New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2018.312

Total domination on anti fuzzy graph

R.Muthuraj[1] *and A.Sasireka*[2]

[1]Department of Mathematics, H.H.The Rajah's College, Pudukkottai-622 001, TamilNadu, India
[2]Department of Mathematics, PSNA College of Engineering and Technology, Dindigul-624 622, TamilNadu, India

Received: 2 April 2018, Accepted: 10 September 2018
Published online: 25 October 2018.

Abstract: In this paper, the concept of total dominating set and total domination number on anti fuzzy graph is introduced. The bounds on total domination number of an anti fuzzy graph are obtained. Using strong adjacency matrix, an algorithm is derived for finding minimal total dominating set of anti fuzzy graph G_A. These concepts are applied on anti cartesian product of anti fuzzy graphs and obtained the results on them.

Keywords: Anti fuzzy graph, total domination number, v-nodal anti fuzzy graph, uninodal anti fuzzy graph.

1 Introduction

Domination plays a vital role in graph theory. When vagueness occurs within objects and its relation, the concept of fuzzy graph is developed. Sometimes uncertainty occurs with only the relation between the objects; in such cases an anti fuzzy graph is introduced. R.Seethalakshmi and R.B.Gnanajothi[8] introduced the definition of anti fuzzy graph. R.Muthuraj and A.Sasireka[6] defined some types of anti fuzzy graph. Moderson J.N. and Peng[3] defined the cartesian product of two arbitrary fuzzy graphs. R.Muthuraj and A.Sasireka[5, 7] illustrated some operations on anti fuzzy graphs such as anti union, anti join, anti cartesian product and anti composition and also introduced the concept of domination on anti fuzzy graph. A.Somasundaram and S.Somasundaram[10] introduced total domination in fuzzy graphs using effective edges. In this paper, we introduce the concept of total domination number on anti fuzzy graph and obtained the bounds on them. Using strong neighbourhood fuzzy matrix, an algorithm is described to predict the total dominating set for an anti fuzzy graph. These concepts are applied on anti cartesian product of anti fuzzy graphs. The results are examined and some theorems are derived from them.

2 Preliminaries

In this section, basic concepts of anti fuzzy graph are discussed. Notations and more formal definitions which are followed as in [4, 5, 6, 9].

Definition 1. [9] *An anti fuzzy graph G_A= (σ,μ) is a pair of functions $\sigma : V \to [0,1]$ and $\mu : V\times V \to [0,1]$, with $\mu(u,v)\geq \sigma(u) \vee \sigma(v)$ for all $u,v\in V$.*

Note. μ is considered as reflexive and symmetric. In all examples σ is chosen suitably. i.e., undirected anti fuzzy graphs are only considered.

Definition 2. [5] *The order p and size q of an anti fuzzy graph G_A=(V,σ,μ) are defined to be $p=\sum_{x\in V}\sigma(x)$ and $q=\sum_{xy\in E}\mu(x,y)$. It is denoted by O(G) and S(G).*

* Corresponding author e-mail: sasireka.psna@gmail.com

Definition 3. [6] *Two vertices u and v in G_A are called adjacent if (1/2) $[\sigma(u) \vee \sigma(v)] \leq \mu(u,v)$.*

Definition 4. [4] *The anti complement of anti fuzzy graph $G_A=(\sigma,\mu)$ is an anti fuzzy graph $\overline{G_A} = (\bar{\sigma}, \bar{\mu})$ where $\bar{\sigma} = \sigma$ and $\bar{\mu}(u,v) = \mu(u,v) - (\sigma(u) \vee \sigma(v))$ for all u,v in V.*

Definition 5. [9] *An anti fuzzy graph $G_A = (\sigma,\mu)$ is a strong anti fuzzy graph of $\mu(u,v)=\sigma(u)\vee\sigma(v)$ for all $(u,v)\in\mu^*$ and G_A is a complete anti fuzzy graph if $\mu(u,v)= \sigma(u)\vee\sigma(v)$ for all $(u,v)\in\mu^*$ and $u,v \in \sigma^*$. Two vertices u and v are said to be neighbors if $\mu(u,v) > 0$.*

Definition 6. [6] *An edge $e = \{u, v\}$ of an anti fuzzy graph G_A is called an effective edge if $\mu(u,v) = \sigma(u) \vee \sigma(v)$.*

Definition 7. [5] *u is a vertex in an anti fuzzy graph G_A then $N(u) = \{v: (u,v)$ is an effective edge$\}$ is called the open neighborhood of u and $N[u] = N(u) \cup \{u\}$ is called closed neighborhood of u.*

Definition 8. [4] *A path P_A in an anti fuzzy graph is a sequence of distinct vertices $u_0, u_1, u_2, \ldots, u_n$ such that $\mu(u_{i-1},u_i)>0$, $1\leq i \leq n$. Here $n\geq 0$ is called the length of the path P_A. The consecutive pairs (u_{i-1}, u_i) are called the edges of the path.*

Definition 9. [4] *A cycle in G_A is said to be an anti fuzzy cycle if it contains more than one weakest edge. It is denoted by C_A.*

Definition 10. [5] *Let $G_A^* = G_{A_1}^* \times G_{A_2}^* = (V, E')$ be the anti cartesian product of anti fuzzy graphs where $V= V_1 \times V_2$ and $E' = \{(u_1,u_2),(u_1,v_2) / u_1 \in V_1, (u_2,v_2)\in E_2\} \cup (u_1,w_2),(v_1,w_2) / w_2\in V_2, (u_1,v_1)\in E_1 \}$. Then the anti cartesian product of two anti fuzzy graphs, $G_A=G_{A_1} \times G_{A_2}$: $(\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ is an anti fuzzy graph and is defined by*

