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A note on the absoute indexed norlund summability

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Abstract: In the present article, we have established a result on indexed Norlund summability factors by generalizing a theorem of Mishra and Sivastava [5] on Cesaro summabilty factors.

Keywords: Absolute summability, summability factors, infinite series.

1 Introduction

Let the infinite series with sequence of partial sums $\{s_n\}$ be $\sum a_n$. Suppose for the sequence $\{s_n\}$, the nth $(C,1)$- mean is $\{t_n\}$. If

$$\sum_{n=1}^{\infty}(n)^{k-1}|t_n-t_{n-1}|^k<\infty, \tag{1}$$

then $\sum a_n$is said to be summable $|C,1|_k, k\geq 1$. (see [4]). Let

$$Q_n=\sum_{v=0}^{n}q_v\to\infty,\ \ as\ \ n\to\infty\,(Q_{-i}=q_i=0, i\geq 1)\,, \tag{2}$$

where $\{q_n\}$ is a sequence with $q_n\in\mathbf{R}^+$. Let the (N,q_n)-mean of the sequence $\{s_n\}$ be $\{T_n\}$, which is generated by the sequence of coefficients $\{q_n\}$, where

$$T_n=\frac{1}{Q_n}\sum_{v=0}^{\infty}q_{n-v}s_v. \tag{3}$$

If

$$\sum_{n=1}^{\infty}\left(\frac{Q_n}{q_n}\right)^{k-1}|T_n-T_{n-1}|^k<\infty, \tag{4}$$

then $\sum a_n$ is said to be summable $|N,q_n|_k, k\geq 1$ (see [3]).

Clearly,$|N,q_n|_k$-summabiity is same as $|C,1|$-summabiity when $q_n=1\ \forall n$. Mishra and Srivatava [5], established the following result for $|C,1|_k$ summability.

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2 Known theorem

Suppose, (Y_n) be a positive non-decreasing sequence and let there be sequences $\{\beta_n\}$ and $\{\mu_n\}$ such that

$$|\Delta\mu_n| \leq \beta_n; \tag{5}$$

$$\beta_n \to 0 \ \ as \ \ n \to \infty; \tag{6}$$

$$|\mu_n| Y_n = O(1) \ \ as \ \ n \to \infty; \tag{7}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n| Y_n < \infty; \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}|s_n|^k = O(Y_m) \ \ as \ \ m \to \infty, \tag{9}$$

then $\sum_{n=1}^{\infty} a_n\mu_n$ is summable $|C,1|_k, k \geq 1$.

3 Main theorem

Suppose, for a non-decreasing sequence (Y_n), let there be sequences $\{\beta_n\}$ and $\{\mu_n\}$ satisfying the conditions (5) to (9) and $\{q_n\}$ be a sequence with $q_n \in \mathbf{R}^+$ such that

$$Q_n = O(nq_n); \tag{10}$$

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n}|s_n|^k = O(Y_m) \ \ as \ \ m \to \infty; \tag{11}$$

$$\frac{Q_{n-r-1}}{Q_n} = O\left(\frac{q_{n-r-1}}{Q_n}\frac{Q_r}{q_r}\right); \tag{12}$$

$$\sum_{n=r+1}^{m+1} \left(\frac{Q_n}{q_n}\right)^{k-1} \frac{q_{n-r}}{Q_n} = O\left(\frac{q_r}{Q_r}\right), \tag{13}$$

then $\sum_{n=1}^{\infty} a_n\mu_n$ is summable $|N,q_n|_k, k \geq 1$. The condition (11) reduces to condition (9) if $q_n = 1 \ \forall n$. After reading [1], [2] and [6], we have established the following result. To establish our main result we need the following lemma.

4 Lemma

Suppose (Y_n) be a positive non decreasing sequence and let there be sequences $\{\beta_n\}$ and $\{\mu_n\}$ such that the conditions (6) to (10) are satisfied.Then,

$$\beta_n Y_n = O(1) \ \ as \ \ n \to \infty, \tag{14}$$

$$\sum_{n=1}^{\infty} \beta_n Y_n < \infty. \tag{15}$$

5 Proof of the main theorem

Let the (N,q_n)- mean of the series $\sum_{n=1}^{\infty} a_n\mu_n$ be denoted by (τ_n) . Then, by definition, we have

$$\tau_n = \frac{1}{Q_n}\sum_{r=0}^{n} q_{n-r}\sum_{s=0}^{r} a_s\mu_s = \frac{1}{Q_n}\sum_{s=0}^{n} a_s\mu_s \sum_{r=s}^{n} q_{n-r} = \frac{1}{Q_n}\sum_{s=0}^{n} a_s\mu_s Q_{n-s} = \frac{1}{Q_n}\sum_{r=0}^{n} a_r Q_{n-r}\mu_r$$

