A new class of operator ideals and approximation numbers

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Abstract: In this study, we introduce the class of generalized Stolz mappings by generalized approximation numbers. Also we prove that the class of $\ell_1^\infty$-type mappings are included in the class of generalized Stolz mappings by generalized approximation numbers and we define a new quasinorm equivalent with $\|T\|_{\phi,\alpha}$. Further we give a new class of operator ideals by using generalized approximation numbers and symmetric norming function and we show that this class is an operator ideal.

Keywords: Operator ideal, $s$-numbers, symmetric norming function.

1 Introduction

The operator ideal theory has a special importance in functional analysis. One of the most important methods to construct operator ideals is via $s-$ numbers. Pietsch defined the approximation numbers of a bounded linear operator in Banach spaces, in 1963 [13]. Later on, the other examples of $s-$numbers, namely Kolmogorov numbers, Weyl numbers, etc. are introduced to the Banach space setting.

In this paper, by $\mathbb{N}$ and $\mathbb{R}^+$ we denote the set of all natural numbers and non-negative real numbers, respectively.

A bounded linear operator whose dimension of the range space is finite is called a finite rank operator [9].

Let $E$ and $F$ be real or complex Banach spaces and $\mathcal{L}(E,F)$ denotes the space of all bounded linear operators from $E$ to $F$ and $\mathcal{L}$ denotes the space of all bounded linear operators between any two arbitrary Banach spaces.

A map $s = (s_n) : \mathcal{L} \to \mathbb{R}^+$ assigning to every operator $T \in \mathcal{L}$ a non-negative scalar sequence $(s_n(T))_{n \in \mathbb{N}}$ is called an $s-$number sequence if the following conditions are satisfied for all Banach spaces $E, F, E_0$ and $F_0$:

(S1) $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for every $T \in \mathcal{L}(E,F)$,

(S2) $s_{m+n}^{-1}(S+T) \leq s_m(S) + s_n(T)$ for every $S, T \in \mathcal{L}(E,F)$ and $m,n \in \mathbb{N}$,

(S3) $s_n(RST) \leq \|R\|s_n(S)\|T\|$ for some $R \in \mathcal{L}(F,F_0), S \in \mathcal{L}(E,F)$ and $T \in \mathcal{L}(E_0,E)$, where $E_0,F_0$ are arbitrary Banach spaces.
If \( \text{rank}(T) \leq n \), then \( s_n(T) = 0 \),

(55) \( s_n(I : l_2^n \rightarrow l_2^n) = 1 \), where \( I \) denotes the identity operator on the \( n \)-dimensional Hilbert space \( l_2^n \), where \( s_n(T) \) denotes the \( n \)-th \( \alpha \)-number of the operator \( T \) [12].

An example of \( s \)-number sequence is the approximation number, which is defined by Pietsch. The \( n \)-th approximation number, denoted by \( a_n(T) \), is defined as

\[
a_n(T) = \inf \{ \| T - A \| : A \in \mathcal{L}(E, F), \text{rank}(A) < n \},
\]

where \( T \in \mathcal{L}(E, F) \) and \( n \in \mathbb{N} \) [13].

Let the space of all real valued sequences be \( \omega \). A sequence space is any vector subspace of \( \omega \).

The Cesaro sequence space \( \text{ces}_p \) is defined as ([11], [4], [5])

\[
\text{ces}_p = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}, \quad 1 < p < \infty.
\]

Let \( E' \) be the dual of \( E \), which is composed of continuous linear functionals on \( E \). Let \( x' \in E' \) and \( y \in F \), then the map \( x' \otimes y : E \rightarrow F \) is defined by

\[
(x' \otimes y)(x) = x'(x)y, \quad x \in E.
\]

