f^n}{\Gamma(n\alpha+1)} \Big) t^{n\alpha}. \end{aligned} \tag{9}$$

System (9) is non-homogenous system. So, we find the first term by creating the first limit of the summation and get

$$\begin{aligned} a^1 &= bb_1 - (d + b_2 d^0 + b_3 l^0) a^0, \\ d^1 &= (b_2 a^0 - b_4 - d - k) d^0, \\ l^1 &= (b_2 a^0 - b_5 - d - k_1) l^0, \\ f^1 &= b_4 d^0 + b_5 l^0 - d f^0. \end{aligned} \tag{10}$$

Then system (9) become

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} \Big(\frac{a^{n+1}}{\Gamma(n\alpha+1)} + d\frac{a^n}{\Gamma(n\alpha+1)} + b_2 c^n + b_3 c_1^n \Big) t^{n\alpha}, \\ 0 &= \sum_{n=0}^{\infty} \Big(\frac{d^{n+1}}{\Gamma(n\alpha+1)} - b_2 c^n + (b_4+d+k)\frac{d^n}{\Gamma(n\alpha+1)} \Big) t^{n\alpha}, \\ 0 &= \sum_{n=0}^{\infty} \Big(\frac{l^{n+1}}{\Gamma(n\alpha+1)} - b_2 c_1^n + (b_5+d+k_1)\frac{l^n}{\Gamma(n\alpha+1)} \Big) t^{n\alpha}, \\ 0 &= \sum_{n=0}^{\infty} \Big(\frac{f^{n+1}}{\Gamma(n\alpha+1)} - b_4\frac{d^n}{\Gamma(n\alpha+1)} - b_5\frac{l^n}{\Gamma(n\alpha+1)} + d\frac{f^n}{\Gamma(n\alpha+1)} \Big) t^{n\alpha}. \end{aligned} \tag{11}$$

In system (11) $t^{n\alpha}$ is impossible equal zero then the coefficient that equal zero and we get the recurrence relations which we calculate values of constants a^n, d^n, l^n, $n = 1, 2, 3, \ldots$

$$\begin{aligned} a^{n+1} &= -da^n - b_2 c^n \Gamma(n\alpha+1) - b_3 c_1^n \Gamma(n\alpha+1), \\ d^{n+1} &= b_2 c^n \Gamma(n\alpha+1) - (b_4+d+k) d^n, \\ l^{n+1} &= b_2 c_1^n \Gamma(n\alpha+1) - (b_5+d+k_1) l^n, \\ f^{n+1} &= b_4 d^n + b_5 l^n - d f^n. \end{aligned} \tag{12}$$

At n=1,

$$
\begin{aligned}
a^2 &= -b_2 a^0 d^1 - b_3 a^0 l^1 - (d + b_2 d^0 + b_3 l^0) a^1, \\
d^2 &= b_2 d^0 a^1 + (b_2 a^0 - b_4 - d - k) d^1, \\
l^2 &= b_2 l^0 a^1 + (b_2 a^0 - b_5 - d - k_1) l^1, \\
f^2 &= b_4 d^1 + b_5 l^1 - d f^1.
\end{aligned} \tag{13}
$$

At n=2,

$$
\begin{aligned}
a^3 &= -b_2 a^0 d^2 - b_3 a^0 l^2 - (d + b_2 d^0 + b_3 l^0) a^2 - (b_2 d^1 + b_3 l^1) a^1 \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}, \\
d^3 &= b_2 d^0 a^2 + (b_2 a^0 - b_4 - d - k) d^2 + b_2 a^1 d^1 \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}, \\
l^3 &= b_2 l^0 a^2 + (b_2 a^0 - b_5 - d - k_1) l^2 + b_2 a^1 l^1 \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}, \\
f^3 &= b_4 d^2 + b_5 l^2 - d f^2.
\end{aligned} \tag{14}
$$

Similarly, we find $a^4,\ d^4,\ l^4,\ f^4,\ a^5,\ d^5,\ l^5,\ f^5$.
Substituting from (10), (13) and (14) into (7), we have the solution in the infinite series form as the following

$$
\begin{aligned}
x(t) &= a^0 + a^1 \frac{t^\alpha}{\Gamma(\alpha+1)} + a^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + a^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots, \\
y(t) &= d^0 + d^1 \frac{t^\alpha}{\Gamma(\alpha+1)} + d^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + d^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots, \\
z(t) &= l^0 + l^1 \frac{t^\alpha}{\Gamma(\alpha+1)} + l^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + l^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots, \\
w(t) &= f^0 + f^1 \frac{t^\alpha}{\Gamma(\alpha+1)} + f^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots.
\end{aligned}
$$

4 Numerical results and discussion

Through this section, we explain the numerical results of the system (2). By using the initial conditions and values of the parameters which taken from [9] and this values defined as follows
$N_1 = 30,\ N_2 = 10,\ N_3 = 5,\ N_4 = 15,\ b = 0.1,\ b_1 = 60,\ b_2 = 0.01,\ b_3 = 0.1,\ b_4 = 0.02,\ b_5 = 0.11,\ k = 0.01,\ k_1 = 0.2,\ d = 1$.
By using the numerical results, we find GMLFM is a perfect method for solving a fractional system and this is obvious to us over the next figures.The first figure explain the relation between each of the $x(t)$ that expresses of (Susceptible), $y(t)$ that expresses of (Exposed), $z(t)$ that expresses of (Infections) and $w(t)$ that expresses of (Recovered) with t which expresses the time in the weeks at $\alpha = 1$ by using two different methods RK4 and GMLF which we will notice through the quality and efficient GMLF method used in solving SEIR or xyzw model. In the second figure, we illustrate the relation between the main variables $x(t)$, $y(t)$, $z(t)$ and $w(t)$ with the time t by using GMLF method but by using the different values of $\alpha = 1,\ 0.95,\ 0.85,\ 0.75$.