Thus

$$\begin{aligned}
\tau_n-\tau_{n-1} &= \frac{1}{Q_n}\sum_{r=1}^{n}Q_{n-r}a_r\mu_r-\frac{1}{Q_{n-1}}\sum_{r=1}^{n-1}Q_{n-r-1}a_r\mu_r\\
&=\sum_{r=1}^{n}\left(\frac{Q_{n-r}}{Q_n}-\frac{Q_{n-r-1}}{Q_{n-1}}\right)a_r\mu_r\\
&=\frac{1}{Q_nQ_{n-1}}\sum_{r=1}^{n}(Q_{n-r}Q_{n-1}-Q_{n-r-1}Q_n)a_r\mu_r\\
&=\frac{1}{Q_nQ_{n-1}}\left[\sum_{r=1}^{n-1}\Delta\{(Q_{n-r}Q_{n-1}-Q_{n-r-1}Q_n)\,\mu_r\}\right]\sum_{v=1}^{r}a_v,\quad with\quad q_0=0\\
&=\frac{1}{Q_nQ_{n-1}}\left[\sum_{r=1}^{n-1}(q_{n-r}Q_{n-1}-q_{n-r-1}Q_n)\,\mu_rs_r+\sum_{r=1}^{n-1}(Q_{n-r-1}Q_{n-1}-Q_{n-r-2}Q_n)\,\Delta\mu_rY_rs_r\right](ByAbel'stransformation)\\
&=T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4},
\end{aligned}$$

In order to complete the proof of the main theorem by using Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{Q_n}{q_n}\right)^{k-1}|T_{n,j}|^k<\infty\ \ for\ \ j=1,2,3,4.$$

Now, we have

$$\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}|T_{n,1}|^k\\
&\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}|\frac{1}{Q_nQ_{n-1}}\sum_{r=1}^{n-1}q_{n-r}Q_{n-1}\mu_rs_r|^k\\
&\leq\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\frac{1}{Q_n}\left(\sum_{r=1}^{n-1}q_{n-r}|\mu_r|^k|s_r|^k\right)\left(\frac{1}{Q_n}\sum_{r=1}^{n-1}q_{n-r}\right)^{k-1}\quad(Using\ Holder's\ inequality)\\
&=O(1)\sum_{r=1}^{m}|\mu_r|^k|s_r|^k\sum_{n=r+1}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\left(\frac{q_{n-r}}{Q_n}\right)\\
&=O(1)\sum_{r=1}^{m}|\mu_r|^k|s_r|^k\frac{q_r}{Q_r},\ by\ (13)\\
&=O(1)\sum_{r=1}^{m}\frac{q_r}{Q_r}|s_r|^k|\mu_r||\mu_r|^{k-1}\\
&=O(1)\sum_{r=1}^{m-1}\Delta|\mu_r|\sum_{w=1}^{r}\frac{q_w}{Q_w}|s_w|^k+O(1)|\mu_m|\sum_{r=1}^{m}\frac{q_r}{Q_r}|s_r|^k\\
&=O(1)\sum_{r=1}^{m-1}|\Delta\mu_r|Y_r+O(1)|\mu_m|Y_m\ ,\ by\ (11)\\
&=O(1),\ \ as\ \ m\to\infty.\text{(By the lemma and (7))}
\end{aligned}$$

Next,

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}|T_{n,2}|^k &= \sum_{n=1}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\left|\frac{1}{Q_nQ_{n-1}}\sum_{r=1}^{n-1}q_{n-r-1}Q_n\mu_r s_r\right|^k\\ &\leq \sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\frac{1}{Q_{n-1}}\left(\sum_{r=1}^{n-1}q_{n-r-1}|\mu_r|^k|s_r|^k\right)\left(\frac{1}{Q_{n-1}}\sum_{r=1}^{n-1}q_{n-r-1}\right)^{k-1}\\ &= O(1)\sum_{r=1}^{m}|\mu_r|^k|s_r|^k\sum_{n=r+1}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\left(\frac{q_{n-r-1}}{Q_{n-1}}\right)\\ &= O(1)\sum_{r=1}^{m}|\mu_r|^k|s_r|^k\frac{q_r}{Q_r}\\ &= O(1),\ \ as\ \ m\to\infty,\quad As\ in\ proof\ of\ the\ 1st\ part.\end{aligned}$$

Further,

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}|T_{n,3}|^k &= \sum_{n=1}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\left|\frac{1}{Q_nQ_{n-1}}\sum_{r=1}^{n-1}Q_{n-r-1}Q_{n-1}\Delta\mu_r s_r\right|^k\\ &\leq \sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\frac{1}{Q_n}\left(\sum_{r=1}^{n-1}Q_{n-r-1}|\Delta\mu_r||s_r|^k\right)\left(\frac{1}{Q_n}\sum_{r=1}^{n-1}Q_{n-r-1}|\Delta\mu_r|\right)^{k-1}.\end{aligned}$$

