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Common fixed point theorems in Fuzzy metric spaces using property E. A.

Prapoorna Manthena and Rangamma Manchala

Department of Mathematics, University College of Science, Osmania University, Hyderabad, Telangana, India.

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Abstract: In this paper, we prove two common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces using the property E.A.

Keywords: Fuzzy metric space, weakly compatible mappings, E.A. Property.

1 Introduction

The notion of fuzzy sets that was introduced by Zadeh[26] with a view to represent the vagueness in everyday life laid the way to the amplification of fuzzy mathematics. Many authors have enormously developed the theory of fuzzy sets and studied its applications. The theory of fuzzy sets has a wide range of applications in various fields such as medicine, control theory, engineering sciences, etc (See also [3], [12], [15], [22]). George and Veeramani[4], [5] modified the concept of fuzzy metric spaces given by Kramosil and Michalek[11] and defined the hausdorff topology on fuzzy metric spaces and also showed that every metric induces a fuzzy metric.

Fixed point theory in fuzzy metric spaces has been developed by Grabiec[6]. Subrahmanyam[23] gave a generalization of Jungck's[9] common fixed point theorem for commuting mappings in the setting of fuzzy metrics spaces. In the recent literature, weaker conditions of commutativity such as weakly commuting mappings, compatible mappings, R-weakly commuting maps, weakly compatible mappings and several others have been considered.

In 2002, M. Aamri and D. El Moutawakil[1] introduced the property E.A., which generalizes the concept of noncompatible mappings and gave some common fixed point theorems under strict contractive conditions. In 2009, Mujahid Abbas[2], et. al., proved common fixed point theorems for a class of four non compatible mappings in fuzzy metric spaces. In 2010, S. Sedghi et.al.[18], proved a common fixed point theorem for weakly compatible mappings in fuzzy metric spaces using the property E.A. In 2010, D. Mihet[13], proved two common fixed point theorems for a pair of weakly compatible maps in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani, by using E.A. property. For more references on common fixed point theorems in fuzzy metric spaces using E.A. property, see [8], [16], [20], [24].

On the other hand, in 1984, Khan et.al.[10] improved the Banach fixed point theorem in metric spaces by introducing a control function, called an altering distance function. Recently, Shen et.al.[19] introduced the notion of altering distance in fuzzy metric space and obtained fixed point results. In 2015, N. Wairojjana, et.al.[25] proved fixed point results in complete and compact fuzzy metric spaces by imposing a contraction condition and using the idea of altering distance.

* Corresponding author e-mail: m_prapoorna@yahoo.com

By using the same altering distance given in [25] and also replacing the completeness of fuzzy metric space by more natural condition of closeness of the range with the help of E.A. property, we prove common fixed point results for a pair of weakly compatible mappings.

2 Preliminaries

Definition 1. [17] *A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if for all $p,q,r,s \in [0,1]$, the following conditions are satisfied:*

(1) $p * 1 = p$
(2) $p * q = q * p$
(3) $p * q \leq r * s$ *whenever* $p \leq r$ *and* $q \leq s$
(4) $p * (q * r) = (p * q) * r$

Definition 2. [4] *The 3-tuple $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times (0,\infty)$ satisfying the following conditions:*

(1) $M(x,y,t) > 0$,
(2) $M(x,y,t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,
(3) $M(x,y,t) = M(y,x,t)$
(4) $M(x,z,t+s) \leq M(x,y,t) * M(y,z,s)$,
(5) $M(x,y,.) : (0,\infty) \to (0,1]$ *is a continuous function,*

for all $x,y,z \in X$ *and* $t,s > 0$.