$(\sigma_1\times \sigma_2)(u_1, u_2) = \max\{\sigma_1(u_1), \sigma_2(u_2)\}$ for all $(u_1, u_2)\in V$,
$(\mu_1\times\mu_2)((u_1, u_2), (u_1, v_2)) = \max\{\sigma_1(u_1), \mu_2(u_2, v_2)\}$ for all $u_1\in V_1$ and $(u_2,v_2)\in E_2$,
$(\mu_1\times \mu_2)((u_1, w_2),(v_1, w_2)) = \max\{\sigma_2 (w_2), \mu_1(u_1, v_1)\}$ for all $w_2\in V_2$ and $(u_1, v_1)\in E_1$.

Then the anti fuzzy graph $G_A= (\sigma_1 \times \sigma_2,\mu_1 \times \mu_2)$ is said to be the anti cartesian product of anti fuzzy graphs $G_{A_1} = (\sigma_1, \mu_1)$ and $G_{A_2} = (\sigma_2, \mu_2)$.

Definition 11. [6] *Every vertex in an anti fuzzy graph G_A has unique fuzzy values then G_A is said to be v-nodal anti fuzzy graph. i.e. $\sigma(u) = c$ for all $u\in V(G_A)$.*

Definition 12. [6] *Every edge in an anti fuzzy graph G_A has unique fuzzy values then G_A is said to be e-nodal anti fuzzy graph. i.e. $\mu(u,v) = c$ for all $(u,v) \in E(G_A)$.*

Definition 13. [6] *Every vertices and edges in an anti fuzzy graph G_A have the unique fuzzy values then G_A is called as uninodal anti fuzzy graph. If $\sigma(u)=c_1$ and $\mu(u,v)= c_2$ in an anti fuzzy graph G_A then G_A is called as binodal anti fuzzy graph.*

Definition 14. *The strong neighbourhood of an edge e_i in an anti fuzzy graph G_A is $N_s(e_i)=\{e_j \in E(G) / e_j$ is an effective edge with $\vee N(e_i)$ in G_A and adjacent to $e_i\}$.*

Definition 15. *An edge $e = \{u,v\}$ of an anti fuzzy graph G_A is called an weak edge if $\mu(u,v)\neq \sigma(u) \vee \sigma(v)$.*

3 Total domination on anti fuzzy graph

In this section, the definition of total dominating set and total domination number on anti fuzzy graph G_A are defined. These concepts are applied on some types of simple anti fuzzy graph G_A, few elementary bounds on total domination number are described along with the corresponding theorems and results are illustrated.

Definition 16. [7] *A set $D \subseteq V(G_A)$ is said to be a dominating set of an anti fuzzy graph G_A if for every vertex $v \in V(G_A) \setminus D$ there exists u in D such that v is a strong neighborhood of u with $\mu(u,v)=\sigma(u) \vee \sigma(v)$ otherwise it dominates itself.*

A dominating set D with minimum number of vertices is called a minimal dominating set if no proper subset of D is a dominating set.

The maximum fuzzy cardinality taken over all minimal dominating set in G_A is called a domination number of anti fuzzy graph G_A and is denoted by $\gamma(G_A)$ or γ_A. ie, $|D|_f = \sum_{v \in D} \sigma(v)$.

Definition 17. *A set $D \subseteq V(G_A)$ is said to be a total dominating set of an anti fuzzy graph G_A if for every vertex $v \in V(G_A) \setminus D$ is adjacent to atleast one strong neighbourhood vertex in D and the induced subgraph of D has no isolated vertices. i.e., A dominating set D of an anti fuzzy graph G_A is said to be a total dominating set of G_A if there exists a vertex in D is not isolate.*

A total dominating set D of an anti fuzzy graph G_A with minimum number of vertices is called a minimal total dominating set of G_A if no proper subset of D is a dominating set.

The maximum fuzzy cardinality taken over all minimal total dominating set is called total domination number of G_A and it is denoted by $\gamma_t(G_A)$ or γ_{At}.

Definition 18. *G_A is an anti fuzzy graph and $u,v \in V(G_A)$. If v is said to be a support vertex to u then v is adjacent to atleast one end vertex u in G_A.*

Note. Every support vertices in G_A contained in total dominating set of G_A. It may dominate more than one vertex in $V \setminus D$ and also in D.

Example 1.

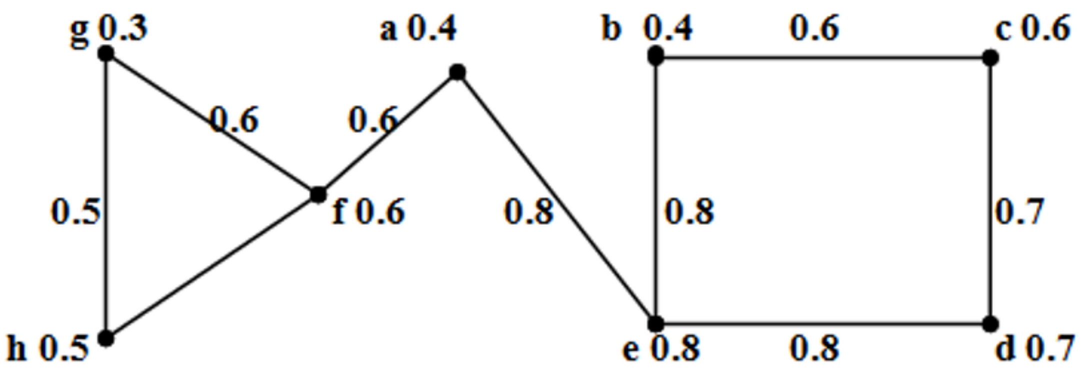


Fig. 1: An Anti Fuzzy Graph G_A

In Figure 1, the total dominating sets are {d,e,f,g}, {d,e,f,h} and {d,e,f,a}. The corresponding total domination numbers are 2.6, 2.3 and 2.4 respectively. Therefore, minimal total dominating set of G_A is {d,e,f,g} and $\gamma_t(G_A)$=2.6

