\(\alpha_{(\gamma,\gamma')}\)-semiregular, \(\alpha_{(\gamma,\gamma')}\)-\(\theta\)-semiopen Sets and \((\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})\)-\(\theta\)-semi Continuous Functions

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Abstract: The purpose of the present paper is to introduce and study two strong forms of \(\alpha_{(\gamma,\gamma')}\)-semiopen sets called \(\alpha_{(\gamma,\gamma')}\)-semiregular sets and \(\alpha_{(\gamma,\gamma')}\)-\(\theta\)-semiopen sets. And also introduce a new class of functions called \((\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})\)-\(\theta\)-semi continuous functions and obtain several properties of such functions.

Keywords: \(\alpha_{(\gamma,\gamma')}\)-open sets, \(\alpha_{(\gamma,\gamma')}\)-semiregular sets, \(\alpha_{(\gamma,\gamma')}\)-\(\theta\)-semiopen sets, \((\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})\)-\(\theta\)-semi continuous functions.

1 Introduction

Njastad \([4]\) defined \(\alpha\)-open sets in a space \(X\) and discussed many of its properties. Ibrahim \([3]\) introduced and discussed an operation on a topology \(\alpha O(X)\) into the power set \(P(X)\) and introduced \(\alpha_{(\gamma,\gamma')}\)-open sets in topological spaces and studied some of its basic properties \([1]\). And also in \([2]\) the author introduced the notion of \(\alpha_{(\gamma,\gamma')}\)-semiopen sets in a topological space and studied some of its properties. In this paper, the author introduce and study the notion of \(\alpha_{(\gamma,\gamma')}\)-\(\theta\)-semiclosed sets, and introduce \((\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')}\)-\(\theta\)-semi continuous functions and investigate some important properties.

2 Preliminaries

Throughout the present paper, \((X, \tau)\) and \((Y, \sigma)\) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. The closure and the interior of a subset \(A\) of \(X\) are denoted by \(\text{Cl}(A)\) and \(\text{Int}(A)\), respectively.

**Definition 1.** \([4]\) A subset \(A\) of a topological space \((X, \tau)\) is called \(\alpha\)-open, if \(A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))\).

The family of all \(\alpha\)-open sets in a topological space \((X, \tau)\) is denoted by \(\alpha O(X, \tau)\) (or \(\alpha O(X)\)).

**Definition 2.** \([3]\) Let \((X, \tau)\) be a topological space. An operation \(\gamma\) on the topology \(\alpha O(X)\) is a mapping from \(\alpha O(X)\) into the power set \(P(X)\) of \(X\) such that \(V \subseteq \gamma\gamma\) for each \(V \in \alpha O(X)\), where \(\gamma\gamma\) denotes the value of \(\gamma\) at \(V\). It is denoted by \(\gamma: \alpha O(X) \rightarrow P(X)\).

**Definition 3.** \([3]\) An operation \(\gamma\) on \(\alpha O(X, \tau)\) is said to be \(\alpha\)-regular if for every \(\alpha\)-open sets \(U\) and \(V\) containing \(x \in X\), there exists an \(\alpha\)-open set \(W\) of \(X\) containing \(x\) such that \(W\gamma \subseteq U\gamma \cap V\gamma\).

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\textbf{Definition 4.}[1] Let \((X, \tau)\) be a topological space and \(\gamma, \gamma'\) be operations on \(\alpha O(X, \tau)\). A subset \(A\) of \(X\) is said to be \(\alpha_{(\gamma, \gamma')}\)-open if for each \(x \in A\) there exist \(\alpha\)-open sets \(U\) and \(V\) of \(X\) containing \(x\) such that \(U \cap V \subseteq A\). A subset of \((X, \tau)\) is said to be \(\alpha_{(\gamma, \gamma')}\)-closed if its complement is \(\alpha_{(\gamma, \gamma')}\)-open. The family of all \(\alpha_{(\gamma, \gamma')}\)-open sets of \((X, \tau)\) is denoted by \(\alpha O(X, \tau)_{(\gamma, \gamma')}\).

**Proposition 1.** [1] Let \(\gamma\) and \(\gamma'\) be \(\alpha\)-regular operations. If \(A\) and \(B\) are \(\alpha_{(\gamma, \gamma')}\)-open, then \(A \cap B\) is \(\alpha_{(\gamma, \gamma')}\)-open.

\textbf{Definition 5.}[2] A subset \(A\) of \(X\) is said to be \(\alpha_{(\gamma, \gamma')}\)-semiopen, if there exists an \(\alpha_{(\gamma, \gamma')}\)-open set \(U\) of \(X\) such that \(U \subseteq A \subseteq \alpha_{(\gamma, \gamma')}\text{Cl}(U)\). A subset \(A\) of \(X\) is \(\alpha_{(\gamma, \gamma')}\)-semiclosed if and only if \(X \setminus A\) is \(\alpha_{(\gamma, \gamma')}\)-semiopen.

The family of all \(\alpha_{(\gamma, \gamma')}\)-semiopen sets of a topological space \((X, \tau)\) is denoted by \(\alpha SO(X, \tau)_{(\gamma, \gamma')}\), the family of all \(\alpha_{(\gamma, \gamma')}\)-semiclosed sets of \((X, \tau)\) containing \(x\) is denoted by \(\alpha SO(X, x)_{(\gamma, \gamma')}\). Also the family of all \(\alpha_{(\gamma, \gamma')}\)-semiclosed sets of a topological space \((X, \tau)\) is denoted by \(\alpha SC(X, \tau)_{(\gamma, \gamma')}\).

**Definition 6.** Let \(A\) be a subset of a topological space \((X, \tau)\). Then:

1. \(\alpha_{(\gamma, \gamma')}\text{-Cl}(A) = \cap \{F : F\) is \(\alpha_{(\gamma, \gamma')}\)-closed and \(A \subseteq F\} \Gamma[1]\).
2. \(\alpha_{(\gamma, \gamma')}\text{-Int}(A) = \cup \{U : U\) is \(\alpha_{(\gamma, \gamma')}\)-open and \(U \subseteq A\} \Gamma[1]\).
3. \(\alpha_{(\gamma, \gamma')}\text{-sCl}(A) = \cap \{F : F\) is \(\alpha_{(\gamma, \gamma')}\)-semiclosed and \(A \subseteq F\} \Gamma[2]\).
4. \(\alpha_{(\gamma, \gamma')}\text{-sInt}(A) = \cup \{U : U\) is \(\alpha_{(\gamma, \gamma')}\)-semiopen and \(U \subseteq A\} \Gamma[2]\).

