Curvature tensor in tangette bundles of semi-riemannian manifold

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Abstract: In the conducted study, some theorems have been written by calculating \( R_{ijkl} \) coefficient of \( H^R \) curvature tensor and \( S_{ijkl} \) coefficient of \( H^S \) torsion tensor according to affine connection in tangentte bundles of Semi-Riemannian manifold. Besides, \( H^R_{ij} \) Ricci tensor has been examined and \( H^R_{ij} \) coefficient has been calculated. Finally, \( S = H^H H^R_{ij} \) scalar curvature has been examined and some theorems have been associated with this.

Keywords: Semi-Riemannian manifold, curvature tensor, torsion tensor, Ricci tensor, scalar curvature.

1 Introduction

In this study, \( S_{ijkl} \) coefficients of \( H^R \) Curvature Tensor and \( R_{ijkl} \) and \( H^S \) Torsion Tensor were calculated according to the affine connection in the tangent bundle of the Semi-Riemannian manifold. Defined on \( M \) manifold.

(i) \( g(X,Y) = g(Y,X), \forall X,Y \in \mathfrak{g}(M) \) (balancing).

(ii) \( g(X,X) \geq 0, \forall X \in \mathfrak{g}(M) \) ve \( g(X,X) = 0 \Leftrightarrow X = 0 \). (Positive definition).

(0,2)-type \( g \) tensor field fulfilling the conditions is called as Riemannian metric or metric tensor. In this case, \((M_0,g)\) pair is called as Riemannian manifold.

Let \( Mn \) be an \( n \)-dimensional differentiable manifold of and \( C^\infty \) class and \( T^1q(Mn) \) the tensor bundle over \( Mn \) of tensor of type \((1,q)\). If \( xi \) are local coordinates in a neighborhood \( U \) of point \( x \in Mn \), then a tensor \( t \) at \( x \) which is an element of \( T^1q(Mn) \) is expressible in the form \((x_i, t_{ij1}...iq)\), where \( t_{ij1}...iq \) are components of \( t \) with respect to the natural frame. It may be considered \((t'_1, t'_2...t'_q) = (x'_i, x'_j)\), \( =i=1,...,n, \tilde{i}=n+1,...,n(1+nq), I=1,...,n(1+nq) \) as local coordinates in a neighborhood \( \pi^{-1} \) is the natural projection \( T^1q(Mn) \) onto \( Mn \).

Let then \( Mn \) be a Riemannian manifold with non-degenerate metric \( g \) whose components in a coordinate neighborhood \( U \) are \( g_{ij} \) and denote by \( \Gamma_{jhi} \) the Christoffel symbols which are formed with\( g_{ij} \).

We indicate by \( \mathfrak{g}(M_0) \) the module over \( F(Mn) \) \( (F(Mn) \) is the ring of \( C^\infty \) functions in \( Mn \) all tensor fields of \( C^\infty \) class and of type \((r,s)\) in \( Mn \). Let \( X \in \mathfrak{g}(M_0) \) and \( w \in \mathfrak{g}(M_0) \). Then \( C^\infty X \in \mathfrak{g}(T^1q(M_0)) \) (complete lift) \( H^X \in \mathfrak{g}(T^1q(M_0)) \) (horizontal lift) and \( Vw \in \mathfrak{g}(T^1q(M_0)) \) (vertical lift) have, respectively, components. \([1,2,3]\)

For the curvature tensor of connection \( \forall X,Y,Z \in \mathfrak{g}(M_0) \)

\[
R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1}
\]

it is defined as above \([4,5,6,7]\). Instead of \( R(X,Y,Z), R(X,Y)Z \) can also be used.

\[
R(X,Y,Z) = -R(Y,X,Z). \tag{2}
\]
is understood from (1). It is easy to see that \( R \) fulfills linearity condition in terms of \( X, Y \) and \( Z \) variables. However, if \( R(X, Y, Z) \in \mathcal{S}_0^1(M_n), R \in \mathcal{S}_1^1(M_n) \).