Theorem 1. *If a total dominating set D is a minimal total dominating set of a connected anti fuzzy graph G_A then each u∈D has atleast one the following conditions hold,*

(1) *There exists a vertex $v \in V \setminus D$ such that $N[v] \cap D = \{u\}$.*
(2) *$< D \setminus \{v\} >$ contains isolated vertex.*

Proof. Let G_A be a simple connected anti fuzzy graph and D be a total dominating set of G_A. Let u∈D and v∈V\D. consider that N(v)∩D={u}. Suppose D′=D\{u} is a total dominating set of G_A. Then v is not dominated by any vertex in D which is contradict to D′ is total dominating set. Thus the vertex u is only strong neighbourhood to v and v is dominated by u. Therefore, N[v]∩ D = {u} and D is minimal total dominating set. Thus every vertex has atleast one neighbour in D. If u is removed from D then the neighbour of u is isolate in D.

Conversely, if the two conditions hold then we prove that D is a minimal total dominating set of anti fuzzy graph G_A. Suppose D is not minimal total dominating set. Then there exists $u \in D$ such that $D\setminus\{u\}$ is a total dominating set. Thus u is a strong neighbour to atleast one vertex in $D\setminus\{u\}$. Therefore, $D\setminus\{u\}$ may contains isolated vertex. If $D\setminus\{u\}$ is a total dominating set then every vertex in $V\setminus D$ is a strong neighbour to atleast one vertex in D\{u}. Therefore the second condition fails. Hence D is a minimal total dominating set.

Proposition 1. *For any anti fuzzy graph G_A, $\gamma_t(G_A) \leq 2p/3$ where $G_A \neq C_A$.*

Theorem 2. *Let G_A be a simple connected anti fuzzy graph and $\overline{G_A}$ be an anti complement of G_A then $\gamma_t(G_A)+\gamma_t(\overline{G_A}) \leq 5p/3$.*

Proof. Let us consider that G_A is a simple connected anti fuzzy graph. $\overline{G_A}$ may not have any isolated vertices. Then $\gamma_t(G_A)$= 2p/3 and $\gamma_t(\overline{G_A}) \leq$ p. Therefore, $\gamma_t(G_A)+\gamma_t(\overline{G_A}) \leq 2p/3 + p$, $\gamma_t(G_A)+\gamma_t(\overline{G_A}) \leq 5p/3$.

Theorem 3. *For an uninodal anti fuzzy graph G_A, $\gamma_t(G_A)= \begin{cases} \leq \frac{p}{2}, & for \;\; p \leq q \\ \geq \frac{p}{2}, & for \;\; p > q \end{cases}$*

Proof. Let G_A be an uninodal anti fuzzy graph with order p and size q.

If p=q then the number of vertices and edges in an anti fuzzy graph are equal. Thus every vertex has atleast one strong neighbour in G_A. It yields that the vertex may dominate itself or dominate to atleast one vertex in G_A. Therefore, $\gamma_{At} \leq p/2$.

If p<q then the number of vertices is less than the number of edges in an anti fuzzy graph G_A which implies that every vertex is adjacent to more than two vertices in G_A. Hence the number of elements in total dominating set is less than or equal to m/2. Therefore, $\gamma_{At} \leq p/2$.

If p>q then the vertices have minimum number of neighbourhoods in G_A. That is, the vertices are adjacent to atmost m/2 vertices and not an isolated vertex in G_A. Hence the maximum number of vertices dominated itself and gives $\gamma_{At} \geq p/2$.

Theorem 4. *If G_A is an uninodal anti fuzzy graph with no isolated vertices then $\gamma_t(G_A)$ = p - $\Delta(G_A)$, where $\Delta(G_A)$ is maximum degree of G_A.*

Proof. Let G_A be an uninodal anti fuzzy graph with no isolated vertices and D be the total dominating set in G_A. Every vertex in D dominates atmost Δ(G)-1 vertices in V(G_A)\D and dominate atleast one vertex in D.

Hence,

$$p = \Delta(G) - 1 + |D| + 1$$
$$p = \Delta(G) + \gamma_t(G_A)$$
$$\gamma_t(G_A) = p - \Delta(G)$$

Theorem 5. *For complete uninodal anti fuzzy graph G_A, $\gamma_t(G_A) = 2\sigma(u_1)$, for all $u_1 \in V(G_A.)$*

Proof. Let G_A be a complete uninodal anti fuzzy graph and D be the total dominating set of G_A. By the definition of complete anti fuzzy graph, every pair of vertices are adjacent to each other and by the definition of uninodal anti fuzzy graph, there exists vertices and edges assign the same fuzzy values. In complete uninodal anti fuzzy graph, every single vertex can dominate the remaining vertices. But in total dominating set no vertex is isolated. Hence D=$\{u_1,u_2\}$.