Lemma 1. [6] *$M(x,y,.)$ is nondecreasing for all $x,y \in X$.*

Definition 3.[4], [7] *Let $(X,M,*)$ be a fuzzy metric space.*

(I) *A sequence $\{x_n\}$ in X is a M-Cauchy sequence if for all $\varepsilon \in (0,1)$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n,x_m,t) > 1-\varepsilon$ for all $n,m \geq n_0$.*
(II) *A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n,x,t) = 1$, $t > 0$.*
(III) *A fuzzy metric space X is M-complete if every M-Cauchy sequence in X is convergent.*

Definition 4. [21] *Two self mappings P and Q on a fuzzy metric space $(X,M,*)$ are said to be compatible if $\lim_{n\to\infty} M(PQx_n,QPx_n,t) = 1$, for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = x \in X$.*

Definition 5. [21] *Two self maps P and Q of a fuzzy metric space $(X,M,*)$ are said to be weakly compatible if they commute at their coincidence points; i.e., $Px = Qx$ for some $x \in X$ implies that $PQx = QPx$.*

Remark. Two compatible self mappings are weakly compatible, but the converse is not true (see example 2.16 of [21]).

Definition 6. [2] *A Pair of self maps (P,Q) on a fuzzy metric space $(X,M,*)$ satisfies the property E.A. if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} M(Px_n,x,t) = \lim_{n\to\infty} M(Qx_n,x,t) = 1$ for some $x \in X$ and all $t > 0$.*

Remark. It is noted that weak compatibility and E.A. property are independent to each other (see [14], Example 2.1, Example 2.2).

3 Main results

Definition 7. [25]. *A function $\phi : [0,1] \to [0,1]$ is called an altering distance function if it satisfies the following properties:*

(1) *ϕ is strictly decreasing and continuous;*
(2) *$\phi(\lambda) = 0$ if and only if $\lambda = 1$.*

It is obvious that $\lim_{\lambda \to 1-} \phi(\lambda) = \phi(1) = 0$.

Theorem 1. *Let $(X,M,*)$ be a fuzzy metric space and T,S be weakly compatible self-maps of X satisfying the following property*

$$\phi(M(Tx,Ty,t)) \leq k_1(t)\min\{\phi(M(Sx,Sy,t)),\phi(M(Sx,Tx,t)),\phi(M(Sx,Ty,2t)),\phi(M(Sy,Ty,t))\}+k_2(t)(\phi(M(Tx,Sy,2t))) \quad (1)$$

where $x,y,\in X, k_1,k_2 : (0,\infty) \to (0,1), t > 0$ and ϕ is an altering distance function. If T and S satisfy the property E.A. and the range of S is a closed subspace of X, then T and S have a unique common fixed point in X.

Proof. Suppose that T and S satisfy the property E.A., then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z \in X \quad (2)$$

Since $S(X)$ is a closed subspace of X, there exists $u \in X$ such that

$$z = Su. \quad (3)$$

For $x = x_n, y = u$, (1) becomes

$$\begin{aligned}\phi(M(Tx_n,Tu,t)) \leq\; & k_1(t)\min(\phi(M(Sx_n,Su,t)),\phi(M(Sx_n,Tx_n,t)),\phi(M(Sx_n,Tu,2t)),\phi(M(Su,Tu,t)))\\ & +k_2(t)(\phi(M(Tx_n,Su,2t)))\end{aligned}$$

Taking limit $n \to \infty$ and using (2) and (3), we get

$$\begin{aligned}&\phi(M(z,Tu,t)) \leq k_1(t)\min(\phi(M(z,Su,t)),\phi(M(z,z,t)),\phi(M(z,Tu,2t)),\phi(M(Su,Tu,t)))+k_2(t)(\phi(M(z,Su,2t)))\\ &\implies \phi(M(z,Tu,t)) \leq k_1(t)\min(\phi(1),\phi(1),\phi(M(z,Tu,2t)),\phi(M(z,Tu,t)))+k_2(t)(\phi(1))\\ &\implies \phi(M(z,Tu,t)) = 0 \text{ which implies } M(z,Tu,t) = 1, \quad \text{i.e.,} \quad Tu = z. \end{aligned} \quad (4)$$