Pietsch [13] defined an operator \( T \in \mathcal{L}(E, F) \) to be \( l_p \) type operator if \( \sum_{n=1}^{\infty} \left( a_n(T) \right)^p < \infty \) for \( 0 < p < \infty \). Then, Constantin [6], generalized the class of \( l_p \) type operators to the class of \( ces - p \) type operators by using the Cesaro sequence spaces, where an operator \( T \in \mathcal{L}(E, F) \) is called \( ces - p \) type if \( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} a_k(T) \right)^p < \infty, \quad 1 < p < \infty \). As a generalization of \( l_p \) type operators, \( A - p \) type operators and Stolz mappings were examined in [7], [8]. Also in [9], [10], [11] Maji and Srivastava studied the class \( A^{(s)} - p \) of \( s \)-type \( ces_p \) operators using \( s \)-number sequence and Cesaro sequence spaces and they introduced a new class \( A_{p,q}^{(s)} \) of \( s \)-type \( ces(p,q) \) operators by using weighted Cesaro sequence space for \( 1 < p < \infty \). In [23], the class of \( s \)-type \( Z(u,v;l_p) \) operators is defined and worked on some properties of this class.

Now let give the definitions of operator ideal and quasi-norm:

A subcollection \( \mathcal{I} \) of \( \mathcal{L} \) is called an operator ideal if each component \( \mathcal{I}(E, F) = \mathcal{I} \cap \mathcal{L}(E, F) \) satisfies the following conditions:

\((OI - 1)\) if \( x' \in E', y \in F \), then \( x' \otimes y \in \mathcal{I}(E, F) \),

\((OI - 2)\) if \( S, T \in \mathcal{I}(E, F) \), then \( S + T \in \mathcal{I}(E, F) \),

\((OI - 3)\) if \( S \in \mathcal{I}(E, F) \), \( T \in \mathcal{L}(E_0, E) \) and \( R \in \mathcal{L}(F, F_0) \), then \( RST \in \mathcal{I}(E_0, F_0) \)[14].

A function \( \alpha : \mathcal{I} \rightarrow \mathbb{R}^+ \) is said to be a quasi-norm on the operator ideal \( \mathcal{I} \) if the following conditions hold:

\((QN - 1)\) If \( x' \in E', y \in F \), then \( \alpha(x' \otimes y) = \|x'\| \|y\| \);
A function \( s: \mathfrak{S} \rightarrow \mathbb{R}^+ \) is called a generalized approximation number if there exists a constant \( C \geq 1 \) such that \( \alpha(S + T) \leq C[\alpha(S) + \alpha(T)] \).

In particular if \( C = 1 \) then \( \alpha \) becomes a norm on the operator ideal \( \mathfrak{S} \).

An ideal \( \mathfrak{S} \) with a quasi-norm \( \alpha \), denoted by \([\mathfrak{S}, \alpha]\) is said to be a quasi-Banach operator ideal if each component \( \mathfrak{S}(E, F) \) is complete under the quasi-norm \( \alpha \).

A map \( s^\alpha = (s^\alpha_n): \mathfrak{S}(E, F) \rightarrow \mathbb{R}^+ \) assigning to every operator \( T \in \mathfrak{S}(E, F) \) a non-negative scalar sequence \( \{s^\alpha_n(T)\}_{n \in \mathbb{N}} \) is called a generalized \( s \)-number sequence if the following conditions are satisfied for all Banach spaces \( E, F \) [2,3]:

- \((S^\alpha_1)\) \( \alpha(T) = s^\alpha_n(T) \geq s^\alpha_k(T) \geq \ldots \geq 0 \) for all \( T \in \mathfrak{S}(E, F) \),

- \((S^\alpha_2)\) \( s^\alpha_{m+n-1}(S + T) \leq s^\alpha_m(S) + s^\alpha_n(T) \) for every \( S, T \in \mathfrak{S}(E, F) \) and \( m, n \in \mathbb{N} \),

- \((S^\alpha_3)\) \( s^\alpha_n(RST) \leq \|R\| s^\alpha_n(S) \|T\| \) for some \( R \in \mathfrak{L}(F, F), S \in \mathfrak{S}(E, F) \) and \( T \in \mathfrak{L}(E, E) \),

- \((S^\alpha_4)\) If \( \dim(T) \leq n \), then \( s^\alpha_n(T) = 0 \).

