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Integral inequalities for $n-$times differentiable mappings

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Abstract: In this paper, using integral representations for $n-$times differentiable mappings, we establish new generalizations of certain Hermite-Hadamard type inequality for convex functions by using fairly elementary analysis. Also a parallel development is made base on concavity.

Keywords: Hermite-Hadamard Integral Inequality, Hölder Inequality, Jensen Inequality, Convex Functions.

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a,b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

Both the inequalities hold in reversed direction if f is concave. A letter was sent by Hermite (1822-1901) to the Journal Mathesis on November 22, 1881 and it was published in Mathesis 3 (1883, p: 82). This letter involved an inequality which is well-known in the literature as Hermite-Hadamard integral inequality. Since its discovery in 1883, Hermite-Hadamard inequality has been considered as very useful inequality in mathematical analysis.These inequalities have been used in numerous settings. In addition, a great number of inequalities of special means can be obtained for a particular choice of the function f. The rich geometrical significance of Hermite-Hadamard's inequality enables literature to grow by providing its new proofs, extensions, refinements and generalizations, see for example ([2], [6], [7], [10]-[14], [16], [18], [20]-[25]).

Definition 1. *The function* $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *is said to be convex if the following inequality holds:*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

for all $x,y \in [a,b]$ *and* $t \in [0,1]$.

It is said that f is concave if $(-f)$ is convex. This definition is originated from Jensen's results in [9] and has led to the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have been seen in mathematical literature since antiquity and naturally many significant results related to them were obtained.
Cerone *et. al.* (see [4]) proved the following generalization for $n-$time differentiable functions.

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Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then for all $x\in[a,b]$ we have the identity:*

$$\int_a^b f(t)dt=\sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)+(-1)^n\int_a^b K_n(x,t)f^{(n)}(t)dt,$$

where the kernel $K_n:[a,b]^2\to\mathbb{R}$ is given by

$$K_n(x,t)=\begin{cases}\frac{(t-a)^n}{n!} & if\ t\in[a,x]\\ \frac{(t-b)^n}{n!} & if\ t\in(x,b],\end{cases}$$

$x\in[a,b]$ and n is natural number, $n\geq 1$.

For other recent results regarding the $n-$time differentiable functions see [3]-[5], [8], [10], [13], [15], [17], [24] where further references are given.

In [19], Özdemir *et. al.* proved the following Hadamard type inequalities.

Theorem 1. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f''\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f''|^q$, $q\geq 1$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left\{\left(\frac{2}{(s+1)(s+2)(s+3)}|f''(a)|^q+\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{2}{(s+1)(s+2)(s+3)}|f''(b)|^q\right)^{\frac{1}{q}}\right\}.$$

Corollary 1. *In Theorem 1 , if we choose $s=1$ we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{48}\left(\frac{3}{4}\right)^{\frac{1}{q}}\left\{\left(\frac{|f''(a)|^q}{3}+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{|f''(b)|^q}{3}\right)^{\frac{1}{q}}\right\}. \tag{2}$$

In [1], Alomari and Darus obtained the following theorem and corollary.

Theorem 2. *Let $f:I\subseteq[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^{\frac{p}{p-1}}$ is concave on $[a,b]$, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{(b-x)^2}{(b-a)(p+1)^{1/p}}\left|f'\left(\frac{b+x}{2}\right)\right|+\frac{(x-a)^2}{(b-a)(p+1)^{1/p}}\left|f'\left(\frac{a+x}{2}\right)\right|$$

for each $x\in[a,b]$, where $p>1$.

Corollary 2. *In Theorem 2, choose $x=\frac{a+b}{2}$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{b-a}{4(p+1)^{1/p}}\left[\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'\left(\frac{3a+b}{4}\right)\right|\right] \tag{3}$$

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for each $x \in [a,b]$, *where* $p > 1$.

The present paper mainly aims to establish several new inequalities for $n-$ time differantiable mappings related to the celebrated Hermite-Hadamard integral inequality.