\section{\(\alpha_{(\gamma, \gamma')}\)-semiregular Sets and \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-semiopen Sets}

\textbf{Definition 7.} A subset \(A\) of a topological space \((X, \tau)\) is said to be \(\alpha_{(\gamma, \gamma')}\)-semiregular, if it is both \(\alpha_{(\gamma, \gamma')}\)-semiopen and \(\alpha_{(\gamma, \gamma')}\)-semiclosed.

The family of all \(\alpha_{(\gamma, \gamma')}\)-semiregular sets in \(X\) is denoted by \(\alpha SR(X)_{(\gamma, \gamma')}\).

\textbf{Lemma 1.} The following properties hold for a subset \(A\) of a topological space \((X, \tau)\):

1. If \(A \in \alpha SO(X)_{(\gamma, \gamma')}\), then \(\alpha_{(\gamma, \gamma')}\text{-sCl}(A) \in \alpha SR(X)_{(\gamma, \gamma')}\).
2. If \(A \in \alpha SC(X)_{(\gamma, \gamma')}\), then \(\alpha_{(\gamma, \gamma')}\text{-sInt}(A) \in \alpha SR(X)_{(\gamma, \gamma')}\).

\textbf{Proof.} (1) Since \(\alpha_{(\gamma, \gamma')}\text{-sCl}(A)\) is \(\alpha_{(\gamma, \gamma')}\)-semiclosed, we have to show that \(\alpha_{(\gamma, \gamma')}\text{-sCl}(A) \in \alpha SO(X)_{(\gamma, \gamma')}\). Since \(A \in \alpha SO(X)_{(\gamma, \gamma')}\), then for \(\alpha_{(\gamma, \gamma')}\)-open set \(U\) of \(X\), \(U \subseteq A \subseteq \alpha_{(\gamma, \gamma')}\text{Cl}(U)\). Therefore we have,

\[
U \subseteq \alpha_{(\gamma, \gamma')}\text{-sCl}(A) \subseteq \alpha_{(\gamma, \gamma')}\text{-Cl}(A) \subseteq \alpha_{(\gamma, \gamma')}\text{-Cl}(\alpha_{(\gamma, \gamma')}\text{-Cl}(U)) = \alpha_{(\gamma, \gamma')}\text{-Cl}(U)
\]

This follows from (1).

\textbf{Definition 8.} A point \(x \in X\) is said to be \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-semiadherent point of a subset \(A\) of \(X\) if \(\alpha_{(\gamma, \gamma')}\text{-sCl}(U) \cap A \neq \emptyset\) for every \(\alpha_{(\gamma, \gamma')}\)-semiopen set \(U\) containing \(x\). The set of all \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-semiadherent points of \(A\) is called the \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-semiclosure of \(A\) and is denoted by \(\alpha_{(\gamma, \gamma')}\text{-sCl}_{\theta}(A)\). A subset \(A\) is called \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-semiclosed if \(\alpha_{(\gamma, \gamma')}\text{-sCl}_{\theta}(A) = A\). A subset \(A\) is called \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-semiopen if and only if \(X \setminus A\) is \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-semiclosed.

\textbf{Definition 9.} A point \(x \in X\) is said to be \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-adherent point of a subset \(A\) of \(X\) if \(\alpha_{(\gamma, \gamma')}\text{-Cl}(U) \cap A \neq \emptyset\) for every \(\alpha_{(\gamma, \gamma')}\)-open set \(U\) containing \(x\). The set of all \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-adherent points of \(A\) is called the \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-closure of \(A\) and is denoted by \(\alpha_{(\gamma, \gamma')}\text{-Cl}_{\theta}(A)\). A subset \(A\) is called \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-closed if \(\alpha_{(\gamma, \gamma')}\text{-Cl}_{\theta}(A) = A\). The complement of an \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-closed set is called an \(\alpha_{(\gamma, \gamma')}\)-\(\theta\)-open set.

\textbf{Corollary 1.} Let \(x \in X\) and \(A \subseteq X\). If \(x \in \alpha_{(\gamma, \gamma')}\text{-sCl}_{\theta}(A)\), then \(x \in \alpha_{(\gamma, \gamma')}\text{-Cl}_{\theta}(A)\).
Proof. Let \( x \in \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \). Then, \( \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(U) \cap A \neq \emptyset \) for every \( \alpha_{(\gamma',\gamma)}-\text{semiopen} \) set \( U \) containing \( x \). Since \( \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \subseteq \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(U) \), we have \( \emptyset \neq \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \cap A \subseteq \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \cap A \). Hence, \( \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \cap A \neq \emptyset \) for every \( \alpha_{(\gamma',\gamma)}-\text{open} \) set \( U \) containing \( x \). Therefore, \( x \in \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \).

Lemma 2. The following properties hold for a subset \( A \) of a topological space \((X, \tau)\):

1. If \( A \in \alpha\text{SO}(X)_{(\gamma',\gamma)} \), then \( \alpha_{(\gamma',\gamma)}-\text{Cl}(A) = \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \).
2. If \( A \in \alpha\text{SR}(X)_{(\gamma',\gamma)} \) if and only if \( A \) is both \( \alpha_{(\gamma',\gamma)}-\text{theta-closed} \) and \( \alpha_{(\gamma',\gamma)}-\text{theta-semiopen} \).
3. If \( A \in \alpha\text{O}(X)_{(\gamma',\gamma)} \), then \( \alpha_{(\gamma',\gamma)}-\text{Cl}(A) = \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \).