Therefore, $\gamma_t(G_A) = \sigma(u_1) + \sigma(u_2) = \sigma(u_1) + \sigma(u_1)\{since G_A is uninodal anti fuzzy graph\} = 2\sigma(u_1.)$

Theorem 6. *For complete uninodal anti fuzzy graph G_A, and $\overline{G_A}$ is a anti complement of G_A then $\gamma_t(G_A)+\gamma_t(\overline{G_A})$= p + 2 $\sigma(u_1)$, for all $u_1 \in V(G_A)$*

Proof. In complete anti fuzzy graph G_A, $\gamma_t(G_A)$= $2\sigma u_1$ {Theorem 5}, In anti complement of $\overline{G_A}$, all vertices are isolated then $\gamma_t(\overline{G_A})$= p. Therefore, $\gamma_t(G_A)+\gamma_t(\overline{G_A})$= p +2 $\sigma(u_1.)$

Note.

(1) In G_A, a pendent vertex incident with only weakest edge then the vertex dominates itself and also it may be a member of total dominating set.
(2) Every vertex in total dominating set may have strong neighbours with effective edge.
(3) $\delta \leq p - \Delta(G_A.)$

Proposition 2. (1) *For any v-nodal anti fuzzy graph, $\gamma_{At} \leq \frac{p}{3}$.*
(2) *For any e-nodal anti fuzzy graph, $\gamma_{At} \leq \frac{4p}{3}$.*

Theorem 7. *If G_A is an anti fuzzy cycle then $\gamma_{At} \leq \frac{p+q-\Delta(G_A)}{2}$.*

Proof. G_A is an anti fuzzy cycle then there is a vertex in an anti fuzzy cycle, dominate atmost two neighbours in G_A. To construct a total dominating set, choose a vertex(say u) which have maximum fuzzy value and this vertex dominate their two neighbours. To make non-isolated vertices in D, choose the strong neighbourhood of vertex u in $V \setminus D$.

Hence $\gamma_{At} \leq \frac{p+q-\Delta(G_A)}{2}$.

Theorem 8. *If G_A is an anti fuzzy path then $\gamma_t(G_A) \leq p - \delta$, where δ is minimum degree of G_A.*

Proof. Let G_A be an anti fuzzy path with order p and δ be the minimum degree of G_A. Let D be the minimal total dominating set of G_A. In an anti fuzzy path, except starting and ending vertices(say u_1 and u_n) have maximum two neighbours and it may be the set M. Therefore, the total dominating set contains maximum number of vertices from M. The vertices u_1 and u_n dominated by some vertex in M and it contains degree δ . Hence $\gamma_t(G_A \leq p - \delta$.

Theorem 9. *Let G_A be an anti fuzzy graph and $v \in V(G_A)$ with $\sigma(v) = \wedge\sigma(V(G_A))$. If $N_s(v) = u$ and $|N_s(v)| \leq 1$ then u belongs to total dominating set of G_A.*

Proof. Let G_A be an anti fuzzy graph and v$\in$V(G_A) and D be a total dominating set of G_A. Consider that u$\in$V(G_A) and N_s(v)=u. If $|N(v)| \leq 1$ then v is adjacent to u only or it may be isolate in G_A. Hence u$\in$D.

Corollary 1. If G_A is an anti fuzzy graph then $\gamma(G_A) \leq \gamma_t(G_A)$.

Proof. Let G_A be an anti fuzzy graph. Consider that D is a dominating set and D_t is a total dominating set of G_A. Every vertices in D dominates atleast one vertex in V\D also the vertices in D may be isolated. But by the definition of total dominating set, no vertex is isolated. Therefore the total dominating set contains atleast one vertex is greater than or equal to the dominating set of anti fuzzy graph G_A. ie., $|D| \leq |D_t|$. Hence $\gamma(G_A) \leq \gamma_t(G_A)$.

Theorem 10. *Let an anti fuzzy graph G_A contain more than one component without isolated vertices then the total dominating set contains atleast two vertices in each components and $\gamma_t(G_A) \leq p\text{-}\delta$ where p= $\sum\sigma(u)$, $u \in V(G_A)$ and δ = min {minimum degree of $C(G_A)$}*

Proof. Let G_A be an anti fuzzy graph and D be a total dominating set of G_A. Consider that G_A contains atleast one component. ie., every component of G_A contains atleast two vertices, Since G_A has no isolated vertices. To construct a total dominating set, there are no isolated vertices in D. Therefore atleast two vertices in each component should be contained in D. Hence $\gamma_t(G_A) \leq p\text{-}\delta$.

Corollary 2. If G_A contains m components then $\gamma_t(G_A) \leq \sum (p\text{-}\delta)_i$, where i=1 to m components.

Theorem 11. *If G_A is an anti fuzzy tree then $\gamma_t(G_A) \leq p\text{-}\sum\sigma(u_i)$, where u_i are leaf in G_A.*

Proof. Let G_A be an anti fuzzy tree and D be a total dominating set of G_A. Then the root and subroot vertices of G_A contained in D. Because these vertices have more adjacencies in G_A with maximum fuzzy value which gives except the leaf of a tree remaining all vertices contained in total dominating set. Hence $\gamma_t(G_A) \leq p - \sum\sigma(u_i)$

Theorem 12. *If G_A is star then $\gamma_t(G_A)=\sigma(r) + \vee(\sigma(u_i))$ where u_i is a leaf and r is root of G_A.*

Proof. If G_A is a star and D is minimal total dominating set of G_A. In G_A, the root vertex(say r) should be adjacent to all remaining vertices in G_A with maximum fuzzy value. Thus it dominates itself and all vertices. Therefore, D={r}. But D is a minimal total dominating set. Therefore the vertex r cannot be isolate in D. Hence a support vertex is chosen which has maximum fuzzy value (say u_i). Hence D= {r, u_i} and $\gamma_t(G_A)=\sigma(r) + \vee(\sigma(u_i))$.