From (3) and (4), we have

$$Tu = Su = z. \quad (5)$$

Since T, S are weakly compatible, we get

$$Tz = Sz \quad (6)$$

Now we shall show that z is a fixed point of T. Suppose let us assume that $Tz \neq z$.
In view of (5), (1), (6) and using properties of ϕ, we get

$$\begin{aligned}
\phi(M(Tz,z,t)) &= \phi(M(Tz,Tu,t)) \\
&\leq k_1(t)\min(\phi(M(Sz,Su,t)),\phi(M(Sz,Tz,t)),\phi(M(Sz,Tu,2t)),\phi(M(Su,Tu,t))) + k_2(t)(\phi(M(Tz,Su,2t))) \\
&= k_1(t)\min(\phi(M(Tz,z,t)),\phi(1),\phi(M(Tz,z,2t)),\phi(M(z,z,t))) + k_2(t)(\phi(M(Tz,z,2t))) \\
&= k_2(t)(\phi(M(Tz,z,2t))) \\
&< \phi(M(Tz,z,2t)) < \phi(M(Tz,z,t)),\ t>0
\end{aligned}$$

which is a contradiction. Therefore, $Tz = z$. Thus,

$$Tz = z = Sz \quad \text{i.e.,} \quad z \text{ is a common fixed point of } T \text{ and } S. \tag{7}$$

For uniqueness, let $w \in X$ be another common fixed point of T and S such that

$$Tw = Sw = w \text{ and } w \neq z. \tag{8}$$

Then in view of (7), (8), (1) and properties of ϕ, we have

$$\begin{aligned}
\phi(M(z,w,t)) &= \phi(M(Tz,Tw,t)) \\
&\leq k_1(t)\min(\phi(M(Sz,Sw,t)),\phi(M(Sz,Tz,t)),\phi(M(Sz,Tw,2t)),\phi(M(Sw,Tw,t))) + k_2(t)(\phi(M(Tz,Sw,2t))) \\
&= k_1(t)\min(\phi(M(z,w,t)),\phi(1),\phi(M(z,w,2t)),\phi(1)), + k_2(t)(\phi(M(z,w,2t))) \\
&= k_2(t)(\phi(M(z,w,2t))) \\
&< \phi(M(z,w,2t)) \\
&< \phi(M(z,w,t)); \quad t>0.
\end{aligned}$$

which is a contradiction and thus, z is the unique common fixed point of T and S.

Example 1. Let $X = [0,1]$ and $*$ be the minimum t-norm. Define $M(x,y,t) = \frac{t}{t+|x-y|}$ for any x, y $\in$ X and $t > 0$. Let $T,S : X \to X$ be defined by $Tx = 1$ and $Sx = x$.
Let $\varphi(\tau) = 1-\tau$ where $\tau \in [0,1]$ and the functions k_1,k_2 be defined by

$$k_1(t) = k_2(t) = \begin{cases} \frac{e^2}{t+e^2}, & \text{if } t \in (0,1]; \\ \frac{t}{t+\frac{1}{e^2}}, & \text{if } t \in (1,\infty) \end{cases}$$

For a sequence $\{x_n\}$ defined by $x_n = 1-\frac{1}{n}$, we have $\lim_{n\to\infty} Tx_n = 1$ and $\lim_{n\to\infty} Sx_n = 1$. Hence, T,S satisfies E.A. property. It is also easy to see that the pair (T,S) is weakly compatible. Here, all the hypothesis of theorem(1) holds and hence $z = 1$ is the unique common fixed point of T and S in X.