Consequently, generalized approximation numbers \( \{a^\alpha_n(T)\} \) are the examples of generalized \( s \)-numbers, where

\[ a^\alpha_n(T) = \inf \{ \alpha(T - K) : K \in \mathfrak{S}, \dim K < n \} \] [3].

By using the generalized approximation numbers we define the class of \( L^\alpha_p \) -type operators as

\[ L^\alpha_p(E, F) = \left\{ T \in \mathfrak{S}(E, F) : \sum_{n=1}^\infty (a^\alpha_n(T))^p < \infty \right\} \]

for \( 0 < p < \infty \).

Let \( \ell_\infty \) be the space of all bounded real sequences and \( K \subset \ell_\infty \) be the set of all sequences \( x \) such that \( \text{card} \{i \in \mathbb{N}, x_i \neq 0\} < n \) and \( x_1 \geq x_2 \geq \ldots \geq 0 \).

A function \( \phi: K \rightarrow \mathbb{R} \) is called symmetric norming function, if the following conditions satisfied

- \((\phi 1)\) \( \phi(x) > 0, \quad \forall x \neq 0 \),

- \((\phi 2)\) \( \phi(\alpha x) = \alpha \phi(x), \quad x \in K, \alpha \geq 0 \)

- \((\phi 3)\) \( \phi(x + y) \leq \phi(x) + \phi(y) \)

- \((\phi 4)\) \( \phi(1, 0, 0, \ldots) = 1 \)

- \((\phi 5)\) if \( \sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i, \quad k = 1, 2, \ldots \) then \( \phi(x) \leq \phi(y) \).
It is given that ([15], [16]) for all symmetric norming functions \( \phi \), the function \( \phi(p) \) defined as
\[
\phi(p) : (x_j) \in K \rightarrow (\phi (\{x_j^p\}))^{\frac{1}{p}}, \quad 1 \leq p \leq \infty
\]
is also a symmetric norming function. For more details on symmetric norming functions we refer to [2], [15], [17], [18], [19], [20].

By using the properties of symmetric norming function and the sequence \( (a_n(T)) \), the class \( \mathcal{L}_p(E,F) \) is defined in [18] and [21] as follows
\[
\mathcal{L}_p(E,F) = \{ T \in \mathcal{L}(E,F) : \phi (\{a_n(T)\}) < \infty \}.
\]

By using the relations \( a_n(T_1 + T_2) \leq a_n^p(T_1) + a_n^p(T_2), n = 1, 2, \ldots \) and \( a_n^p(\beta T) = |\beta| a_n^p(T), (\beta \text{ is a scalar}) \) and the properties of the function \( \phi \), \( \| T \|_{\phi}^\alpha = \phi (\{a_n^p(T)\}) \) and \( \| T \|_{\phi(p)}^\alpha = \phi(p) (\{a_n^p(T)\}) \) are quasinorms.

In this study, we introduce the class of generalized Stolz mappings by generalized approximation numbers. Also we prove that the class of \( \ell_p^\alpha \)-type mappings are included in the class of generalized Stolz mappings by generalized approximation numbers and we define a new quasinorm equivalent with \( \| T \|_{\phi(p)}^\alpha \). Further we introduce a new class of operator ideals by using approximation numbers and symmetric norming function. We prove that this class is an operator ideal.