4 Algorithms to find total dominating set of anti Fuzzy graph

In this section, an algorithm is derived for finding the minimal total dominating set of an anti fuzzy graph G_A using strong adjacency matrix.

Definition 19. [7] *An anti fuzzy graph $G_A=(\sigma,\mu)$ with the fuzzy relation μ to be reflexive and symmetric is completely determined by the adjacency fuzzy matrix and it is denoted by M_μ. Where $(M_\mu)_{i,j} = \begin{cases} \mu(v_i,v_j), for \ \ i \neq j \\ \sigma(v_i), for \ \ i = j \end{cases}$ If σ^* contains n elements then M_μ is a square matrix of order n.*

Definition 20. [7] *Let G_A be a simple connected anti fuzzy graph. Then the strong adjacency matrix $(M_\mu)'$ is defined as,*

$$(M_\mu)' = \begin{cases} \mu(v_i,v_j), \ \ for \ \ i \neq j \text{ and } v_i \text{ is a strong neighbourhood to } v_j \\ \sigma(v_i), \ \ for \ \ i = j \end{cases}$$

Algorithm 1. Let us define the algorithm for finding minimal total dominating set in an anti fuzzy graph G_A with m vertices. Let D be a minimal total dominating set of G_A.

Step 1: For G_A, to construct the $(M_\mu)'$ and p=sum of main diagonal elements.
Step 2: Let D = $\emptyset$
Step 3: Find $\sum R(u_i) and \sum C(u_i)$

Step 4: In $(M_\mu)'$, choose the maximum value in $R(u_i)$ or $C(u_i)$. If tie occurs then choose the vertex which have maximum $\sigma(u_i)$ in main diagonal otherwise break it arbitrarily.

Step 5: Select the corresponding vertex in the selected row as a dominated vertex and write the vertex in the set D.

Step 6: Delete the selected (labeled vertex) row and column with horizontal and vertical line. The resulting reduced matrix is called as $(M_\mu)'_1$.

Step 7: If the reduced matrix $(M_\mu)'_1$ has any rows and columns goto step 3, until to get only diagonal elements in the reduced matrix.

Step 8: Check the connectivity of the vertices in D. If there exists any isolated vertex then move to step 9. Otherwise STOP.

Step 9: Choose the maximum value in $\sum R(u_i)$ from the reduced matrix, check the corresponding vertex is adjacent to the isolate vertex in V(D) by using $(M_\mu)'$. If the adjacencies between vertices not occur then choose the next maximum value. Repeat this step until D has no isolate vertex.

Example 2. For the Figure 2, The following steps for finding the total dominating set using algorithm are as follows,

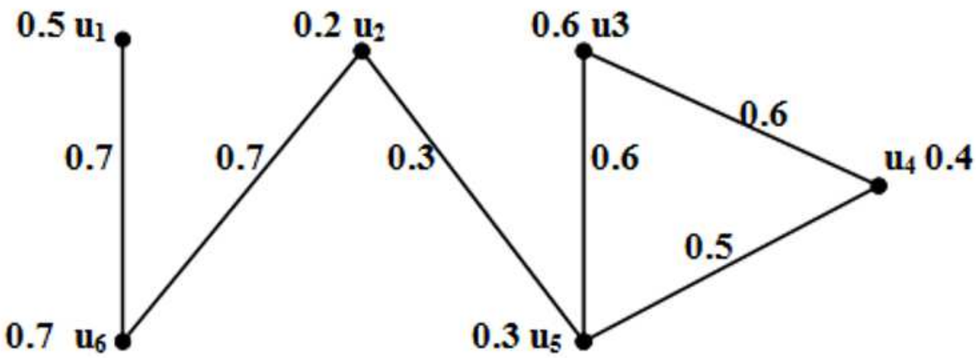


Fig. 2: An Anti Fuzzy Graph G_A

Step 1: For the given anti fuzzy graph G_A, the strong adjacency matrix $(M_\mu)'$ is given as

$$(M_\mu)' = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0.7 \\ 0 & 0.2 & 0 & 0 & 0 & 0.7 \\ 0 & 0 & 0.6 & 0.6 & 0.6 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.3 & 0 \\ 0.7 & 0.7 & 0 & 0 & 0 & 0.7 \end{bmatrix} \end{array}$$

Step 2: Let the total dominating set, D= $\emptyset$

Step 3:

$$u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \quad \sum R(u_i)$$

$$(\mathrm{M}_{\mu})' = \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0.7 \\ 0 & 0.2 & 0 & 0 & 0 & 0.7 \\ 0 & 0 & 0.6 & 0.6 & 0.6 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.3 & 0 \\ 0.7 & 0.7 & 0 & 0 & 0 & 0.7 \end{bmatrix} \begin{array}{l} 1.2 \\ 0.9 \\ 1.8 \\ 1.0 \\ 0.9 \\ 2.1 \rightarrow \end{array}$$
$$\sum C(u_i) \quad 1.2 \quad 0.9 \quad 1.8 \quad 1.0 \quad 0.9 \quad 2.1 \quad [7.9]$$

Step 4: In $(\mathrm{M}\mu)'$, $\mathrm{R}(u_6)$ has maximum value. Then select u_6 and D={ u_6}

Step 5: Delete the selected labeled vertex u_6 in row and column wise at $(\mathrm{M}\mu)'$. The resulting reduced matrix say $(\mathrm{M}\mu)'_1$.