Proof. (1) Clearly \( \alpha_{(\gamma',\gamma)}-\text{Cl}(A) \subseteq \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \). Suppose that \( x \notin \alpha_{(\gamma',\gamma)}-\text{Cl}(A) \). Then, for some \( \alpha_{(\gamma',\gamma)}-\text{semiopen} \) set \( U \) containing \( x \), \( A \cap U = \emptyset \) and hence \( A \cap \alpha_{(\gamma',\gamma)}-\text{Cl}(U) = \emptyset \), since \( A \in \alpha\text{SO}(X)_{(\gamma',\gamma)} \). This shows that \( x \notin \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \).

(2) Let \( A \in \alpha\text{SR}(X)_{(\gamma',\gamma)} \). Then, \( A \in \alpha\text{SO}(X)_{(\gamma',\gamma)} \), by (1), we have \( A = \alpha_{(\gamma',\gamma)}-\text{Cl}(A) = \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \). Therefore, \( A \) is \( \alpha_{(\gamma',\gamma)}-\text{theta-closed} \). Since \( X \setminus A \in \alpha\text{SR}(X)_{(\gamma',\gamma)} \), by the argument above, \( X \setminus A \) is \( \alpha_{(\gamma',\gamma)}-\text{theta-closed} \) and hence \( A \) is \( \alpha_{(\gamma',\gamma)}-\text{theta-semiopen} \). The converse is obvious.

(3) This similar to (1).

Theorem 1. Let \((X, \tau)\) be a topological space and \( A \subseteq X \). Then, \( A \) is \( \alpha_{(\gamma',\gamma)}-\text{theta-semiopen} \) in \( X \) if and only if for each \( x \in A \) there exists \( U \in \alpha\text{SO}(X,x)_{(\gamma',\gamma)} \) such that \( \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \cap (X \setminus A) = \emptyset \).

Proof. Let \( A \) be \( \alpha_{(\gamma',\gamma)}-\text{theta-semiopen} \) and \( x \in A \). Then, \( X \setminus A \) is \( \alpha_{(\gamma',\gamma)}-\text{theta-closed} \) and \( X \setminus A = \alpha_{(\gamma',\gamma)}-\text{Cl}(X \setminus A) \). Hence, \( x \notin \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(X \setminus A) \). Therefore, there exists \( U \in \alpha\text{SO}(X,x)_{(\gamma',\gamma)} \) such that \( \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \cap (X \setminus A) = \emptyset \). Hence, \( A = \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(X \setminus A) \) and \( A \) is \( \alpha_{(\gamma',\gamma)}-\text{theta-semiopen} \).

Conversely, let \( A \subseteq X \) and \( x \in A \). From hypothesis, there exists \( U \in \alpha\text{SO}(X,x)_{(\gamma',\gamma)} \) such that \( \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \subseteq A \). Therefore, \( \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \cap (X \setminus A) = \emptyset \). Hence, \( X \setminus A = \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(X \setminus A) \) and \( A \) is \( \alpha_{(\gamma',\gamma)}-\text{theta-semiopen} \).

Theorem 2. For a subset \( A \) of a topological space \((X, \tau)\), we have \( \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) = \cap\{V : A \subseteq V \in \alpha\text{SR}(X)_{(\gamma',\gamma)}\} \).

Proof. Let \( x \notin \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \). Then, there exists an \( \alpha_{(\gamma',\gamma)}-\text{semiopen} \) set \( U \) containing \( x \) such that \( \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \cap A = \emptyset \). Then \( X \setminus \alpha_{(\gamma',\gamma)}-\text{Cl}(U) = V \) (say). Thus \( V \in \alpha\text{SR}(X)_{(\gamma',\gamma)} \), such that \( x \notin V \). Hence \( x \notin \cap\{V : A \subseteq V \in \alpha\text{SR}(X)_{(\gamma',\gamma)}\} \). Again, if \( x \notin \cap\{V : A \subseteq V \in \alpha\text{SR}(X)_{(\gamma',\gamma)}\} \), then there exists \( V \in \alpha\text{SR}(X)_{(\gamma',\gamma)} \) containing \( A \) such that \( x \notin V \). Then \( X \setminus V = (X \setminus U) \) (say) is an \( \alpha_{(\gamma',\gamma)}-\text{semiopen} \) set containing \( x \) such that \( \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \cap V = \emptyset \). This shows that \( \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \cap A = \emptyset \), so \( x \notin \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \).

Corollary 2. A subset \( A \) of \( X \) is \( \alpha_{(\gamma',\gamma)}-\text{theta-closed} \) if and only if \( A = \cap\{V : A \subseteq V \in \alpha\text{SR}(X)_{(\gamma',\gamma)}\} \).

Proof. Obvious.

Theorem 3. Let \( A \) and \( B \) be any two subsets of a space \( X \). Then, the following properties hold:

1. \( x \in \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \) if and only if \( U \cap A \neq \emptyset \) for each \( U \in \alpha\text{SR}(X)_{(\gamma',\gamma)} \) containing \( x \).
2. If \( A \subseteq B \), then \( \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \subseteq \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(B) \).

Proof. Clear.

Theorem 4. For any subset \( A \) of \( X \), \( \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(\alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A)) = \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \).

Proof. Obviously, \( \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \subseteq \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(\alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A)) \). Now, let \( x \in \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(\alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A)) \). Since \( \alpha_{(\gamma',\gamma)}-\text{Cl}(U) \in \alpha\text{SO}(X,x)_{(\gamma',\gamma)} \), then \( \alpha_{(\gamma',\gamma)}-\text{Cl}(\alpha_{(\gamma',\gamma)}-\text{Cl}(U)) \cap A = \emptyset \). Thus, \( x \notin \alpha_{(\gamma',\gamma)}-\text{Cl}_{0}(A) \).
Corollary 3. For any \( A \subseteq X \), \( \alpha_{(Y')}\text{-sCl}(A) \) is \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \).

Proof. Obvious.

Theorem 5. Intersection of arbitrary collection of \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \) sets in \( X \) is \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \).

