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Curvature tensor in tangette bundles of semi-riemannian manifold

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Abstract: In the conducted study, some theorems have been written by calculating $\tilde{R}^h_{kij}$ coefficient of ${}^HR$ curvature tensor and $\tilde{S}^h_{ij}$ coefficient of ${}^HS$ torsion tensor according to affine connection in tangente bundles of Semi-Riemannian manifold. Besides, ${}^HR_{ij}$ Ricci tensor has been examined and ${}^HR_{ij}$ coefficient has been calculated. Finally, $S = {}^Hg^{IH}{}^HR_{IJ}$ scalar curvature has been examined and some theorems have been associated with this.

Keywords: Semi-Riemannian manifold, curvature tensor, torsion tensor, Ricci tensor, scalar curvature.

1 Introduction

In this study, $\tilde{S}^h_{ij}$ coefficients of ${}^HR$ Curvature Tensor and $\tilde{R}^h_{kij}$ and ${}^HS$ Torsion Tensor were calculated according to the affine connection in the tangent bundle of the Semi-Riemannianian manifold. Defined on M manifold.
(i) $g(X,Y) = g(Y,X), \forall X,Y \in \Im^1_0(M)$,(balancing).
(ii) $g(X,X) \geq 0, \forall X \in \Im^1_0(M)$ ve $g(X,X) = 0 \Leftrightarrow X = 0$, (Positive definition).
(0,2)-typed g tensor field fulfilling the conditions is called as Riemannian metric or metric tensor. In this case, (M_n, g) pair is called as Riemannian manifold.

Let Mn be an n-dimensional differentiable manifold of and C^∞ class and $T^1q(Mn)$ the tensor bundle over Mn of tensor of type (1,q). If xi are local coordinates in a neighborhood U of point $x \in Mn$, then a tensor t at x which is an element of $T^1q(Mn)$ is expressible in the form (xi, $tji_1 \cdots iq$), where $tj\underline{i}_1 \dots iq$ are components of t with respect to the natural frame. It may be considered $(x^i, t^j_{i1} \dots i_q) = (x^i, x^{\bar{i}}) = (x^J)$, i=l,...,n, $\bar{i} = n+1, \dots, n(1+n^q)$, I=1,...,n(l+nq) as local coordinates in a neighborhood π^{-1} is the natural projection $Tq^1(Mn)$ onto Mn).

Let then Mn be a Riemannian manifold with non-degenerate metric g whose components in a coordinate neighborhood U are gji and denote by Γhji the Christoffel symbols which are formed withgji.

We indicate by $\Im^r_s(M_n)$ the module over $F(Mn)$ ($F(Mn)$ is the ring of C^∞ functions in Mn all tensor fields of C^∞ class and of type (r,s) in Mn. Let $X \in \Im^1_0(M_n)$ and $w \in \Im^1_q(M_n)$. Then ${}^CX \in \Im^1_0(T^1_q(M_n))$ (complete lift) ${}^HX \in \Im^1_0(T^1_q(M_n))$ (horizontal lift) and ${}^Vw \in \Im^1_0(T^1_q(M_n))$ (vertical lift) have, respectively, components. [1,2,3]

For the curvature tensor of connection $\forall X,Y,Z \in \Im^1_0(M_n)$

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1}$$

it is defined as above [4,5,6,7]. Instead of $R(X,Y,Z)$, $R(X,Y)Z$ can also be used.

$$R(X,Y,Z) = -R(Y,X,Z), \tag{2}$$

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is understood from (1). It is easy to see that Rfulfills linearity condition in terms of X, Y and Z variables. However, if $R(X,Y,Z) \in \Im_0^1(M_n)$, $R \in \Im_3^1(M_n)$.