2 Main Results

In order to reach our aim, the following lemma is necessary:

Lemma 2. *For* $n \geq 1$, *let* $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *be n-time differentiable functions. If* $f^{(n)} \in L[a,b]$, *then*

$$\begin{aligned}\int_a^b f(t)dt &= \sum_{k=0}^{n-1} \left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1} f^{(k)}\left(\frac{a+b}{2}\right) \\ &+ (-1)^n \frac{(b-a)^{n+1}}{2^{n+1} n!} \left\{\int_0^1 t^n f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right) dt \right. \\ &\left. + \int_0^1 (t-1)^n f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right) dt \right\}.\end{aligned} \tag{4}$$

Proof. The proof is by mathematical induction. For $n = 1$, we have the equality in paper [2]. Assume that (4) holds for "$n = m$" and let us prove it for "$n = m+1$". That is, we have to obtain the equality,

$$\begin{aligned}\int_a^b f(x)dx &= \sum_{k=0}^{m} \left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1} f^{(k)}\left(\frac{a+b}{2}\right) \\ &+ (-1)^{m+1} \frac{(b-a)^{m+2}}{2^{m+2}(m+1)!} \left\{\int_0^1 t^{m+1} f^{(m+1)}\left(t\frac{a+b}{2}+(1-t)a\right) dt \right. \\ &\left. + \int_0^1 (t-1)^{m+1} f^{(m+1)}\left(tb+(1-t)\frac{a+b}{2}\right) dt \right\}.\end{aligned} \tag{5}$$

To handle(5), if we choose

$$\begin{aligned}I = \frac{(b-a)^{m+2}}{2^{m+2}(m+1)!} &\left\{\int_0^1 t^{m+1} f^{(m+1)}\left(t\frac{a+b}{2}+(1-t)a\right) dt \right. \\ &\left. + \int_0^1 (t-1)^{m+1} f^{(m+1)}\left(tb+(1-t)\frac{a+b}{2}\right) dt \right\}\end{aligned}$$

and integrating by parts gives

$$\begin{aligned}I &= \frac{(b-a)^{m+2}}{2^{m+2}(m+1)!} \left\{ t^{m+1} \frac{2}{b-a} f^{(m)}\left(t\frac{a+b}{2}+(1-t)a\right)\Big|_0^1 - \frac{2(m+1)}{b-a}\int_0^1 t^m f^{(m)}\left(t\frac{a+b}{2}+(1-t)a\right) dt \right. \\ &\left. + (t-1)^{m+1} \frac{2}{b-a} f^{(m)}\left(tb+(1-t)\frac{a+b}{2}\right)\Big|_0^1 - \frac{2(m+1)}{b-a}\int_0^1 (t-1)^m f^{(m)}\left(tb+(1-t)\frac{a+b}{2}\right) dt \right\} \\ &= \frac{(b-a)^{m+1}}{2^{m+1}(m+1)!} f^{(m)}\left(\frac{a+b}{2}\right) - \frac{(b-a)^{m+1}}{2^{m+1} m!}\int_0^1 t^m f^{(m)}\left(t\frac{a+b}{2}+(1-t)a\right) dt \\ &+ \frac{(-1)^{m+2}(b-a)^{m+1}}{2^{m+1}(m+1)!} f^{(m)}\left(\frac{a+b}{2}\right) - \frac{(b-a)^{m+1}}{2^{m+1} m!}\int_0^1 (t-1)^m f^{(m)}\left(tb+(1-t)\frac{a+b}{2}\right) dt.\end{aligned}$$

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Hence

$$\frac{(b-a)^{m+1}}{2^{m+1}m!}\left\{\int_0^1 t^m f^{(m)}\left(t\frac{a+b}{2}+(1-t)a\right)dt+\int_0^1 (t-1)^m f^{(m)}\left(tb+(1-t)\frac{a+b}{2}\right)dt\right\}$$
$$=\left(\frac{1+(-1)^{m+2}}{2^{m+1}(m+1)!}\right)(b-a)^{m+1}f^{(m)}\left(\frac{a+b}{2}\right)-I$$

Now, using the mathematical induction hypothesis, upon rearrangement we obtain the following equality:

$$\frac{1}{(-1)^m}\int_a^b f(x)dx=\frac{1}{(-1)^m}\sum_{k=0}^{m-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right) \tag{6}$$
$$+\left(\frac{1+(-1)^{m+2}}{2^{m+1}(m+1)!}\right)(b-a)^{m+1}f^{(m)}\left(\frac{a+b}{2}\right)-I.$$

Multiplying the both sides of (6) by $(-1)^n$ and substituting I in the right part of (6), we obtain

$$\int_a^b f(x)dx=\sum_{k=0}^{m-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)$$
$$+(-1)^{m+1}\frac{(b-a)^{m+2}}{2^{m+2}(m+1)!}\left\{\int_0^1 t^{m+1}f^{(m+1)}\left(t\frac{a+b}{2}+(1-t)a\right)dt\right.$$
$$\left.+\int_0^1 (t-1)^{m+1}f^{(m+1)}\left(tb+(1-t)\frac{a+b}{2}\right)dt\right\}.$$

Thus, the identity (5) and the Lemma 2 is proved.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be $n-$time differentiable function and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ $(n\geq 1)$ is convex on $[a,b]$, then we obtain:*

$$\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{7}$$
$$\leq\frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}}\times\left\{\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q}{2}\right]^{\frac{1}{q}}+\left[\frac{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right]^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2 and Hölder integral inequality, it follows that

$$\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\left(\int_0^1 t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 (1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

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Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then we can write

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1} f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\left(\frac{1}{np+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q + (1-t)\left|f^{(n)}(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{1}{np+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f^{(n)}(b)\right|^q + (1-t)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
&= \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}} \times \left\{\left[\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q}{2}\right]^{\frac{1}{q}} + \left[\frac{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q + \left|f^{(n)}(b)\right|^q}{2}\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

which completes the proof.