In accompany with (1) if it is considered that \( X = \partial_i, Y = \partial_j, Z = \partial_k, \) the coordinates of \( R \) on the natural framework are expressed as following [8],

\[
R^k_{ijk} = \partial_i \Gamma^k_{jk} - \partial_j \Gamma^k_{ik} + \Gamma^m_{ik} \Gamma^k_{jm} - \Gamma^m_{jk} \Gamma^k_{im}.
\]  

(3)

The torsion tensor of the connection is defined as,

\[
2S(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \forall X, Y \in \mathcal{S}_1^1(M_n)
\]

In this expression, if it is considered that \( X = \partial_i, Y = \partial_j, \) the coordinates of \( S \) on the natural framework are,

\[
S^{ij}_{kj} = \frac{1}{2}(\Gamma^{ij}_{kj} - \Gamma^{kj}_{ij}).
\]

(4)

It is seen that \( S^{ij}_{kj} = -S^{ij}_{kj}. \) Using the (4) equation, coefficients of \( S^{ij}_{kj} \) were calculated. Ricci tensor is the tensor defined by utilizing \( R \) curvature tensor,

\[
R_{ij} = R^k_{ikj}.
\]

If curvature tensor formula is used,

\[
R_{ij} = R^k_{ikj} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^l_{ik} \Gamma^k_{lj} - \Gamma^l_{jk} \Gamma^k_{il}.
\]

That means, if \( \Gamma^k_{kj} = \partial_k \ln e, R_{ij} = R^k_{ikj} \) Ricci tensor is symmetrical. This means that Ricci tensor can be indicated as \( R_{ij} = R_{ji}. \) In tension free spaces, if the equation \( R^k_{ij} = \frac{1}{2} \left( R^k_{ij} + R^k_{ji} + R^k_{kj} \right) = 0, \) is used,

\[
R^k_{rsk} = R_{rs} - R_{sr},
\]

is obtained [4]. \( M_n \) is a \( n \)-dimensioned Riemannian manifold among \( C^\infty \) class, let \( g \) metric be regular, symmetrical and let the connection be Levi-Civita connection. On \( M_n, \) if the \( s \) index on \( R^k_{ij} \) curvature tensor is moved down to the place after \( k, (0,4) \)-typed tensor indicated below is obtained.

\[
R_{ik} = g_{ik} R^l_{ljk} \Leftrightarrow R(X, Y, Z, W) = g \left( R(X, Y) Z, W \right)
\]

\[
R_{ij} = R^k_{ij} = g^{il} R_{ljk} = g^{il} R_{ljs} \text{ tensor is called as Ricci tensor [9].}
\]

Full contraction operation is conducted with \( g^{ij} \) tensor and Ricci tensor and

\[
R = g^{ij} R_{ij},
\]

The \( R \) curvature here is called as scaler curvature. If the pseudo-Riemannian metric indicated as \( g \) on \( M_n \) is defined as \( ds^2 = g_{ij} dx^i dx^j \), the pseudo-Riemannian metric indicated as \( H g \) on \( T(M_n) \) is \( 2 g_{ij} \delta y^i dx^j \) [10]. Here, \( \delta y^j = dy^j + \Gamma^j_{ik} dx^k, \) moreover, both \( \Gamma^j_{ik} \) and \( M_n \) affine connection coefficients and the covariant components of \( H g \) metric on tangent bundle are [11,12]:

\[
H g^{ij} = \begin{pmatrix}
0 & g^{ih} \\
\delta g^{ij} - (\Gamma^h_{ij} g^{ih} + \Gamma^i_{j} g^{ih})
\end{pmatrix}
\]

Using \( H g \) coefficients and \( H g^{ij} = \begin{pmatrix}
0 & g^{ih} \\
\delta g^{ij} - (\Gamma^h_{ij} g^{ih} + \Gamma^i_{j} g^{ih})
\end{pmatrix} \) equation, the coefficients of \( S = H g^{ij} H R_{ij} \) were calculated.

2. \( H R \) Curvature and \( H g \) Torsion tensors on the tangent bundle of semi-Riemannian manifold according to affine connection

\( H \Gamma^k_{ij} \) is the symbol of Cristoffel defined with \( H g \). Using \( H \Gamma^k_{ij} = \frac{1}{2} H g^{js} (\partial_j H g_{sk} + \partial_k H g_{js} - \partial_s H g_{jk}) \) formula, each coefficient of \( H \Gamma^k_{ij} \) was calculated [11]. We get,
\[ H \Gamma_{jk}^i = \Gamma_{jk}^i + \frac{1}{2} g^{ik} \nabla_j g_{kl} h^l, \quad H \Gamma_{jk}^i = 0, \quad H \Gamma_{jk} = 0, \quad H \Gamma_{ji} = -H \Gamma_{ij} \]

\[ H \Gamma_{jk}^i = \Gamma_{jk}^i - \frac{1}{2} g^{ik} \nabla_j g_{kl} h^l. \]

Theorem 1. \( \nabla \cdot M_a \) is an affine connection in \( \nabla, M_a \) manifold, the necessary and sufficient condition for \( C^\infty \) full lift and \( H^\infty \) horizontal lift to be equal is that \( \nabla \) is a metric connection.