In accompany with (1) if it is considered that$X = \partial_i, Y = \partial_j, Z = \partial_k$, the coordinates of R on the natural framework are expressed as following [8],

$$R^s_{ijk} = \partial_i \Gamma^s_{jk} - \partial_j \Gamma^s_{ik} + \Gamma^s_{im}\Gamma^m_{jk} - \Gamma^s_{jm}\Gamma^m_{ik}. \tag{3}$$

The torsion tensor of the connection is defined as,

$$2S(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \forall X,Y \in \Im_0^1(M_n)$$

In this expression, if it is considered that$X = \partial_i, Y = \partial_j$, the coordinates of S on the natural framework are,

$$S^k_{ij} = \frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}). \tag{4}$$

It is seen that $S^k_{ij} = -S^k_{ji}$. Using the (4) equation, coefficients of $\tilde{S}^h_{ij}$ were calculated. Ricci tensor is the tensor defined by utilizing R curvature tensor,

$$R_{ij} = R^k_{kij}.$$

If curvature tensor formula is used,

$$R_{ij} = R^k_{kij} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^l_{ki}\Gamma^k_{lj}.$$

That means, if $\Gamma^s_{ks} = \partial_k \ln e$, $R_{ij} = R^k_{kij}$ Ricci tensor is symmetrical. This means that Ricci tensor can be indicated as $R_{ij} = R_{ji}$. In tension free spaces, if the equation $R^s_{[ijk]} = \frac{1}{3}\left(R^s_{ijk} + R^s_{jki} + R^s_{kij}\right) = 0$, is used,

$$R^k_{rsk} = R_{rs} - R_{sr},$$

is obtained [4]. M_n is a n-dimensioned Riemannian manifold among C^∞ class, letg_{ij} metric be regular, symmetrical and let the connection be Levi-Civita connection. On M_n, if the s index on R^s_{ijk} curvature tensor is moved down to the place after k, (0,4)-typed tensor indicated below is obtained.

$$R_{ijkt} = g_{st} R^s_{ijk} \Leftrightarrow R(X,Y,Z,W) = g\left(R(X,Y)Z,W\right)$$

$R_{ij} = R^s_{sij} = g^{ts} R_{tijs} = g^{ts} R_{itsj}$ tensor is called as Ricci tensor [9].

Full contraction operation is conducted with g^{ij} tensor and Ricci tensor and

$$R = g^{ij} R_{ij},$$

The R curvature here is called as scaler curvature. If the pseudo-Riemannian metric indicated as g on M_n is defined as $ds^2 = g_{ji} dx^j dx^i$, the pseudo-Riemannian metric indicated as ${}^H g$ on $T(M_n)$ is $2g_{ji}\delta y^j dx^i$ [10]. Here, $\delta y^j = dy^j + \tilde{\Gamma}^j_{lk} y^l dx^k$, moreover, both Γ^j_{lk} and M_n ∇affine connection coefficients and the covariant components of ${}^H g$metric on tangent cluster are [11,12];

$${}^H g^{IH} = \begin{pmatrix} 0 & g^{ih} \\ g^{ih} & -(\Gamma^h_j g^{ji} + \Gamma^i_s g^{sh}) \end{pmatrix}$$

Using ${}^H R_{ij}$ coefficients and ${}^H g^{IH} = \begin{pmatrix} 0 & g^{ih} \\ g^{ih} & -(\Gamma^h_j g^{ji} + \Gamma^i_s g^{sh}) \end{pmatrix}$ equation, the coefficients of $S = {}^H g^{IH\,H} R_{IJ}$ were calculated.