Proof. Let \( \{A_i : i \in I\} \) be any collection of \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \) sets in a topological space \((X, \tau)\) and \( A = \bigcap_{i \in I} A_i \). Now, using Definition 8, \( x \in \alpha_{(Y')}\text{-sCl}(A) \), in consequence, \( x \in \alpha_{(Y')}\text{-sCl}(A_i) \) for all \( i \in I \). Therefore, \( x \in A_i \) for all \( i \in I \).

Corollary 4. For any \( A \subseteq X \), \( \alpha_{(Y')}\text{-sCl}(A) \) is the intersection of all \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \) sets each containing \( A \).

Proof. Obvious.

Corollary 5. Let \( A \) and \( A_i \) \((i \in I)\) be any subsets of a space \( X \). Then, the following properties hold:

1. \( A \) is \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \) in \( X \) if and only if for each \( x \in A \) there exists \( U \in \alpha\text{SR}(X)_{(Y')} \) such that \( x \in U \subseteq A \).
2. If \( A_i \) is \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \) in \( X \) for each \( i \in I \), then \( \bigcup_{i \in I} A_i \) is \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \) in \( X \).

Proof. Obvious.

Remark. The following example shows that the union of \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \) sets may fail to be \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \).

Example 1. Let \( X = \{1, 2, 3\} \) and \( \tau = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, X\} \) be a topology on \( X \). For each \( A \in \alpha\text{O}(X, \tau) \), we define two operations \( \gamma \) and \( \gamma' \), respectively, by \( A^\gamma = \text{Int}(\text{Cl}(A)) \) and \( A^{\gamma'} = \begin{cases} X, & \text{if } A = \{1, 3\} \\ A, & \text{if } A \neq \{1, 3\}. \end{cases} \)

Then, the subsets \( A = \{1\} \) and \( B = \{3\} \) are \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \), but their union \( \{1, 3\} = A \cup B \) is not \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \).

Example 2. Let \( X = \{1, 2, 3\} \) and \( \tau = \{\emptyset, \{1\}, \{3\}, \{1, 3\}, X\} \) be a topology on \( X \). For each \( A \in \alpha\text{O}(X, \tau) \), we define two operations \( \gamma \) and \( \gamma' \), respectively, by \( A^\gamma = \begin{cases} A, & \text{if } A \neq \{1, 3\} \\ X, & \text{if } A = \{1, 3\}. \end{cases} \)

and \( A^{\gamma'} = A \). The subsets \( \{2\} \) is \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \), but not \( \alpha_{(Y')}\text{-}\text{semiirregular} \).

Remark. From Lemma 2 (2), we have \( \alpha_{(Y')}\text{-}\text{semiirregular} \) set is \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \) set. In the above example, \( \{2\} \) is \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \), but not \( \alpha_{(Y')}\text{-}\text{semiirregular} \).

Example 3. Let \( X = \{1, 2, 3\} \) and \( \tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, X\} \) be a topology on \( X \). For each \( A \in \alpha\text{O}(X, \tau) \), we define two operations \( \gamma \) and \( \gamma' \), respectively, by \( A^\gamma = \begin{cases} A, & \text{if } A \neq \{1\} \\ X, & \text{if } A = \{1\}. \end{cases} \)

and \( A^{\gamma'} = \begin{cases} A, & \text{if } A \neq \{2\} \\ X, & \text{if } A = \{2\}. \end{cases} \)

Then, \( \{1, 2\} \) is \( \alpha_{(Y')}\text{-}\text{semiopen} \) set but not an \( \alpha_{(Y')}\text{-}\theta\text{-semiopen} \) set.
Throughout this section, let \( f : (X, \tau) \to (Y, \sigma) \) be a function and \( \gamma', \gamma : \alpha O(X, \tau) \to P(X) \) be operations on \( \alpha O(X, \tau) \) and \( \beta, \beta' : \alpha O(Y, \sigma) \to P(Y) \) be operations on \( \alpha O(Y, \sigma) \).

**Definition 10.** A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \( (\alpha_{(\gamma', \gamma)}, \alpha_{(\beta, \beta')}) \)-\( \theta \)-semi continuous if for each point \( x \in X \) and each \( \alpha_{(\beta, \beta')}. \)semiopen set \( V \) of \( Y \) containing \( f(x) \), there exists an \( \alpha_{(\gamma', \gamma)} \)-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq \alpha_{(\beta, \beta')}. \)Cl\((V)\).

**Example 4.** Let \( X = \{r, m, n\}, Y = \{1, 2, 3\} \), \( \tau = \{\phi, \{r\}, \{m\}, \{r, m\}, \{r, n\}, X\} \) and \( \sigma = \{\phi, \{3\}, \{1, 2\}, Y\} \). For each \( A \in \alpha O(X, \tau) \) and \( B \in \alpha O(Y, \sigma) \), we define the operations \( \gamma : \alpha O(X, \tau) \to P(X) \), \( \gamma' : \alpha O(X, \tau) \to P(X) \), \( \beta : \alpha O(Y, \sigma) \to P(Y) \) and \( \beta' : \alpha O(Y, \sigma) \to P(Y) \), respectively, by

\[
A^\gamma = \begin{cases} A, & \text{if } n \notin A, \\ X, & \text{if } n \in A, \end{cases}
\]

\[
A'^\gamma = \begin{cases} A, & \text{if } m \in A \text{ or } A = \phi \\ A \cup \{m\}, & \text{if } m \notin A, \end{cases}
\]

\[
B^\beta = \begin{cases} Y, & \text{if } 2 \notin B \\ B, & \text{if } 2 \in B \text{ or } B = \phi, \end{cases}
\]

\[
B'^\beta = \begin{cases} Y, & \text{if } 1 \notin B \\ B, & \text{if } 1 \in B \text{ or } B = \phi. \end{cases}
\]

Define a function \( f : (X, \tau) \to (Y, \sigma) \) as follows:

\[
f(x) = \begin{cases} 1, & \text{if } x = r \\ 1, & \text{if } x = m \\ 3, & \text{if } x = n. \end{cases}
\]

Clearly, \( \alpha O(X, \tau)|_{(\gamma', \gamma)} = \{\phi, \{m\}, \{r, m\}, X\} \) and \( \alpha SO(Y, \sigma)|_{(\beta, \beta')} = \{\phi, \{1, 2\}, Y\} \). Then, \( f \) is \( (\alpha_{(\gamma', \gamma)}, \alpha_{(\beta, \beta')}) \)-\( \theta \)-semi continuous.