Since,

$$\left(\frac{1}{Q_n}\sum_{r=1}^{n-1}Q_{n-r-1}|\Delta\mu_r|\right)\leq\sum_{r=1}^{n-1}|\Delta\mu_n|\leq n|\Delta\mu_r|\leq n\beta_n.$$

Therefore,

$$\begin{aligned}&\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}|T_{n,3}|^k\\ &\leq O(1)\sum_{r=1}^{m}(r\beta_r)^{k-1}|\Delta\mu_r||s_r|^k\sum_{n=r+1}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\frac{Q_{n-r-1}}{Q_n}\\ &= O(1)\sum_{r=1}^{m}|\Delta\mu_r||s_r|^k\frac{q_r}{Q_r}\\ &\leq O(1)\sum_{r=1}^{m}\beta_r|s_r|^k\frac{q_r}{Q_r}\\ &= O(1)\sum_{r=1}^{m-1}\Delta(\beta_r)\sum_{w=1}^{r}\frac{q_w}{Q_w}|s_w|^k+O(1)(\beta_m)\sum_{r=1}^{m}\frac{q_r}{Q_r}|s_r|^k\\ &= O(1)\sum_{r=1}^{m-1}|\Delta\beta_r|Y_r+O(1)(\beta_m)Y_m\\ &= O(1)\end{aligned}$$

Now,

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}|T_{n,4}|^k &= \sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\left|\frac{1}{Q_nQ_{n-1}}\sum_{r=1}^{n-1}Q_{n-r-2}Q_n\Delta\mu_r s_r\right|^k \\
&\leq \sum_{n=2}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\frac{1}{Q_{n-1}}\left(\sum_{r=1}^{n-1}Q_{n-r-2}|\Delta\mu_r||s_r|^k\right)\frac{1}{Q_{n-1}}\sum_{r=1}^{n-1}Q_{n-r-2}|\Delta\mu_r|^{k-1} \\
&= O(1)\sum_{r=1}^{m}(r\beta_r)^{k-1}|\Delta\mu_r||s_r|^k\sum_{n=r+1}^{m+1}\left(\frac{Q_n}{q_n}\right)^{k-1}\left(\frac{Q_{n-r-1}}{Q_n}\right), \quad (as\ \ above) \\
&= O(1)\sum_{r=1}^{m}|\Delta\mu_r||s_r|^k\frac{q_r}{Q_r} \\
&= O(1), \quad (as\ \ above)
\end{aligned}
$$

This completes the proof of the theorem.

Conclusion

If (Y_n) is a positive non-decreasing sequence and let there be sequences $\{\beta_n\}$ and $\{\mu_n\}$ such that the conditions (5) to (9) along with the conditions (14) and (15) are satisfied then the series $\sum_{n=1}^{\infty} a_n\mu_n$ is summable $|N,q_n|_k, k \geq 1$, under the conditions (10) to (13). Thus, our result generalizes the result of Mishra and Srivastava [5].

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Bor, H., A note on two summability methods, Proc. Amer. Math. Soc. 98(1986), 81-84.
[2] Bor, H., A note on absolute summability factors, Int. J. Math. And Math. Sci., 17(3), (1994), 479-482.
[3] Flett, T.M., On an extension of absolute summability and some theorems of Little wood and Palay, Proc. Lond. Math. Soc. 7(1957), 113-141.
[4] Hardy, G.H., Divergent series, Oxford University press,(1949).
[5] Mishra, K.N and Srivastava, R.S.L., On absolute Cesaro summability factors of infinite series, Portugaliae Math.,42(1),(1983-84), 53-61.
[6] Misra, U.K., Misra, M. and Padhy, B.P., On the local property of indexed Norlund summability of a factored Fourier series, Int. J. Res. and Rev. in App. Sci., 5(1),(2010), 52-58.

[7] Mishra, L.N. , On existence and behavior of solutions to some nonlinear integral equations with Applications, Ph.D. Thesis (2017), National Institute of Technology, Silchar , Assam, India.

[8] Mishra, L. N., Mishra, V.N., Khatri, K., Deepmala, On The Trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L^r, \xi(t))(r \geq 1)-$ class by matrix $(C^1.N_p)$ Operator of conjugate series of its Fourier series, Applied Mathematics and Computation, Vol. 237 (2014) 252-263.

[9] Mishra, V.N., Some Problems on Approximations of Functions in Banach Spaces, Ph.D. Thesis (2007), Indian Institute of Technology, Roorkee, Uttarakhand, India.

[10] Mishra, V.N. , Mishra, L.N. , Trigonometric Approximation of Signals (Functions) in $L_p(p \geq 1)$'norm, International Journal of Contemporary Mathematical Sciences, Vol. 7, no. 19, (2012), pp. 909-918.