2 ${}^H R$ Curvature and ${}^H S$ Torsion tensors on the tangent bundle of semi-Riemannian manifold according to affine connection

${}^H \Gamma^I_{JK}$ is the symbol of Cristoffel defined with ${}^H g$. Using ${}^H \Gamma^I_{JK} = \frac{1}{2} {}^H g^{IS}(\partial_J {}^H g_{SK} + \partial_K {}^H g_{JS} - \partial_S {}^H g_{JK})$ formula, each coefficient of ${}^H \Gamma^I_{JK}$was calculated [11]. We get,

$$ {}^H\Gamma^i_{jk} = \Gamma^i_{jk} + \frac{1}{2} g^{is}\nabla_s g_{jk},\ {}^H\Gamma^{i}_{\bar{j}\bar{k}} = 0,\ {}^H\Gamma^{i}_{j\bar{k}} = 0,\ {}^H\Gamma^{i}_{\bar{j}k} = 0,\ {}^H\Gamma^{\bar{i}}_{\bar{j}\bar{k}} = 0,\ {}^H\Gamma^{\bar{i}}_{j\bar{k}} = \Gamma^i_{jk} - \frac{1}{2} g^{is}\nabla_j g_{sk},\ {}^H\Gamma^{\bar{i}}_{\bar{j}k} = \Gamma^i_{jk} - \frac{1}{2} g^{is}\nabla_k g_{js}, $$

$$ {}^H\Gamma^{\bar{i}}_{jk} = y^l \partial_l \Gamma^i_{jk} - y^l(\nabla_l g^{is}) g_{is}\Gamma^i_{jk},\ -\frac{1}{2} g^{is} y^l (\partial_j(\nabla_l g_{sk}) + \partial_k(\nabla_l g_{js}) - \partial_s(\nabla_l g_{jk})),\ +\frac{1}{2} y^l (\nabla_l g^{is})(\nabla_s g_{jk}) + \frac{1}{2} y^l \partial_l g^{is}\nabla_s g_{jk}. $$

Theorem 1. *∇, M_n is an affine connection in ∇, M_n manifold, the necessary and sufficient condition for ${}^C\nabla$full lift and ${}^H\nabla$horizontal lift to be equal is that ∇ is a metric connection.*

Using $\tilde{R}^H_{KJI} = \partial_K {}^H\Gamma^H_{JI} - \partial_J {}^H\Gamma^H_{KI} + {}^H\Gamma^H_{KT}\,{}^H\Gamma^T_{JI} - {}^H\Gamma^H_{JI}\,{}^H\Gamma^T_{KI}$ formula and each of the coefficients of ${}^H\Gamma^I_{JK}$, the $\tilde{R}^h_{kji}$ coefficients of ${}^H R$curvature tensor were calculated. It is found that