**Theorem 6.** The following statements are equivalent for a function \( f : (X, \tau) \to (Y, \sigma) \):

1. \( f \) is \( (\alpha_{(\gamma', \gamma)}, \alpha_{(\beta, \beta')}) \)-\( \theta \)-semi continuous.
2. For each \( x \in X \) and \( V \in \alpha \text{SR}(Y)|_{(\beta, \beta')} \) containing \( f(x) \), there exists an \( \alpha_{(\gamma', \gamma)} \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).
3. \( f^{-1}(V) \) is \( \alpha_{(\gamma', \gamma)} \)-clopen (That is, \( \alpha_{(\gamma', \gamma)} \)-open as well as \( \alpha_{(\gamma', \gamma)} \)-closed) in \( X \) for every \( V \in \alpha \text{SR}(Y)|_{(\beta, \beta')} \).
4. \( f^{-1}(V) \subseteq \alpha_{(\gamma', \gamma)} \cdot \text{Int}(f^{-1}(\alpha_{(\beta, \beta')}. \text{Cl}(V))) \) for every \( V \in \alpha \text{SO}(Y)|_{(\beta, \beta')} \).
5. \( \alpha_{(\gamma', \gamma)} \cdot \text{Cl}(f^{-1}(\alpha_{(\beta, \beta')}. \text{Int}(V))) \subseteq f^{-1}(V) \) for every \( \alpha_{(\beta, \beta')}. \text{semi closed} \text{ set } V \text{ of } Y \).
6. \( \alpha_{(\gamma', \gamma)} \cdot \text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\alpha_{(\beta, \beta')}. \text{Cl}(V)) \) for every \( V \in \alpha \text{SO}(Y)|_{(\beta, \beta')} \).
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**Proposition 2.** The following statements are equivalent for a function \(f: (X, \tau) \rightarrow (Y, \sigma)\):

1. \(f\) is \((\alpha_{(Y, \gamma)}, \alpha_{(\beta, \gamma)})\)-\(\theta\)-semi-continuous.
2. For each \(x \in X\) and \(V \in \alpha\text{SR}(Y)_{(\beta, \gamma)}\) containing \(f(x)\), there exists an \(\alpha_{(Y, \gamma)}\)-clopen set \(U\) containing \(x\) such that \(f(U) \subseteq \alpha_{(\beta, \gamma)}\text{-sCl}(V)\).
3. For each \(x \in X\) and \(V \in \alpha\text{SO}(Y)_{(\beta, \gamma)}\) containing \(f(x)\), there exists an \(\alpha_{(Y, \gamma)}\)-clopen set \(U\) containing \(x\) such that \(f(U) \subseteq \alpha_{(\beta, \gamma)}\text{-sCl}(V)\).

**Proof.** (1) \(\Rightarrow\) (2): Let \(x \in X\) and \(V \in \alpha\text{SR}(Y)_{(\beta, \gamma)}\) containing \(f(x)\). By (1), there exists an \(\alpha_{(Y, \gamma)}\)-open set \(U\) containing \(x\) such that \(f(U) \subseteq \alpha_{(\beta, \gamma)}\text{-sCl}(V) = V\).

(2) \(\Rightarrow\) (3): Let \(V \in \alpha\text{SR}(Y)_{(\beta, \gamma)}\) and \(x \in f^{-1}(V)\). Then, \(f(U) \subseteq V\) for some \(\alpha_{(Y, \gamma)}\)-open set \(U\) of \(X\) containing \(x\), hence \(x \in U \subseteq f^{-1}(V)\). This shows that \(f^{-1}(V)\) is \(\alpha_{(Y, \gamma)}\)-open in \(X\). Since \(Y \setminus V \in \alpha\text{SR}(Y)_{(\beta, \gamma)}\), \(f^{-1}(Y \setminus V)\) is also \(\alpha_{(Y, \gamma)}\)-open and hence \(f^{-1}(V)\) is \(\alpha_{(Y, \gamma)}\)-clopen in \(X\).

(3) \(\Rightarrow\) (4): Let \(V \in \alpha\text{SO}(Y)_{(\beta, \gamma)}\). Since \(V \subseteq \alpha_{(\beta, \gamma)}\text{-sCl}(V)\) and by Lemma 1, we have \(\alpha_{(\beta, \gamma)}\text{-sCl}(V) \in \alpha\text{SR}(Y)_{(\beta, \gamma)}\).

(4) \(\Rightarrow\) (5): Let \(V\) be an \(\alpha_{(\beta, \gamma)}\)-semiclosed subset of \(Y\). By (4), we have \(f^{-1}(Y \setminus V) \subseteq \alpha_{(Y, \gamma)}\text{-Int}(f^{-1}(\alpha_{(\beta, \gamma)}\text{-sCl}(V)))\).

(5) \(\Rightarrow\) (6): Let \(V \in \alpha\text{SO}(Y)_{(\beta, \gamma)}\). By Lemma 1, \(\alpha_{(\beta, \gamma)}\text{-sCl}(V) \in \alpha\text{SR}(Y)_{(\beta, \gamma)}\). By (5), we obtain \(\alpha_{(Y, \gamma)}\text{-Cl}(f^{-1}(V)) \subseteq \alpha_{(Y, \gamma)}\text{-Cl}(f^{-1}(\alpha_{(\beta, \gamma)}\text{-sCl}(V))) = \alpha_{(Y, \gamma)}\text{-Cl}(f^{-1}(\alpha_{(\beta, \gamma)}\text{-sInt}(V)))\).

(6) \(\Rightarrow\) (1): Let \(x \in X\) and \(V \in \alpha\text{SO}(Y, f(x))_{(\beta, \gamma)}\). By Lemma 1, we have \(\alpha_{(\beta, \gamma)}\text{-sCl}(V) \in \alpha\text{SR}(Y)_{(\beta, \gamma)}\) and \(f(x) \notin Y \setminus \alpha_{(\beta, \gamma)}\text{-sCl}(V)\).