2 Main results

Throughout this paper \( (u_n) \) and \( (w_n) \) sequences satisfies the following conditions:

Let \( (u_n) \) and \( (w_n) \) be sequences of non-negative real numbers such that \( u_1 \geq u_2 \geq \ldots \geq u_n \geq \ldots \) and \( w_1 \leq w_2 \leq \ldots \leq w_n \leq \ldots \) and \( w_n \leq n \leq \frac{W_n}{u_n} \). Let \( T \in \mathcal{L}(E,F) \) then the class of generalized Stolz mappings \( L_{\text{GSTOL},p}(E,F) \), in [22] defined as
\[
L_{\text{GSTOL},p}(E,F) = \left\{ T : \sum_{n=1}^{\infty} \left[ \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i(T) \right]^p < \infty \right\}, \quad 0 < p < \infty.
\]

Now we define the class of generalized Stolz mappings by generalized approximation numbers \( L_{\text{GSTOL},p}^\alpha(X) \) as:

Let \( T \in \mathcal{L}(E,F) \) and \( \alpha \) be an ideal norm:
\[
L_{\text{GSTOL},p}^\alpha(X) = \left\{ T : \sum_{n=1}^{\infty} \left[ \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i(T) \right]^p < \infty \right\}, \quad 0 < p < \infty.
\]

Then we introduce a new class of operator ideals with the help of symmetric norming function as follows:
\[
\mathfrak{G}_{\phi(p)}^\alpha(E,F) = \left\{ T \in \mathcal{L}(E,F) : \phi(p) \left( \left\{ \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i(T) \right\} \right) < \infty \right\}.
\]

In the following theorem, we prove that \( \ell_p^\alpha \)- type mappings are included in the class of generalized Stolz mappings by generalized approximation numbers.
Theorem 1. If \( \lim_{n \to \infty} u_n = u \neq 0 \), the class of \( \ell^p_{\alpha} \)–type mappings are included in the class of generalized Stolz mappings by generalized approximation numbers \( 1 < p < \infty \).

Proof. Let \( T \in \mathcal{S}(E,F) \) and \((u_n)\) and \((w_n)\) be sequences of non-negative real numbers such that \( u_1 \geq u_2 \geq \ldots \geq u_n \geq \ldots \) and \( w_1 \leq w_2 \leq \ldots \leq w_n \leq \ldots \) and \( w_n \leq u_n \) for \( n \to \infty \) and \( \lim u_n \neq 0 \). Then we can write

\[
\sum_{n=1}^{\infty} \left( \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i^\alpha(T) \right)^p \leq \sum_{n=1}^{\infty} \left( \frac{u_1}{n u_n} \sum_{i=1}^{n} a_i^\alpha(T) \right)^p \leq \left( \frac{u_1}{u} \right)^p \sum_{n=1}^{\infty} \left( \frac{n}{n} \sum_{i=1}^{n} a_i^\alpha(T) \right)^p.
\]

Since \( \sum_{n=1}^{\infty} (a_i^\alpha(T))^p < \infty \), we obtain from Hardy’s inequality that

\[
\left( \frac{u_1}{u} \right)^p \sum_{n=1}^{\infty} \left( \frac{n}{n} \sum_{i=1}^{n} a_i^\alpha(T) \right)^p \leq \left( \frac{u_1}{u} \right)^p \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} (a_i^\alpha(T))^p < \infty.
\]

It follows that

\[
\sum_{n=1}^{\infty} \left( \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i^\alpha(T) \right)^p < \infty.
\]

Hence the class of \( \ell^p_{\alpha} \)–type mappings are included in the class of generalized Stolz mappings by generalized approximation numbers \( 1 < p < \infty \).

Theorem 2. Let \( \lim_{n \to \infty} u_n \neq 0 \) then the quasi-norm \( \| T \|_{\phi(p)}^\alpha \) is equivalent with

\[
\| T \|_{\phi(p)}^\alpha = \phi(p) \left( \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i^\alpha(T) \right) \quad (1 < p < \infty).
\]

Proof. Since the sequences \((u_n)\) and \((a_i^\alpha(T))\) are decreasing we can write

\[
\frac{1}{n u_n} a_i^\alpha(T) \leq \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i^\alpha(T) \leq \frac{1}{n u_n} \sum_{i=1}^{n} a_i^\alpha(T).
\]