$$ \begin{aligned} \tilde{R}^h_{kji} =& \partial_k {}^H\Gamma^h_{ji} - \partial_j {}^H\Gamma^h_{ki} + {}^H\Gamma^h_{kT}\,{}^H\Gamma^T_{ji} - {}^H\Gamma^h_{jT}\,{}^H\Gamma^T_{ki} \\ =& \partial_k {}^H\Gamma^h_{ji} - \partial_j {}^H\Gamma^h_{ki} + \left({}^H\Gamma^h_{kt}\,{}^H\Gamma^t_{ji} + {}^H\Gamma^h_{k\bar{t}}\,{}^H\Gamma^{\bar{t}}_{ji}\right) - \left({}^H\Gamma^h_{jt}\,{}^H\Gamma^t_{ki} + {}^H\Gamma^h_{j\bar{t}}\,{}^H\Gamma^{\bar{t}}_{ki}\right) \\ =& \partial_k\left(\Gamma^h_{ji} + \frac{1}{2} g^{hs}\nabla_s g_{ji}\right) - \partial_j\left(\Gamma^h_{ki} + \frac{1}{2} g^{hs}\nabla_s g_{ki}\right) + \left[\left(\Gamma^h_{kt} + \frac{1}{2} g^{hs}\nabla_s g_{kt}\right).\left(\Gamma^t_{ji} + \frac{1}{2} g^{ts}\nabla_s g_{ji}\right)\right] \\ &- \left[\left(\Gamma^h_{jt} + \frac{1}{2} g^{hs}\nabla_s g_{jt}\right).\left(\Gamma^t_{ki} + \frac{1}{2} g^{ts}\nabla_s g_{ki}\right)\right] \\ =& \partial_k\Gamma^h_{ji} + \frac{1}{2}\partial_k(g^{hs}\nabla_s g_{ji}) - \partial_j\Gamma^h_{ki} - \frac{1}{2}\partial_j(g^{hs}\nabla_s g_{ki}) + \Gamma^h_{kt}\Gamma^t_{ji} + \frac{1}{2}\Gamma^h_{kt} g^{ts}\nabla_s g_{ji} + \frac{1}{2}\Gamma^t_{ji} g^{hs}\nabla_s g_{kt}) \\ &+ \frac{1}{4} g^{hs} g^{ts}\nabla_s g_{kt}\nabla_s g_{ji} - \Gamma^h_{jt}\Gamma^t_{ki} - \frac{1}{2}\Gamma^h_{jt} g^{ts}\nabla_s g_{ki} - \frac{1}{2}\Gamma^t_{ki} g^{hs}\nabla_s g_{jt} - \frac{1}{4} g^{hs} g^{ts}\nabla_g g_{jt}\nabla_s g_{ki} \\ =& R^h_{kji} + \frac{1}{2}\left(\partial_k g^{hs}\nabla_s g_{ji}\right) - \partial_j(g^{hs}\nabla_s g_{ki}) + \frac{1}{2} g^{ts}\left(\Gamma^h_{kt}\nabla_s g_{ji} - \Gamma^h_{jt}\nabla_s g_{ki}\right) \\ &+ \frac{1}{2} g^{hs}\left(\Gamma^t_{ji}\nabla_s g_{gt} - \Gamma^t_{ki}\nabla_s g_{jt}\right) + \frac{1}{4} g^{hs} g^{ts}\left(\nabla_s g_{kt}\nabla_s g_{ji} - \nabla_s g_{jt} + \nabla_s g_{ki}\right) \end{aligned} $$

$$ \begin{aligned} \tilde{R}^{\bar{h}}_{kji} =& R^h_{kji} \\ \tilde{R}^{\bar{h}}_{kj\bar{i}} =& R^h_{kji} - \frac{1}{2}(\partial_k(g^{hs}\nabla_j g_{is}) - \partial_j(g^{hs}\nabla_k g_{is})) - \frac{1}{2} g^{ts}(\Gamma^h_{kt}\nabla_i g_{js} - \Gamma^h_{jt}\nabla_i g_{ks}) \\ &- \frac{1}{2} g^{hs}(\Gamma^t_{ji}\nabla_t g_{ks} - \Gamma^t_{ki}\nabla_t g_{js}) + \frac{1}{4} g^{hs} g^{ts}(\nabla_t g_{ks}\nabla_i g_{js} - \nabla_t g_{js}\nabla_i g_{ks}) \\ \tilde{R}^{\bar{h}}_{kj\bar{i}} =& \partial_k\left(\Gamma^h_{ji} - \frac{1}{2} g^{hs}\nabla_j g_{si}\right) - \left(\partial_j\Gamma^h_{ki} - \nabla_j g^{hs}\right) g_{hs}\Gamma^h_{ki} \\ &- \frac{1}{2} g^{hs} y^\ell\left(\partial_k(\nabla_\ell g_{si})\,\partial_i(\nabla_\ell g_{ks}) - \partial_s(\nabla_\ell g_{ki})\right) \\ &+ \frac{1}{2} y^\ell\left(\nabla_\ell g^{hs}\right)(\nabla_s g_{ki}) + \frac{1}{2} y^\ell\partial_\ell g^{hs}\nabla_s g_{ki}] \end{aligned} $$