**Theorem 7.** The following statements are equivalent for a function \(f: (X, \tau) \rightarrow (Y, \sigma)\):

1. \(f\) is \((\alpha_{(Y, \gamma)}, \alpha_{(\beta, \gamma)})\)-\(\theta\)-semi-continuous.
2. \(\alpha_{(Y, \gamma)}\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\alpha_{(\beta, \gamma)}\text{-sCl}(B))\) for every subset \(B\) of \(Y\).
3. \(f(\alpha_{(Y, \gamma)}\text{-Cl}(A)) \subseteq \alpha_{(\beta, \gamma)}\text{-sCl}(f(A))\) for every subset \(A\) of \(X\).

(4) \(\Rightarrow\) (5): \(f^{-1}(F)\) is \(\alpha_{(Y, \gamma)}\)-closed in \(X\) for every \(\alpha_{(\beta, \gamma)}\)-\(\theta\)-semiclosed set \(F\) of \(Y\).

(5) \(\Rightarrow\) (1): \(f\) is \((\alpha_{(Y, \gamma)}, \alpha_{(\beta, \gamma)})\)-\(\theta\)-semi-continuous.

**Proof.** (1) \(\Rightarrow\) (2): Let \(B\) be any subset of \(Y\) and \(x \notin f^{-1}(\alpha_{(\beta, \gamma)}\text{-sCl}(B))\). Then, \(f(x) \notin \alpha_{(\beta, \gamma)}\text{-sCl}(B)\) and there exists \(V \in \alpha\text{SO}(Y, f(x))_{(\beta, \gamma)}\) such that \(\alpha_{(\beta, \gamma)}\text{-sCl}(V) \cap B = \emptyset\). By (1), there exists an \(\alpha_{(Y, \gamma)}\)-open set \(U\) containing \(x\) such that \(f(U) \subseteq \alpha_{(\beta, \gamma)}\text{-sCl}(V)\). Hence, \(f(U) \cap B = \emptyset\) and \(U \cap f^{-1}(B) = \emptyset\). Consequently, we obtain \(x \notin \alpha_{(Y, \gamma)}\text{-Cl}(f^{-1}(B))\).

(2) \(\Rightarrow\) (3): Let \(A\) be any subset of \(X\). By (2), we have \(\alpha_{(Y, \gamma)}\text{-Cl}(A) \subseteq \alpha_{(Y, \gamma)}\text{-Cl}(f^{-1}(f(A))) \subseteq f^{-1}(\alpha_{(\beta, \gamma)}\text{-sCl}(f(A)))\) and hence \(f(\alpha_{(Y, \gamma)}\text{-Cl}(A)) \subseteq \alpha_{(\beta, \gamma)}\text{-sCl}(f(A))\).
Proposition 3. The following statements are equivalent for a function \( f : (X, \tau) \to (Y, \sigma) \):

1. \( f \) is \((\alpha_{(Y, \gamma)}, \alpha)(\beta, \beta')\)-\(\theta\)-semi continuous.
2. \( \alpha_{(Y, \gamma)} - \text{Cl}_\theta(f^{-1}(f(A))) \subseteq f^{-1}(\alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(A))) \) for every subset \( A \) in \( X \).
3. \( f_{\alpha_{(Y, \gamma)}}(\text{Cl}_\theta(\alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(A)))) \subseteq f^{-1}(\alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(A))) \) for every \( \alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(A))) \) in \( Y \).
4. \((\beta, \beta')\)-closed in \( X \) for every \( \alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(A))) \) in \( Y \).
5. \( f^{-1}(V) \) is \((\alpha_{(Y, \gamma)}, \alpha)(\beta, \beta')\)-\(\theta\)-semi open in \( X \) for every \( \alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(A))) \) in \( Y \).

Proof. (1) \(\Rightarrow\) (2): Let \( B \) be any subset of \( Y \) and \( x \in f^{-1}(\alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(B))) \). Then, \( f(x) \notin \alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(B)) \) and there exists \( V \in \alpha SO(Y, f(x)) \) such that \( \alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(V)) \) and \( B = \phi \). By Proposition 2 (3), there exists an \( \alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(B))) \) in \( Y \).

(2) \(\Rightarrow\) (3): Let \( A \) be any subset of \( X \). By (2), we have \( f(A) \subseteq f^{-1}(\alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(A))) \) and hence \( f(\alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(A))) \subseteq f^{-1}(\alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(A))). \) Consequently, we obtain \( x \notin \alpha_{(Y, \gamma)} - \text{Cl}_\theta(f^{-1}(f(B))) \).

(3) \(\Rightarrow\) (4): Let \( B \) be any subset of \( Y \). Then, by (3), we have \( f(\alpha_{(Y, \gamma)} - \text{Cl}_\theta(f^{-1}(f(B))) \subseteq f^{-1}(\alpha_{(Y, \gamma)} - \text{Cl}_\theta(f^{-1}(f(B))) \subseteq f^{-1}(\alpha_{(Y, \gamma)} - \text{Cl}_\theta(f(B))) \).

(4) \(\Rightarrow\) (5): Obvious.

Proposition 4. The following statements are equivalent for a function \( f : (X, \tau) \to (Y, \sigma) \):

1. \( f \) is \((\alpha_{(Y, \gamma)}, \alpha)(\beta, \beta')\)-\(\theta\)-semi continuous.
2. \( \alpha_{(Y, \gamma)} - Cl(f^{-1}(\alpha_{(Y, \gamma)} - sInt(\alpha_{(Y, \gamma)} - Cl(B)))) \subseteq f^{-1}(\alpha_{(Y, \gamma)} - sInt(\alpha_{(Y, \gamma)} - Cl(B))) \), for every subset \( B \) in \( Y \).
3. \( f^{-1}(\alpha_{(Y, \gamma)} - sInt(\alpha_{(Y, \gamma)} - Cl(B))) \subseteq \alpha_{(Y, \gamma)} - Cl(f^{-1}(\alpha_{(Y, \gamma)} - sInt(\alpha_{(Y, \gamma)} - Cl(B))) \), for every subset \( B \) in \( Y \).