Step 6: If $(\mathrm{M}\mu)'_1$ is a strong adjacency matrix then go to step 3.

Step 3A:

$$\begin{array}{cc} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & \sum R1(u_i) \end{array} \\ (\mathrm{M}_{\mu})'_1 = \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{array} & \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0.6 & 0.6 \\ 0 & 0 & 0.6 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.3 \end{bmatrix} \begin{array}{l} 0.5 \\ 0.2 \\ 1.8 \rightarrow \\ 1.0 \\ 0.9 \end{array} \end{array}$$
$$\sum C1(u_i) \quad 0.5 \quad 0.2 \quad 1.8 \quad 1.0 \quad 0.9 \quad [4.4]$$

Step 4A: In $(\mathrm{M}_{\mu})'_1$, $\mathrm{R1}(u_3)$ has maximum value. Then select u_3 and D={u_6, u_3}

Step 5A: Delete the selected labeled vertex u_3 in row and column wise at $(\mathrm{M}_{\mu})'_1$. The resulting reduced matrix say $(\mathrm{M}_{\mu})'_2$.

Step 7: $(\mathrm{M}_{\mu})'_2$ has four rows with diagonal values only.

$$\begin{array}{cc} & \begin{array}{ccccc} u_1 & u_2 & u_4 & u_5 & \sum R2(u_i) \end{array} \\ (\mathrm{M}_{\mu})'_2. = \begin{array}{c} u_1 \\ u_2 \\ u_4 \\ u_5 \end{array} & \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix} \begin{array}{l} 0.5 \rightarrow \\ 0.2 \\ 0.4 \\ 0.3 \end{array} \end{array}$$
$$\sum C2(u_i) \quad 0.5 \quad 0.2 \quad 0.4 \quad 0.3 \quad [1.4]$$

Step 8: Now D has only isolate vertices. In $(\mathrm{M}_{\mu})'_2$, $\mathrm{R2}(u_1)$ has maximum value. From the Figure 2, it has adjacent with u_6 in D. Then select u_1 and put in D. then D={u_6, u_3,u_1}.

Still D has isolate vertex u_3.

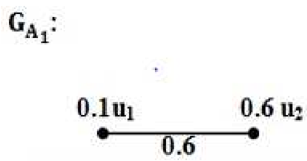
Step 8A: In $(\mathrm{M}_{\mu})'_2$, the next maximum value occur at $\mathrm{R2}(u_4)$ and it has adjacent with u_3 in D. Then select u_4 and put in D. Then D={ u_6, u_3,u_1, u_4}.

Now D has no isolate vertex. STOP.

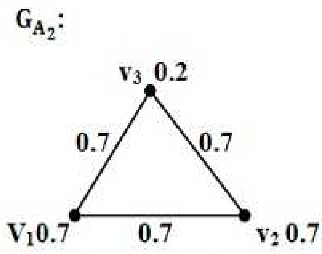
The resulting set D = { u_6, u_3,u_1, u_4} is a total dominating set of the given anti fuzzy graph G_A and which is also minimal. The total domination number is γ_{At}=2.2

5 Total domination on anti cartesian product of anti fuzzy graphs

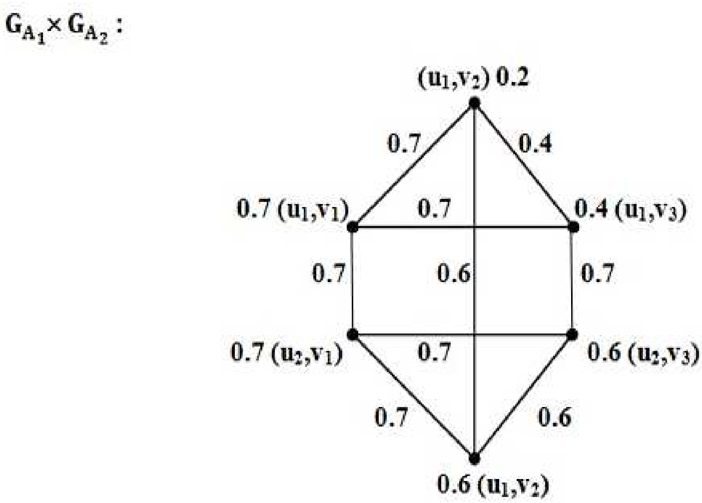
In this section, we apply total domination number on anti cartesian product on same types of anti fuzzy graphs such as cycle, path and complete anti fuzzy graph. To the resulting anti fuzzy graph obtain the bounds on them. In this paper, to derive the theorems consider that the number of vertices in G_{A_1} (say m_1) should be greater than or equal to the number of vertices in G_{A_2} (say m_2).



(a) Complete anti fuzzy graph K_2



(b) Complete anti fuzzy graph K_3



(c) Anti cartesian product of $K_2 \times K_3$

Fig. 3: Anti fuzzy graphs.

Theorem 13.[5] *Let G_A be an anti cartesian product of anti fuzzy graphs G_{A_1} and G_{A_2} where $G_{A_1} = (\sigma_1,\mu_1)$ and $G_{A_2} = (\sigma_2, \mu_2)$ then $G_A = (\sigma_1 \times \sigma_2,\mu_1 \times \mu_2)$ is an anti fuzzy graph.*

Note. In G_A, the vertices of total dominating set D incident with atleast one effective edges of G_A and every vertex in D is a strong neighbourhood to atleast one vertex in G_A.