Summing from \( n = 1 \) to \( k \), we get

\[
\sum_{n=1}^{k} (u_n a_n^\alpha(T))^p \leq \sum_{n=1}^{k} \left( \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i^\alpha(T) \right)^p \leq \sum_{n=1}^{k} \left( \frac{u_1}{n u_n} \sum_{i=1}^{n} a_i^\alpha(T) \right)^p.
\]

If \( \lim_{n \to \infty} u_n = u \neq 0 \) then we obtain

\[
u^p \sum_{n=1}^{k} (a_n^\alpha(T))^p \leq \sum_{n=1}^{k} \left( \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i^\alpha(T) \right)^p \leq \left( \frac{u_1}{u} \right)^p \sum_{n=1}^{k} \left( \frac{n}{n} \sum_{i=1}^{n} a_i^\alpha(T) \right)^p
\]

for every \( k \in \mathbb{N} \). By using Hardy’s inequality

\[
u^p \sum_{n=1}^{k} (a_n^\alpha(T))^p \leq \sum_{n=1}^{k} \left( \frac{1}{w_n} \sum_{i=1}^{n} u_i a_i^\alpha(T) \right)^p \leq \left( \frac{u_1}{u} \right)^p \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{k} (a_n^\alpha(T))^p.
\]
for every $k \in \mathbb{N}$. From the properties of the function $\phi$ we have that
\[
u \|T\|_{\phi(p)}^\alpha \leq \|\hat{T}\|_{\phi(p)}^\alpha \leq \left( \frac{u_1}{u} \right) \left( \frac{p}{p - 1} \right) \|T\|_{\phi(p)}^\alpha.
\]

To prove the next theorem we need the following lemma:

**Lemma 1.** [2] Generalized approximation numbers verify the inequality:
\[
k \sum_{n=1}^{k} a_n^q (S + T) \leq 2 \sum_{n=1}^{k} (a_n^q (S) + a_n^q (T)), \quad k = 1, 2, \ldots.
\]  

**Proof.**
\[
k \sum_{n=1}^{k} a_n^q (S + T) \leq 2 \sum_{n=1}^{k} a_n^q (S + T) = k \sum_{n=1}^{k} a_{2n-1}^q (S + T) + \sum_{n=1}^{k} a_n^q (S + T)
\]
\[
\leq 2 \sum_{n=1}^{k} a_{2n-1}^q (S + T) \leq 2 \sum_{n=1}^{k} (a_n^q (S) + a_n^q (T)).
\]

**Theorem 3.** If $\phi_{(p)} \left( \left\{ \frac{1}{w_n} \right\} \right) < \infty$, then the class $\mathcal{I}^{\gamma}_{\phi_{(p)}} (E, F)$ is a quasi-normed operator ideal by
\[
\|T\|_{\phi_{(p)}}^{\alpha, \gamma} = \frac{\phi_{(p)} \left( \left\{ \frac{1}{w_n} \right\} \right)}{u_1 \phi_{(p)} \left( \left\{ \frac{1}{w_n} \right\} \right)}, \quad (1 < p < \infty).
\]

**Proof.** We prove the properties of an operator ideal and the ideal quasi-norm. Let $E$ and $F$ be any two Banach spaces. Let $x' \in E'$, $y \in F$ then $x' \otimes y$ is a rank one operator. So
\[
a_n^q (x' \otimes y) = 0 \quad \text{for all } n \geq 2.
\]

By using the properties of symmetric norming function and the generalized approximation number we can get:
\[
\|x' \otimes y\|_{\phi_{(p)}}^{\alpha, \gamma} = \frac{\phi_{(p)} \left( \left\{ \frac{1}{w_n} \right\} \right)}{u_1 \phi_{(p)} \left( \left\{ \frac{1}{w_n} \right\} \right)} = \frac{\phi_{(p)} \left( \left\{ \frac{1}{w_n} \right\} \right)}{u_1 \phi_{(p)} \left( \left\{ \frac{1}{w_n} \right\} \right)}.
\]