Proof. (1) \(\Rightarrow\) (2): Let \( B \) be any subset of \( Y \). Then, \( \alpha_{(Y, \gamma)} - Cl(B) \) is \((\beta, \beta')\)-semi closed in \( Y \) and by Theorem 6 (5), \( \alpha_{(Y, \gamma)} - Cl(Cl^{-1}(\alpha_{(Y, \gamma)} - Cl(B))) \subseteq f^{-1}(\alpha_{(Y, \gamma)} - Cl(B)). \)

(2) \(\Rightarrow\) (3): Let \( B \) be any subset of \( Y \) and \( x \in f^{-1}(\alpha_{(Y, \gamma)} - sInt(B)) \). Then, we have \( x \notin f^{-1}(\alpha_{(Y, \gamma)} - sCl(Y \setminus B)) \) and \( (\alpha_{(Y, \gamma)} - Cl(Y \setminus B)) \) in \( Y \). So, \( x \notin f(Y \setminus B) \) and \( \alpha_{(Y, \gamma)} - Cl(Y \setminus B)) \) in \( Y \). Hence, \( f^{-1}(\alpha_{(Y, \gamma)} - sInt(B)) \subseteq f^{-1}(\alpha_{(Y, \gamma)} - Cl(Y \setminus B)) \).

(3) \(\Rightarrow\) (4): Let \( V \) be any \((\beta, \beta')\)-semi open set in \( Y \). Suppose that \( z \notin f^{-1}(\alpha_{(Y, \gamma)} - Cl(V)) \). Then, \( f(z) \notin \alpha_{(Y, \gamma)} - Cl(V) \) and there exists an \( \alpha_{(Y, \gamma)} - Cl(V) \) in \( Y \). Hence, \( \alpha_{(Y, \gamma)} - Cl(f^{-1}(\alpha_{(Y, \gamma)} - Cl(V))) \subseteq f^{-1}(\alpha_{(Y, \gamma)} - Cl(V)) \).

(4) \(\Rightarrow\) (5): Obvious.
(1) \( f \) is \((\alpha_{\gamma}, \alpha_{\beta'})\)-\(\theta\)-semi continuous.

(2) \( \alpha_{(Y', \gamma)} - \text{Cl}(f^{-1}(\alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(B))) \subseteq f^{-1}(\alpha_{\beta'} - \text{sCl}(B)), \) for every subset \( B \) of \( Y \).

(3) \( \alpha_{Y, Y'} - \text{Cl}(f^{-1}(\alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(B))) \subseteq f^{-1}(\alpha_{\beta'} - \text{sCl}(B)), \) for every subset \( B \) of \( Y \).

(4) \( \alpha_{Y, Y'} - \text{Cl}(f^{-1}(\alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(O))) \subseteq f^{-1}(\alpha_{\beta'} - \text{sCl}(O)), \) for every \( \alpha_{\beta'} - \text{semiopen set} \) \( O \) of \( Y \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( B \) be any subset of \( Y \). Then, \( \alpha_{(\beta, \beta')} - \text{Cl}(B) \) is \( \alpha_{(\beta, \beta')} - \text{semiopen} \) in \( Y \). Then by Theorem 6 (5), if \( x \in \alpha_{(Y', \gamma)} - \text{Cl}(f^{-1}(\alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(B)))), \) then \( x \in f^{-1}(\alpha_{\beta'} - \text{sCl}(B)) \).

(2) \( \Rightarrow \) (3): This is obvious since \( \alpha_{\beta'} - \text{Cl}(B) \subseteq \alpha_{\beta'} - \text{sCl}(B) \) for every subset \( B \).

(3) \( \Rightarrow \) (4): By Lemma 2 (1), we have \( \alpha_{\beta'} - \text{sCl}(O) = \alpha_{\beta'} - \text{sCl}(O) \) for every \( \alpha_{\beta'} - \text{semiopen set} \) \( O \).

(4) \( \Rightarrow \) (1): Let \( V \) be any \( \alpha_{\beta'} - \text{semiopen} \) set in \( Y \) and \( x \in \alpha_{(Y', \gamma)} - \text{Cl}(f^{-1}(V)) \). Then, \( V \) is \( \alpha_{\beta'} - \text{semiopen} \) and \( x \in \alpha_{(Y', \gamma)} - \text{Cl}(f^{-1}(\alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(V)))) \).

It follows from Theorem 6, that \( f \) is \((\alpha_{(Y', \gamma)}, \alpha_{\beta'})\)-\(\theta\)-semi continuous.

**Corollary 6.** A function \( f : (X, \tau) \to (Y, \sigma) \) is \((\alpha_{(Y', \gamma)}, \alpha_{\beta'})\)-\(\theta\)-semi continuous if and only if \( f^{-1}(\alpha_{\beta'} - \text{sCl}(V)) \) is \( \alpha_{(Y', \gamma)} - \text{open} \) set in \( X \), for each \( \alpha_{\beta'} - \text{semiopen set} \) \( V \) in \( Y \).

**Proof.** Since \( V \in \alpha\text{SO}(Y)_{\beta'} \), then, \( \alpha_{\beta'} - \text{Cl}(V) \in \alpha\text{SR}(Y)_{\beta'} \), so by Theorem 6 (3), \( f^{-1}(\alpha_{\beta'} - \text{sCl}(V)) \) is \( \alpha_{(Y', \gamma)} - \text{open} \), which is \( \alpha_{\beta'} - \text{open} \).

Conversely, if \( V \in \alpha\text{SO}(Y)_{\beta'} \), then by hypothesis, \( f^{-1}(V) \subseteq f^{-1}(\alpha_{\beta'} - \text{sCl}(V)) = \alpha_{(Y', \gamma)} - \text{Int}(f^{-1}(\alpha_{\beta'} - \text{sCl}(V))) \), so by Theorem 6 (4), \( f \) is \((\alpha_{(Y', \gamma)}, \alpha_{\beta'})\)-\(\theta\)-semi continuous.