$$ \begin{aligned} \tilde{R}^{\bar{h}}_{kji} =& (\partial_k(y^l\partial_l\Gamma^h_{ji} - y^l(\nabla_l g^{hs}) g_{hs}\Gamma^h_{ji} - \frac{1}{2} g^{hs} y^l(\partial_j(\nabla_l g_{si}) + (\partial_i(\nabla_l g_{js}) - (\partial_s(\nabla_l g_{ji}) \\ &+ \frac{1}{2} y^l(\nabla_l g^{hs})(\nabla_s g_{ji}) + \frac{1}{2} y^l\partial_l g^{hs}\nabla_s g_{ji})) - (\partial_j(\Gamma^h_{ki} - \frac{1}{2} g^{hs}\nabla_k g_{si})) \\ &+ ((\Gamma^h_{kt} - \frac{1}{2} g^{hs}\nabla_k g_{st})(\Gamma^t_{ji} - \frac{1}{2} g^{ts}\nabla_j g_{si}) - ((\Gamma^h_{jt} - \frac{1}{2} g^{hs}\nabla_t g_{sj})(\Gamma^t_{ki} - \frac{1}{2} g^{ts}\nabla_i g_{ks})) \end{aligned} $$

and

$$ \tilde{R}^{h}_{\bar{k}ji} = \tilde{R}^{h}_{k\bar{j}i} = \tilde{R}^{h}_{\bar{k}\bar{j}i} = \tilde{R}^{h}_{kj\bar{i}} = \tilde{R}^{h}_{\bar{k}j\bar{i}} = \tilde{R}^{h}_{k\bar{j}\bar{i}} = \tilde{R}^{h}_{\bar{k}\bar{j}\bar{i}} = \tilde{R}^{\bar{h}}_{\bar{k}\bar{j}i} = \tilde{R}^{\bar{h}}_{\bar{k}j\bar{i}} = \tilde{R}^{\bar{h}}_{k\bar{j}\bar{i}} = \tilde{R}^{\bar{h}}_{\bar{k}\bar{j}\bar{i}} = 0. $$

Result. Let (M_n, g) be semi-Riemannian manifold. According to metric connection, $\tilde{R}^h_{ijk}$coefficients of R tensor are as following on $(\nabla g = 0)$ tangent bundle,

$$\tilde{R}^{h}_{kji} = R^{h}_{kji}, \tilde{R}^{\bar{h}}_{kji} = R^{h}_{kji}, \tilde{R}^{\bar{h}}_{kj\bar{i}} = R^{h}_{kji}, \tilde{R}^{\bar{h}}_{\bar{k}ji} = R^{h}_{kji},$$

$$\tilde{R}^{\bar{h}}_{kj\bar{i}} = \partial_k \Gamma^{h}_{ji} - \partial_j \Gamma^{h}_{ki} g_{hs} \Gamma^{h}_{ki},$$

and the others are zero.
The torsion tensor of the connection is,

$$S^{k}_{ij} = \frac{1}{2}(\Gamma^{k}_{ij} - \Gamma^{k}_{ji}) \tag{5}$$

Using (5) equation, $\tilde{S}^{h}_{ij}$ coefficients were calculated. They are

$$\begin{aligned}
\tilde{S}^{h}_{ji} &= \frac{1}{2}\left({}^{H}\Gamma^{h}_{ji} - {}^{H}\Gamma^{h}_{ij}\right)\\
&= \frac{1}{2}\left(\Gamma^{h}_{ji} + \frac{1}{2}g^{hs}\nabla_s g_{ji} - \Gamma^{h}_{ij} - \frac{1}{2}g^{hs}\nabla_s g_{ij}\right)\\
&= \frac{1}{2}\left(\Gamma^{h}_{ji} - \Gamma^{h}_{ij} + \frac{1}{2}g^{hs}\nabla_s g_{ji} - \frac{1}{2}g^{hs}\nabla_s g_{ij}\right)\\
&= S^{h}_{ji} + \frac{1}{4}g^{h}\nabla_s\left(g_{ji} - g_{ij}\right)
\end{aligned}$$