Using \( \tilde{R}^H_{Kji} = \partial_k H \Gamma_{ji}^H - \partial_j H \Gamma_{ki}^H + H^T \Gamma_{ji}^H \Gamma_{ki}^H - \partial_i H \Gamma_{jkl}^H \Gamma_{kl}^H \) formula and each of the coefficients of \( H \Gamma_{jk} \), the \( \tilde{R}^H_{kji} \) coefficients of \( H \) curvature tensor were calculated. It is found that

\[ \tilde{R}^H_{kji} = \frac{1}{2} g^{ik} \nabla_j g_{kl} h^l, \quad \tilde{R}^H_{k} = \frac{1}{2} g^{ik} \nabla_j g_{kl} h^l. \]

Result. Let \((M_a, g)\) be semi-Riemannian manifold. According to metric connection, \( \tilde{R}^H_{kji} \) coefficients of \( R \) tensor are as
following on \((\nabla g = 0)\) tangent bundle,
\[
\tilde{R}^h_{kji} = R^h_{kji}, \quad \tilde{R}^h_{li} = R^h_{li}, \quad \tilde{R}^h_{kli} = R^h_{kli}, \quad \tilde{R}^h_{ki} = R^h_{ki},
\]
and the others are zero.
The torsion tensor of the connection is,
\[
S^h_{ji} = \frac{1}{2} (\Gamma^h_{ji} - \Gamma^h_{ij})
\]
Using (5) equation, \(S^h_{ji}\) coefficients were calculated. They are
\[
\begin{align*}
\tilde{S}^h_{ji} &= \frac{1}{2} \left( H \Gamma^h_{ji} - H \Gamma^h_{ij} \right) \\
&= \frac{1}{2} \left( \Gamma^h_{ji} + \frac{1}{2} g^{hs} \nabla_s g_{ji} - \Gamma^h_{ij} - \frac{1}{2} g^{hs} \nabla_s g_{ij} \right) \\
&= \frac{1}{2} \left( \Gamma^h_{ij} - \Gamma^h_{ij} + \frac{1}{2} g^{hs} \nabla_s g_{ji} - \frac{1}{2} g^{hs} \nabla_s g_{ij} \right) \\
&= \tilde{S}^h_{ji} + \frac{1}{4} g^{hs} \nabla_s (g_{ji} - g_{ij})
\end{align*}
\]
\[
\begin{align*}
\tilde{S}^h_{ji} &= S^h_{ji}, \\
\tilde{S}^h_{ij} &= S^h_{ij}, \\
\tilde{S}^h_{ji} &= \tilde{S}^h_{ij} = \tilde{S}^h_{ji} = 0.
\end{align*}
\]

**Theorem 2.** Torsion-free space which has metric connection on \((M_n, g)\) semi-Riemannian manifold tangent bundle is \(S^h = 0\).