Theorem 2. *Let $(X,M,*)$ be a fuzzy metric space and T,S be weakly compatible self-maps of X satisfying the following property:*

$$\phi(M(Tx,Ty,t)) \leq k_1(t)\phi(M(Sx,Tx,t) + k_2(t)\phi(M(Sy,Ty,t)) + k_3(t)\phi(M(Sx,Sy,t)) \tag{9}$$

where $x,y \in X$, $k_1,k_2,k_3 : (0,\infty) \to (0,1)$, $t > 0$ and ϕ is an altering distance function. If T and S satisfy property E.A. and the range of S is a closed subspace of X, then T and S have a unique common fixed point in X.

Proof. Suppose that T and S satisfy the property E.A., then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z \in X \tag{10}$$

Since S(X) is a closed subspace of X, there exists u ∈ X such that

$$z = Su. \tag{11}$$

For $x = x_n, y = u$, (9) becomes

$$\phi(M(Tx_n,Tu,t)) \leq k_1(t)\phi(M(Sx_n,Tx_n,t)) + k_2(t)\phi(M(Su,Tu,t)) + k_3(t)\phi(M(Sx_n,Su,t))$$

Taking limit $n \to \infty$, and using (10) and (11), we get

$$\begin{aligned}
&\phi(M(z,Tu,t)) \leq k_1(t)\phi(M(z,z,t)) + k_2(t)\phi(M(z,Tu,t)) + k_3(t)\phi(M(z,z,t))\\
&\implies \phi(M(z,Tu,t)) \leq k_1(t)\phi(1) + k_2(t)\phi(M(z,Tu,t)) + k_3(t)\phi(1)\\
&\implies \phi(M(z,Tu,t)) \leq k_2(t)\phi(M(z,Tu,t))\\
&\implies (1-k_2(t))\phi(M(z,Tu,t)) \leq 0 \text{ which gives } \phi(M(z,Tu,t)) = 0 \implies M(z,Tu,t) = 1\\
&\text{i.e., } Tu = z
\end{aligned} \tag{12}$$

From (11) and (12), we have

$$Tu = Su = z \tag{13}$$

Since T, S are weakly compatible, we get

$$Tz = Sz \tag{14}$$

Now we shall show that z is a fixed point of T. Suppose let us assume that $Tz \neq z$.
In view of (9), (13), (14), we obtain

$$\begin{aligned}
\phi(M(Tz,z,t) &= \phi(M(Tz,Tu,t))\\
&\leq k_1(t)\phi(M(Sz,Tz,t)) + k_2(t)\phi(M(Su,Tu,t)) + k_3(t)\phi(M(Sz,Su,t))\\
&= k_1(t)\phi(1) + k_2(t)\phi(1) + k_3(t)\phi(M(Tz,z,t))\\
&= k_3(t)\phi(M(Tz,z,t))\\
\implies \phi(M(Tz,z,t) &\leq k_3(t)\phi(M(Tz,z,t))
\end{aligned}$$

$\Rightarrow (1-k_3(t)\phi(M(Tz,z,t)) \leq 0$ which gives $\phi(M(Tz,z,t)) = 0$ implies $M(Tz,z,t) = 1$. Thus, $Tz = z$

$$\therefore\ Tz = z = Sz \tag{15}$$

i.e., z is a common fixed point T and S.

For uniqueness, let $w \in X$ be another common fixed point of T and S where

$$Tw = Sw = w \text{ and } w \neq z \tag{16}$$

In view of (15), (16), (9) and from the properties of ϕ, we have

$$\begin{aligned}\phi(M(z,w,t)) &= \phi(M(Tz,Tw,t))\\ &\le k_1(t)\phi(M(Sz,Tz,t)) + k_2(t)\phi(M(Sw,Tw,t)) + k_3(t)\phi(M(Sz,Sw,t))\\ &= k_1(t)\phi(1) + k_2(t)\phi(1) + k_3(t)\phi(M(z,w,t))\\ &= k_3(t)\phi(M(z,w,t)) < \phi(M(z,w,t))\end{aligned}$$

which is a contradiction and thus, z is the unique common fixed point of T and S.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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