Theorem 14. *Let G_{A_1} and G_{A_2} be any two strong anti fuzzy graphs. $G_A = (G_{A_1} \times G_{A_2})$ be an anti cartesian product of anti fuzzy graphs then $\gamma_t(G_A) \neq \gamma_t(\overline{G_A})$.*

Proof. Let G_A be an anti cartesian product of two strong anti fuzzy graphs G_{A_1} and G_{A_2}. Therefore, there exists no isolated vertices in G_A. If each vertex in G_A has minimum number of neighbourhood then $\gamma_t(G_A) \leq p/2$. Thus each vertex in $\overline{G_A}$ has maximum number of neighbourhood then minimum number of vertices in total dominating set. Hence $\gamma_t(G_A) \neq \gamma_t(\overline{G_A})$.

Proposition 3. *Let G_{A_1} and G_{A_2} be anti fuzzy graphs and G_A be the anti cartesian product of anti fuzzy graphs of G_{A_1} and G_{A_2}. Then*

(1) $\gamma_t(G_A) \neq \gamma_t(G_{A_1}) \times \gamma_t(G_{A_2})$
(2) $\gamma_t(G_A) \leq (|V(G_{A_1})| \vee |V(G_{A_2})|) \times (\gamma_t(G_{A_1}) \vee \gamma_t(G_{A_2}))$.
(3) $\gamma_t(G_{A_1} \times G_{A_2}) \geq \gamma_t(G_{A_1}) \times \gamma_t(G_{A_2})$

Example 3. From Figure 3,

For $G_{A_1}, |V(G_{A_1})| = 2, \gamma_t(G_{A_1}) = 0.7,$
For $G_{A_2}, |V(G_{A_2})| = 3, \gamma_t(G_{A_2}) = 1.4,$
For $G_{A_1} \times G_{A_2}, |V(G_{A_1} \times G_{A_2})| = 6. \gamma_t(G_{A_1} \times G_{A_2}) = 1.4,$
$|V(G_{A_1})| \vee |V(G_{A_2})|) \times (\gamma_t(G_{A_1}) \vee \gamma_t(G_{A_2}) = (2 \vee 3) \times (0.7 \vee 1.4) = 3 \times 1.4 = 4.2 \geq \gamma_t(G_A)\{proposition3(2)isverified.\}$

Algorithm 2. For the Figure 3(c), the minimal total dominating set is obtained by using algorithm 1. Applying this algorithm, a strong adjacency matrix for the resulting anti fuzzy graph is constructed.

Step 1: For the given anti fuzzy graph $G_{A_1} \times G_{A_2}$, the strong adjacency matrix $(M_\mu)'$ is given as

$$(M_\mu)' = \begin{array}{c} \\ (u_1,v_1) \\ (u_1,v_2) \\ (u_1,v_3) \\ (u_2,v_1) \\ (u_2,v_2) \\ (u_2,v_3) \end{array} \begin{array}{c} \begin{array}{cccccc} (u_1,v_1) & (u_1,v_2) & (u_1,v_3) & (u_2,v_1) & (u_2,v_2) & (u_2,v_3) \end{array} \\ \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0 & 0 & 0.7 \\ 0.7 & 0.2 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.4 & 0 & 0 & 0 \\ 0.7 & 0 & 0 & 0.7 & 0.7 & 0.7 \\ 0 & 0 & 0 & 0.7 & 0.6 & 0 \\ 0 & 0 & 0 & 0.7 & 0 & 0.7 \end{bmatrix} \end{array} \begin{array}{ccc} \sum R(u_i) & \sum R1(u_i) & \sum R2(u_i) \\ 2.8 & 2.8 & \rightarrow \\ 0.9 & 0.9 & 0.2 \\ 1.1 & 1.1 & 0.4 \\ 2.8 & \rightarrow & - \\ 1.3 & 0.6 & 0.6 \\ 1.4 & 0.7 & 0.7 \end{array}$$

$$\begin{array}{ccccccc} \sum C(u_i) & 2.8 & 0.9 & 1.1 & 2.1 & 1.3 & 2.1 \\ \sum C1(u_i) & 2.1 & 0.9 & 1.1 & - & 0.6 & 1.4 \\ \sum C2(u_i) & - & 0.2 & 0.4 & - & 0.6 & 0.7 \end{array}$$

The total dominating set is $D = \{(u_1,v_1),(u_2,v_1)\}$. Now D has no isolate vertex. STOP.

The resulting set $D = \{(u_1,v_1),(u_2,v_1)\}$ is a total dominating set of the given anti fuzzy graph G_A and which is also minimal. The total domination number is $\gamma_{At} = 1.4$.

Proposition 4. *Let $G_{A_1} \times G_{A_2}$ be an anti cartesian product of anti fuzzy paths of G_{A_1} and G_{A_2}. D be the total dominating set of $G_{A_1} \times G_{A_2}$.*

(1) *If $|V(G_{A_1} \times G_{A_2})| \leq 12$ then G_{A_1} fiber at u_1 or G_{A_2} fiber at u_2 contained in D.*
(2) *If $|V(G_{A_1} \times G_{A_2})| > 12$ then atleast two vertices of G_{A_1} fiber at u_1 or G_{A_2} fiber at u_2 is contained in D.*

Theorem 15. *Let $G_{A_1} \times G_{A_2}$ be an anti cartesian product of anti fuzzy paths of G_{A_1} and G_{A_2} with m_1 and m_2 vertices and order p_1 and p_2. D be the total dominating set then $\gamma_{At} \leq \frac{m_1p_1+m_2p_2}{m_1+m_2}$.*

Proof. Let G_{A_1} and G_{A_2} be an anti fuzzy paths on m_1 and m_2 vertices with order p_1 and p_2 and $G_A = G_{A_1} \times G_{A_2}$ be an anti fuzzy graph on m_1m_2 vertices. Let D be a minimal total dominating set of G_A. The anti cartesian product of anti fuzzy paths looks like a grid graph. By proposition 4, there exists atleast one vertex in G_{A_1} fiber at u_1 or G_{A_2} fiber at u_2. Thus all the vertices in G_A has 3 or 4 neighbours. Hence $\gamma_{At} \leq \frac{m_1p_1+m_2p_2}{m_1+m_2}$.