Corollary 3. *Under the assumptions of Theorem 3, we have*

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1} f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^{n+1}}{2^{n+1}n!} \times \left\{\left[\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q}{2}\right]^{\frac{1}{q}} + \left[\frac{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q + \left|f^{(n)}(b)\right|^q}{2}\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{8}$$

Proof. For $p > 1$, since

$$\lim_{p\to\infty}\left(\frac{1}{np+1}\right)^{\frac{1}{p}} = 1 \quad \text{and} \quad \lim_{p\to 1^+}\left(\frac{1}{np+1}\right)^{\frac{1}{p}} = \frac{1}{n+1},$$

we have

$$\frac{1}{n+1} < \lim_{p\to\infty}\left(\frac{1}{np+1}\right)^{\frac{1}{p}} < 1, \quad p \in (1,\infty).$$

Hence we obtain the inequality (8).

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be $n-$time differentiable function and $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ $(n \geq 1)$ is convex on $[a,b]$, then we get*

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1} f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-\frac{1}{q}} \times \left\{\left[\frac{1}{(p+1)(p+2)}\left|f^{(n)}(a)\right|^q + \frac{1}{p+2}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left. + \left[\frac{1}{p+2}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{(p+1)(p+2)}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}
\end{aligned} \tag{9}$$

where $p > 1$.

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Proof. From Lemma 2 and using the properties of modulus, we get

$$\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\int_0^1 t^n\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\int_0^1(1-t)^n\left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right\}$$
$$=\frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\int_0^1 \frac{t^n t^{\frac{p}{q}}}{t^{\frac{p}{q}}}\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\int_0^1\frac{(1-t)^n(1-t)^{\frac{p}{q}}}{(1-t)^{\frac{p}{q}}}\left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right\}.$$

Using the Hölder integral inequality, we can write

$$\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\left(\int_0^1\left[\frac{t^n}{t^{\frac{p}{q}}}\right]^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^p\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left[\frac{(1-t)^n}{(1-t)^{\frac{p}{q}}}\right]^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^p\left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, we have

$$\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-\frac{1}{q}}\times\left\{\left[\frac{1}{(p+1)(p+2)}\left|f^{(n)}(a)\right|^q+\frac{1}{p+2}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{1}{p+2}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{(p+1)(p+2)}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}$$

which completes the proof of the theorem (4).

Corollary 4. *In Theorem 4, if we choose $n=1$, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{(b-a)}{4}\left(\frac{q-1}{2q-p-1}\right)^{1-\frac{1}{q}}\times\left\{\left[\frac{1}{(p+1)(p+2)}\left|f'(a)\right|^q+\frac{1}{p+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{1}{p+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{(p+1)(p+2)}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.$$

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Corollary 5. *In Theorem 4, if we choose $n=2$, then we obtain*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{q-1}{3q-p-1}\right)^{1-\frac{1}{q}}\times\left\{\left[\frac{1}{(p+1)(p+2)}\left|f''(a)\right|^q+\frac{1}{p+2}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{1}{p+2}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{(p+1)(p+2)}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Theorem 5. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable function. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]\subset\mathbb{R}$, for $q\geq 1$, then the following inequality obtains:*

$$\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\left\{\left[\frac{1}{(n+2)}\left|f^{(n)}(a)\right|^q+\left(\frac{n+1}{n+2}\right)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \tag{10}$$
$$\left.+\left[\left(\frac{n+1}{n+2}\right)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{(n+2)}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Proof. Suppose that $q=1$. From Lemma 2, we have

$$\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^{n+1}}{2^{n+1}(n+2)!}\left[\left|f^{(n)}(a)\right|+2(n+1)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|+\left|f^{(n)}(b)\right|\right].$$

Suppose now that $q>1$. Using the well known Power-mean integral inequality and Lemma 2, we obtain

$$\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\left(\int_0^1 t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^n\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^n\left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, for $q\geq 1$, then we obtain

$$\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{n+1}\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1 t^n\left[t\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right.$$

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$$+\left(\int_0^1 (1-t)^n \left[t\left|f^{(n)}(b)\right|^q+(1-t)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\Bigg\}$$
$$=\frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\left\{\left[\frac{1}{(n+2)}\left|f^{(n)}(a)\right|^q+\left(\frac{n+1}{n+2}\right)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\left(\frac{n+1}{n+2}\right)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{(n+2)}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Hence, the proof of the theorem is completed.