### 3 Analysis of \(H R_{ij}\) Ricci tensor on tangent bundle

Using,
\[
R^k_{ijk} = R_{ikj} = R_{ijk}
\]
and \(\tilde{R}^h_{kji}\) coefficients, \(H R_{ij}\) coefficients were calculated. They are found as
\[
\begin{align*}
H R_{ij} &= R^K_{kji} = R^K_{kji} + \frac{1}{2} (\partial_k (g^{ks} \nabla_s g_{ji}) - (\partial_j (g^{ks} \nabla_s g_{k}) + \frac{1}{2} g^{ks} (\Gamma^s_{ks} \nabla_j g_{ji} - \Gamma^s_{ji} \nabla_s g_{k})) \\
&\quad + \frac{1}{2} g^{ks} (\Gamma^s_{kj} \nabla_i g_{sj} - \Gamma^s_{kj} \nabla_i g_{sj}) + \frac{1}{4} g^{ks} g^{st} (\nabla_s g_{kj} \nabla_i g_{sj} - \nabla_s g_{kj} \nabla_i g_{sj}) \\
&\quad + (\partial_k (y^l \partial_l \Gamma^s_{ij} - y^l (\nabla_l g^{ks}) g_{sj} \Gamma^s_{kj}) - \frac{1}{2} g^{ks} y^l (\partial_l (\nabla_j g_{sj}) + (\partial_j (\nabla_i g_{ij}) + \partial_j (\nabla_i g_{ij}) \\
&\quad + \frac{1}{2} y^l (\nabla_j g^{ks}) (\nabla_i g_{sj}) + \frac{1}{2} g^{ks} (\nabla_j g_{sj}) + \frac{1}{2} g^{ks} (\nabla_j g_{sj}) - \partial_j (\Gamma^s_{kj} - \frac{1}{2} g^{ks} \nabla_s g_{k})) \\
&\quad + (\Gamma^s_{kj} - \frac{1}{2} g^{ks} \nabla_s g_{k}) (\nabla_j g_{sj} - \frac{1}{2} g^{ks} \nabla_s g_{k}) - (\Gamma^s_{kj} - \frac{1}{2} g^{ks} \nabla_s g_{k}) (\nabla_j g_{sj} - \frac{1}{2} g^{ks} \nabla_s g_{k})
\end{align*}
\]
\[ H_{R_{ij}} = R_{k ji}^K = R_{kji}^K + R_{kji}^I = 0 + 0 = 0 \]
\[ H_{R_{ij}} = R_{k ji}^K = R_{kji}^K + R_{kji}^I = 0 + 0 = 0 \]
\[ H_{R_{ij}} = R_{k ji}^K = R_{kji}^K + R_{kji}^I = 0 + 0 = 0. \]

When \( \nabla g = 0 \),
\[ H_{R_{ij}} = R_{k ji}^K + \partial_k \left( g^{jk} \partial_l g_{lj} \right) - \partial_j (g^{ki} \nabla g_{lj}) + \frac{1}{2} g^{jk} (\Gamma^k_{lj} \nabla g_{lj} - \Gamma^k_{lj} \nabla g_{jk}) + \partial_j (g^{ki} \nabla g_{lj}) - \partial_i (g^{kl} \nabla g_{jk}) + \frac{1}{2} g^{jk} (\nabla g_{lj} + \nabla g_{jk} - \nabla g_{kj} - \nabla g_{jk}) = 0 \]
\[ H_{R_{ij}} = H_{R_{ij}} - H_{R_{ij}} = 0. \]

### 4 Analysis of \( S = H g^{HH} R_{ij} \) scaler curvature on tangent bundle

Using \( H_{R_{ij}} \) coefficients and \( H g^{HH} = \left( g^{ab} - (\Gamma^a_j g^h + \Gamma^a_i g^k) \right) \) equation, the coefficients of \( S = H g^{HH} R_{ij} \) were calculated. They are found as,

\[ H g^{ijh} R_{ij} = 0, (R_{kji}^K + \frac{1}{2} (\partial_k (g^{kl} \nabla g_{lj}) - \partial_l (g^{kj} \nabla g_{lk})) + \frac{1}{2} g^{jk} (\Gamma^k_{lj} \nabla g_{lj} - \Gamma^k_{lj} \nabla g_{jk}) + \partial_j (g^{ki} \nabla g_{lj}) - \partial_i (g^{kl} \nabla g_{jk}) + \frac{1}{2} g^{jk} (\nabla g_{lj} + \nabla g_{jk} - \nabla g_{kj} - \nabla g_{jk}) = 0 \]
\[ H g^{ijh} R_{ij} = g^{ih} .0 = 0 \]
\[ H g^{ijh} R_{ij} = g^{gh} .0 = 0 \]
\[ H g^{ijh} R_{ij} = - \left( g^{ih} g^{ij} + g^{kj} g^{gh} \right) .0 = 0 \]

### Theorem 3
Let \((M_n, g)\) be semi-Riemannian manifold. \( H R \) scaler curvature of \( \left( T \left( M_n, H g \right) \right) \) space is zero.

### Competing interests
The authors declare that they have no competing interests.

### Authors’ contributions
All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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