$$\begin{aligned}
\tilde{S}^{\bar{h}}_{\bar{j}i} &= S^{h}_{ji},\\
\tilde{S}^{\bar{h}}_{j\bar{i}} &= S^{h}_{ji},\\
\tilde{S}^{\bar{h}}_{ji} &= S^{h}_{ji} y^{\ell}\left(\partial_\ell - \left(\nabla_\ell g^{hs}\right) g_{hs},\right)\\
\tilde{S}^{h}_{\bar{j}i} &= \tilde{S}^{h}_{j\bar{i}} = \tilde{S}^{h}_{\bar{j}\bar{i}} = \tilde{S}^{\bar{h}}_{\bar{j}\bar{i}} = 0.
\end{aligned}$$

Theorem 2. *Torsion-free space which has metric connection on (M_n, g) semi-Riemannian manifold tangent bundle is* $S^{h} = 0$.

3 Analysis of ${}^{H}R_{ij}$ Ricci tensor on tangent bundle

Using,

$$R^{k}_{rsk} = R_{rs} - R_{sr}$$

and $\tilde{R}^{h}_{kji}$ coefficients, ${}^{H}R_{ij}$ coefficients were calculated. They are found as

$$\begin{aligned}
{}^{H}R_{ji} = R^{K}_{Kji} = R^{k}_{kji} + R^{\bar{k}}_{\bar{k}ji} &= R^{k}_{kji} + \frac{1}{2}(\partial_k(g^{ks}\nabla_s g_{ji}) - (\partial_j(g^{ks}\nabla_s g_{ki}) + \frac{1}{2}g^{ts}(\Gamma^{k}_{kt}\nabla_s g_{ji} - \Gamma^{k}_{jt}\nabla_s g_{ki})\\
&+ \frac{1}{2}g^{ks}(\Gamma^{t}_{ji}\nabla_s g_{kt} - \Gamma^{t}_{ki}\nabla_s g_{jt}) + \frac{1}{4}g^{ks}g^{ts}(\nabla_s g_{kt}\nabla_s g_{ji} - \nabla_s g_{jt}\nabla_s g_{ki})\\
&+ (\partial_k(y^{l}\partial_l\Gamma^{k}_{ji} - y^{l}(\nabla_l g^{ks})g_{ks}\Gamma^{k}_{ji} - \frac{1}{2}g^{ks}y^{l}(\partial_j(\nabla_l g_{si}) + (\partial_i(\nabla_l g_{js}) - (\partial_s(\nabla_l g_{ji})\\
&+ \frac{1}{2}y^{l}(\nabla_l g^{ks})(\nabla_s g_{ji}) + \frac{1}{2}y^{l}\partial_l g^{ks}\nabla_s g_{ji})) - (\partial_j(\Gamma^{k}_{ki} - \frac{1}{2}g^{ks}\nabla_k g_{si}))\\
&+ (\Gamma^{k}_{kt} - \frac{1}{2}g^{ks}\nabla_k g_{st})(\Gamma^{t}_{ji} - \frac{1}{2}g^{ts}\nabla_j g_{si}) - ((\Gamma^{k}_{jt} - \frac{1}{2}g^{ks}\nabla_t g_{sj})(\Gamma^{t}_{ki} - \frac{1}{2}g^{ts}\nabla_i g_{ks}))
\end{aligned}$$

$$
\begin{aligned}
{}^{H}R_{j\bar{i}} &= R^{K}_{Kj\bar{i}} = R^{k}_{kj\bar{i}} + R^{\bar{k}}_{\bar{k}j\bar{i}} = 0+0=0\\
{}^{H}R_{\bar{j}i} &= R^{K}_{K\bar{j}i} = R^{k}_{k\bar{j}i} + R^{\bar{k}}_{\bar{k}\bar{j}i} = 0+0=0\\
{}^{H}R_{\bar{j}\bar{i}} &= R^{K}_{K\bar{j}\bar{i}} = R^{k}_{k\bar{j}\bar{i}} + R^{\bar{k}}_{\bar{k}\bar{j}\bar{i}} = 0+0=0.
\end{aligned}
$$