Theorem 16. *Let G_A be an anti cartesian product of anti fuzzy cycles of G_{A_1} and G_{A_2} with order p_1 and p_2. Then $\gamma_{At} \leq 2\,[\,p_1 + p_2 - \delta(G_A)\,]$*

Proof. Let $G_A = G_{A_1} \times G_{A_2}$ be anti cartesian product of anti fuzzy cycles of G_{A_1} and G_{A_2} with order p_1 and p_2. Let D be a minimal total dominating set of G_A. Every G_{A_1} fiber at u_i or G_{A_2} fiber at u_i has an anti fuzzy cycle and has atleast two vertices in D and also all the vertices in G_A has 3 or 4 neighbours. Hence $\gamma_{At} \leq 2\,[\,p_1 + p_2 - \delta(G_A)\,]$.

Theorem 17. *If G_{A_1} and G_{A_2} are two complete anti fuzzy graphs on m_1 and m_2 vertices with order p_1 and p_2. $G_{A_1} \times G_{A_2}$ is an anti fuzzy graph on $m_1 m_2$ vertices then* $\gamma_t(G_{A_1} \times G_{A_2}) = \begin{cases} p_2, & if\ m_1 > m_2 \\ p_1, & if\ m_1 < m_2 \end{cases}$

Proof. Let G_{A_1} and G_{A_2} be two complete anti fuzzy graphs on m_1 and m_2 vertices with order p_1 and p_2. $G_A = G_{A_1} \times G_{A_2}$ is an anti fuzzy graph on $m_1 m_2$ vertices. Let D be a minimal total dominating set of G_A. By the definition of complete anti fuzzy graph every pair of vertices is adjacent to each other. If $m_1 > m_2$ then in $G_A = (G_{A_1} \times G_{A_2})$, split the vertex set as m_1 component with m_2 vertices. That is, each component in $G_{A_1} \times G_{A_2}$ contain m_2 vertices and every vertices in a component is adjacent to each other and the vertex which having the maximum value dominate all the vertices in that component and also it dominate a single vertex in the remaining components also. Similarly, proceed in all m_1-1 components. The dominant vertices in each component should be adjacent to each other. i.e., $\gamma_t(G_A) = p_2$. Hence $\gamma_t(G_{A_1} \times G_{A_2}) = p_2$.

Similarly, if $m_1 < m_2$ then $\gamma_t(G_{A_1} \times G_{A_2}) = p_1$.

6 Conclusion

In this paper, total domination number is defined on anti fuzzy graphs. The definition of total domination number is applied on various types of anti fuzzy graph and obtained the bounds on them. An algorithm is described to find the minimal total dominating set for given anti fuzzy graph. The total domination number concept is applied on anti cartesian product of anti fuzzy graphs such as anti fuzzy path, anti fuzzy cycle and complete anti fuzzy graph and obtained the bounds on them and the algorithm is applied on anti cartesian product of anti fuzzy graph to get minimal total dominating set.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Kaufmann. A., "Introduction to the theory of Fuzzy Subsets", *Academic Press*, Newyork (1975).

[2] Mordeson. J.N., and Nair. P.S., "Fuzzy graphs and Fuzzy Hypergraphs", *Physica Verlag, Heidelberg*, 1998; second edition 2001.

[3] Mordeson. J.N., and Peng. C.S., "Operations on fuzzy graphs", *Information Sciences*, vol. 79 (1994), pp. 159-170.

[4] Muthuraj. R., and Sasireka. A., "On Anti fuzzy graphs", *Advances in Fuzzy Mathematics*, Vol. 12, No.5 (2017), pp. 1123 - 1135.

[5] Muthuraj. R., and Sasireka. A., "Some Characterization on Operations of Anti Fuzzy Graphs", *International Conference on Mathematical Impacts in Science And Technology*, (MIST -17), November 2017, IJRASET, pp. 109-117.

[6] Muthuraj. R., and Sasireka. A., "Some Types of Nodal and Edge Regular Anti Fuzzy Graph", *international journal of Fuzzy Mathematical Archive*, Vol.14, No. 2, 2017, pp. 365-378.

[7] Muthuraj. R., and Sasireka. A., "Domination on Anti Fuzzy Graph", *international journal of Mathematical Archive*, Vol.9, No. 5, 2018, pp. 82-92.

[8] Rosenfeld A., Fuzzy graphs. In: Zadeh, L.A., Fu, K.S., Shimura, M (Eds.), *Fuzzy Sets and their Applications*, Academic Press, New York, (1975).

[9] Seethalakshmi. R., and Gnanajothi. R.B., "Operations on Anti fuzzy graphs", *Mathematical Sciences International Research Journal*, Vol 5,Issue 2 (2016), pp. 210-214.

[10] Somadsundaram. A., and Somadsundaram. S., "Domination in fuzzy graphs", *Pattern Recognit. Let.*, 19(9)(1998),pp.787-791.

[11] Zadeh. L.A., "Fuzzy sets", *Information Sciences*, No. 8 (1965), pp. 338-353.