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Hence, \( x' \otimes y \in \mathcal{Z}_\alpha^\gamma (E,F) \) and \( ||x' \otimes y||_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} = ||x'|| ||y|| \). Let \( S,T \in \mathcal{Z}_\alpha^\gamma (E,F) \). By using (1) and the properties of symmetric norming function we can calculate;

\[
\|S + T\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} = \frac{\phi_p \left( \left\{ \frac{1}{w_n} \sum_{i=1}^n u_i \alpha_i^a (S + T) \right\} \right)}{u_1 \phi_p \left( \left\{ \frac{1}{w_n} \right\} \right)} \leq \frac{2 \phi_p \left( \left\{ \frac{1}{w_n} \sum_{i=1}^n u_i (\alpha_i^a (S) + \alpha_i^a (T)) \right\} \right)}{u_1 \phi_p \left( \left\{ \frac{1}{w_n} \right\} \right)}
\]

\[
= 2 \left[ \frac{\phi_p \left( \left\{ \frac{1}{w_n} \sum_{i=1}^n u_i \alpha_i^a (S) \right\} \right)}{u_1 \phi_p \left( \left\{ \frac{1}{w_n} \right\} \right)} + \frac{\phi_p \left( \left\{ \frac{1}{w_n} \sum_{i=1}^n u_i \alpha_i^a (T) \right\} \right)}{u_1 \phi_p \left( \left\{ \frac{1}{w_n} \right\} \right)} \right]
\]

\[
= 2 \left[ \|S\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} + \|T\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} \right] < \infty.
\]

Hence \( S + T \in \mathcal{Z}_\alpha^\gamma (E,F) \) and \( \|S + T\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} \leq 2 \left[ \|S\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} + \|T\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} \right] \). Now let \( S \in \mathcal{Z}_\alpha^\gamma (E,F) \), \( T \in \mathcal{Z} (E_0,E) \) and \( R \in \mathcal{Z} (F,F_0) \). Then

\[
\|RST\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} = \frac{\phi_p \left( \left\{ \frac{1}{w_n} \sum_{i=1}^n u_i \alpha_i^a (RST) \right\} \right)}{u_1 \phi_p \left( \left\{ \frac{1}{w_n} \right\} \right)} \leq \frac{\phi_p \left( \left\{ \frac{1}{w_n} \sum_{i=1}^n u_i ||R|| \alpha_i^a (S) ||T|| \right\} \right)}{u_1 \phi_p \left( \left\{ \frac{1}{w_n} \right\} \right)} \]

\[
= ||R|| ||T|| \left[ \frac{\phi_p \left( \left\{ \frac{1}{w_n} \sum_{i=1}^n u_i \alpha_i^a (S) \right\} \right)}{u_1 \phi_p \left( \left\{ \frac{1}{w_n} \right\} \right)} \right]
\]

\[
= ||R|| ||T|| \|S\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} < \infty.
\]

Hence \( RST \in \mathcal{Z}_\alpha^\gamma (E_0,F_0) \) and \( \|RST\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} \leq ||R|| ||T|| \|S\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} \). \( \mathcal{Z}_\alpha^\gamma (E,F) \) is an operator ideal and \( \|\cdot\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} \) is an ideal quasi-norm.

For the particular case, if we choose \( (u_n) = (\frac{2n + 5}{10n}) \), \( (w_n) = (n) \) and \( \phi (x) = \sum_{i=1}^n x_i \), we can get a quasi-normed operator ideal by

\[
\|T\|_{\mathcal{H}_\alpha^\gamma}^{\alpha,\gamma} = \frac{\phi_p \left( \left\{ \frac{1}{n} \sum_{i=1}^n \left( \frac{2n + 5}{10n} \right) \alpha_i^a (T) \right\} \right)}{\left( \frac{1}{n} \right) \phi_p \left( \left\{ \frac{1}{n} \right\} \right)}.
\]

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References