Corollary 6. *In Theorem 5, if we choose $n=1$, we obtain*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right|\leq\frac{(b-a)}{8}\left\{\left(\frac{|f'(a)|^q+2\left|f'\left(\frac{a+b}{2}\right)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right\}.$$

Remark. In Theorem 5, if we choose $n=2$, we get the inequality (2).

Now, we give the following Hadamard type inequality for concave mappings.

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be $n-$time differentiable function and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is concave on $[a,b]$, then we have:*

$$\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{11}$$
$$\leq\frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}}\left\{\left|f^{(n)}\left(\frac{3a+b}{4}\right)\right|+\left|f^{(n)}\left(\frac{a+3b}{4}\right)\right|\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2 and Hölder integral inequality, we can write

$$\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\left(\int_0^1 t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \tag{12}$$
$$\left.+\left(\int_0^1(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f^{(n)}\right|^q$ is concave on $[a,b]$, we can use the Jensen's integral inequality to get

$$\int_0^1\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt=\int_0^1 t^0\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt \tag{13}$$
$$\leq\left(\int_0^1 t^0dt\right)\left|f^{(n)}\left(\frac{\int_0^1\left(t\frac{a+b}{2}+(1-t)a\right)dt}{\int_0^1 t^0dt}\right)\right|^q$$
$$=\left|f^{(n)}\left(\frac{3a+b}{4}\right)\right|^q$$

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and similarly

$$\int_0^1 \left| f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt \le \left| f^{(n)}\left(\frac{a+3b}{4}\right)\right|^q. \tag{14}$$

Therefore, if we use (13) and (14) in the inequality (12), we obtain the inequality of (11).

Remark. In Theorem 6, if we choose $n=1$, we have the inequality (3).

Corollary 7. *In the inequality (11) if we choose* $n=2$, *then we can*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \left\{ \left| f''\left(\frac{3a+b}{4}\right)\right| + \left| f''\left(\frac{a+3b}{4}\right)\right| \right\}.$$

3 Applications to Special Means

We now consider the means for arbitrary real numbers α, β $(\alpha \neq \beta)$. We take

1. *Arithmetic mean* :

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \ \ \alpha,\beta \in \mathbb{R}^+.$$

2. *Logarithmic mean*:

$$L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|, \ \alpha,\beta \neq 0, \ \alpha,\beta \in \mathbb{R}^+.$$

3. *Generalized log − mean*:

$$L_m(\alpha,\beta) = \left[\frac{\beta^{m+1}-\alpha^{m+1}}{(m+1)(\beta-\alpha)}\right]^{\frac{1}{m}}, \quad m \in \mathbb{Z}\backslash\{-1,0\}, \ \alpha,\beta \in \mathbb{R}^+.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

Proposition 1. *Let* $0<a<b$ *and* $m \in \mathbb{N}, m>1$. *Then, we have:*

$$\begin{aligned}
&|A^m(a,b) - L_m^m(a,b)| \\
&\le m\frac{(b-a)}{4(p+2)^{\frac{1}{q}}}\left(\frac{q-1}{2q-p-1}\right)^{1-\frac{1}{q}} \left\{ \left[\frac{a^{nq-q}}{p+1} + \left(\frac{a+b}{2}\right)^{nq-q}\right]^{\frac{1}{q}} \right. \\
&\left. + \left[\left(\frac{a+b}{2}\right)^{nq-q} + \frac{b^{nq-q}}{p+1}\right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Proof. The assertion follows from Corollary 4 applied for $f(x)=x^m$, $x \in \mathbb{R}$.

Proposition 2. *Let* $a,b \in \mathbb{R}^+$, $a<b$. *Then, we have the following inequality;*

$$\begin{aligned}
|A^m(a,b) - L_m^m(a,b)| \le \frac{(b-a)}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}} &\left\{ \left[\frac{1}{a^{2q}} + \frac{2^{2q+1}}{(a+b)^{2q}}\right]^{\frac{1}{q}} \right. \\
&\left. + \left[\frac{2^{2q+1}}{(a+b)^{2q}} + \frac{1}{b^{2q}}\right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Proof. The assertion follows from Corollary 6 applied for $f(x)=\frac{1}{x}$, $x \in [a,b]$.

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