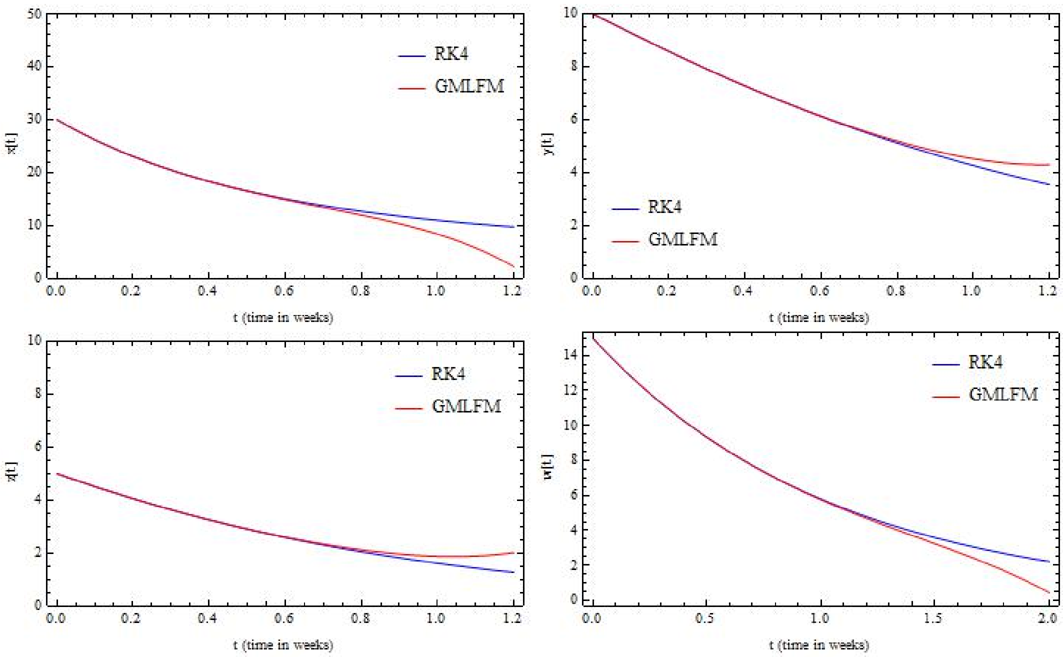

Fig. 1: The comparison plots x,y,z and w for $\alpha = 1$ using "GMLFM" and "RK4" method.

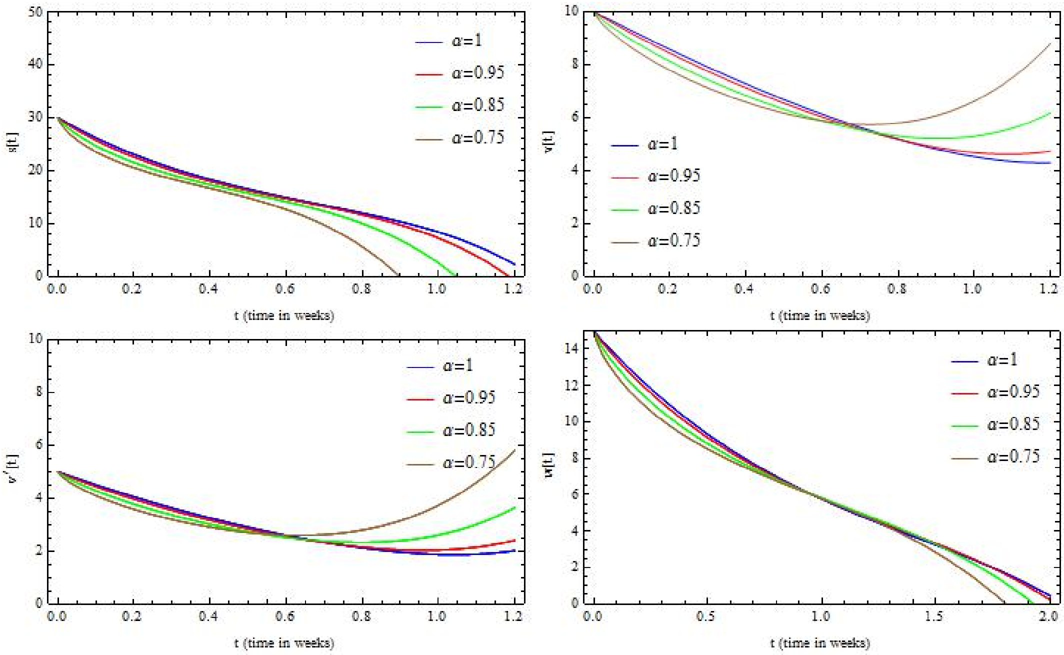

Fig. 2: Plots show that x,y,z and w with different values of α.

5 Conclusion

In this article, we applied an analytical method for non-linear fractional differential equations and this method called the Generalized Mittag-Leffler Function Method (GMLFM) to solve an endemic model (SEIR) of non-fatal disease in a community. As a consequence, our results are found to be remarkably accurate, in agreement with the numerical solutions such as Runge-Kutta 4 method and we find that in the figure 1. This figure shows that the results are efficiency, ease and lightness that obtain by GMLFM. This method is likely to inspire applications of the presented analytical procedure for solving highly non-linear systems in different areas of knowledge.

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