When $\nabla g = 0$,

$$
\begin{aligned}
&{}^{H}R_{ji} = R^{k}_{kji} + \partial_k\left(y^l\partial_l\Gamma^k_{ji}\right) - \partial_j\Gamma^k_{ki} + \Gamma^k_{kt}\Gamma^t_{ji} - \Gamma^k_{tj}\Gamma^t_{ki},\\
&{}^{H}R_{\bar{j}\bar{i}} = {}^{H}R_{j\bar{i}} = {}^{H}R_{\bar{j}i} = 0.
\end{aligned}
$$

4 Analysis of $S = {}^{H}g^{IH\,H}R_{IJ}$ scaler curvature on tangent bundle

Using ${}^{H}R_{ij}$ coefficients and ${}^{H}g^{IH} = \begin{pmatrix} 0 & g^{ih} \\ g^{ih} & -(\Gamma^h_j g^{ji} + \Gamma^i_s g^{sh}) \end{pmatrix}$ equation, the coefficients of $S = {}^{H}g^{IH\,H}R_{IJ}$ were calculated. They are found as,

$$
\begin{aligned}
{}^{H}g^{ij\,H}R_{ij} = 0.(R^k_{kji} &+ \frac{1}{2}(\partial_k(g^{ks}\nabla_s g_{ji}) - (\partial_j(g^{ks}\nabla_s g_{ki}) + \frac{1}{2}g^{ts}(\Gamma^k_{kt}\nabla_s g_{ji} - \Gamma^k_{jt}\nabla_s g_{ki})\\
&+ \frac{1}{2}g^{ks}(\Gamma^t_{ji}\nabla_s g_{kt} - \Gamma^t_{ki}\nabla_s g_{jt}) + \frac{1}{4}g^{ks}g^{ts}(\nabla_s g_{kt}\nabla_s g_{ji} - \nabla_s g_{jt}\nabla_s g_{ki})\\
&+ (\partial_k(y^l\partial_l\Gamma^k_{ji} - y^l(\nabla_l g^{ks})g_{ks}\Gamma^k_{ji} - \frac{1}{2}g^{ks}y^l(\partial_j(\nabla_l g_{si})\\
&+ (\partial_i(\nabla_l g_{js}) - (\partial_s(\nabla_l g_{ji}) + \frac{1}{2}y^l(\nabla_l g^{ks})(\nabla_s g_{ji}) + \frac{1}{2}y^l\partial_l g^{ks}\nabla_s g_{ji})) - (\partial_j(\Gamma^k_{ki} - \frac{1}{2}g^{ks}\nabla_k g_{si}))\\
&+ (\Gamma^k_{kt} - \frac{1}{2}g^{ks}\nabla_k g_{st})(\Gamma^t_{ji} - \frac{1}{2}g^{ts}\nabla_j g_{si}) - ((\Gamma^k_{jt} - \frac{1}{2}g^{ks}\nabla_t g_{sj})(\Gamma^t_{ki} - \frac{1}{2}g^{ts}\nabla_i g_{ks})) = 0\\
{}^{H}g^{\bar{i}j\,H}R_{\bar{i}j} = g^{ih}.0 &= 0\\
{}^{H}g^{i\bar{j}\,H}R_{i\bar{j}} = g^{ih}.0 &= 0\\
{}^{H}g^{\bar{i}\bar{j}\,H}R_{\bar{i}\bar{j}} = -&\left(\Gamma^h_j g^{ji} + \Gamma^i_s g^{sh}\right).0 = 0
\end{aligned}
$$

Theorem 3. *Let (M_n, g) be semi-Riemannian manifold. ${}^{H}R$ scaler curvature of $\left(T\left(M_n, {}^{H}g\right)\right)$ space is zero.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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