**Corollary 7.** A function \( f : (X, \tau) \to (Y, \sigma) \) is \((\alpha_{(Y', \gamma)}, \alpha_{\beta'})\)-\(\theta\)-semi continuous if and only if \( f^{-1}(\alpha_{\beta'} - \text{sInt}(F)) \) is an \( \alpha_{(Y', \gamma)} - \text{closed} \) set in \( X \), for each \( \alpha_{\beta'} - \text{semiopen} \) set \( F \) of \( Y \).

**Proof.** It follows from Corollary 6.

**Corollary 8.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a function. If \( f^{-1}(\alpha_{\beta'} - \text{sCl}(B)) \) is \( \alpha_{(Y', \gamma)} - \text{closed} \) in \( X \) for every subset \( B \) of \( Y \), then \( f \) is \((\alpha_{(Y', \gamma)}, \alpha_{\beta'})\)-\(\theta\)-semi continuous.

**Proof.** Let \( B \subseteq Y \). Since \( f^{-1}(\alpha_{\beta'} - \text{sCl}(B)) \) is \( \alpha_{(Y', \gamma)} - \text{closed} \) in \( X \), then \( \alpha_{(Y', \gamma)} - \text{Cl}(f^{-1}(B)) \subseteq \alpha_{(Y', \gamma)} - \text{Cl}(f^{-1}(\alpha_{\beta'} - \text{sCl}(B))) = f^{-1}(\alpha_{\beta'} - \text{sCl}(B)) \). By Theorem 7, \( f \) is \((\alpha_{(Y', \gamma)}, \alpha_{\beta'})\)-\(\theta\)-semi continuous.

**Proposition 6.** The following statements are equivalent for a function \( f : (X, \tau) \to (Y, \sigma) \):

1. \( f \) is \((\alpha_{(Y', \gamma)}, \alpha_{\beta'})\)-\(\theta\)-semi continuous.
2. \( \alpha_{Y, Y'} - \text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(V))), \) for every \( V \subseteq \alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(V)). \)
3. \( f^{-1}(V) \subseteq \alpha_{(Y', \gamma)} - \text{Int}(f^{-1}(\alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(V))), \) for every \( V \subseteq \alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(V)). \)

**Proof.** (1) \( \Rightarrow \) (2): Let \( V \subseteq \alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(V)) \) such that \( x \in \alpha_{(Y', \gamma)} - \text{Cl}(f^{-1}(V)) \). Suppose that \( x \notin f^{-1}(\alpha_{\beta'} - \text{sCl}(V)) \). Then there exists an \( \alpha_{\beta'} - \text{semiopen} \) set \( W \) containing \( f(x) \) such that \( W \cap V = \phi \). Hence, we have \( W \cap \alpha_{\beta'} - \text{Cl}(V) = \phi \) and hence \( \alpha_{\beta'} - \text{Cl}(W) \cap \alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(V)) = \phi \). Since \( V \subseteq \alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(V)) \) and we have \( V \cap \alpha_{\beta'} - \text{Cl}(W) = \phi \). Since \( f \) is \((\alpha_{(Y', \gamma)}, \alpha_{\beta'})\)-\(\theta\)-semi continuous at \( x \) in \( X \) and \( X \) is an \( \alpha_{\beta'} - \text{semiopen} \) set containing \( f(x) \), there exists \( U \in \alpha\text{O}(X)_{(Y', \gamma)} \) containing \( x \) such that \( f(U) \subseteq \alpha_{\beta'} - \text{Cl}(W) \). Then, \( f(U) \cap V = \phi \) and hence \( U \cap f^{-1}(V) = \phi \). This shows that \( x \notin \alpha_{(Y', \gamma)} - \text{Cl}(f^{-1}(V)) \). This is a contradiction. Therefore, we have \( x \in f^{-1}(\alpha_{\beta'} - \text{sCl}(V)) \).

(2) \( \Rightarrow \) (3): Let \( V \subseteq \alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(V)) \) and \( x \in f^{-1}(V) \). Then, we have \( f^{-1}(V) \subseteq f^{-1}(\alpha_{\beta'} - \text{Int}(\alpha_{\beta'} - \text{sCl}(V))) \subseteq X \setminus f^{-1}(\alpha_{\beta'} - \text{sCl}(Y \setminus \alpha_{\beta'} - \text{sCl}(V))) \). Therefore, \( x \notin f^{-1}(\alpha_{\beta'} - \text{Cl}(Y \setminus \alpha_{\beta'} - \text{sCl}(V))) \). Then by (2), \( x \notin \alpha_{(Y', \gamma)} - \text{Cl}(f^{-1}(Y \setminus \alpha_{\beta'} - \text{sCl}(V))). \) Hence, \( x \notin \alpha_{(Y', \gamma)} - \text{Int}(f^{-1}(\alpha_{\beta'} - \text{sCl}(V))). \)

(3) \( \Rightarrow \) (1): Let \( V \) be any \( \alpha_{\beta'} - \text{semiopen} \) set in \( Y \). Then, \( V = \alpha_{\beta'} - \text{Int}(V) \subseteq \alpha_{\beta'} - \text{sCl}(V) \). Hence, by (3) and Theorem 6, \( f \) is \((\alpha_{(Y', \gamma)}, \alpha_{\beta'})\)-\(\theta\)-semi continuous.
Proposition 7. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \((\alpha, \beta, \beta')\)-\(\Theta\)-semi continuous at \( x \in X \), then for each \( \alpha(\gamma, \gamma') \)-open set \( B \) containing \( f(x) \) and each \( \alpha(\gamma', \gamma) \)-open set \( A \) containing \( x \), there exists a nonempty \( \alpha(\gamma, \gamma') \)-open set \( U \subseteq A \) such that \( U \subseteq \alpha(\gamma, \gamma')-\text{Cl}(f^{-1}(\alpha(\beta, \beta')-sCl(B))) \). Where \( \gamma \) and \( \gamma' \) are \( \alpha \)-regular operations.

Proof. Let \( B \) be any \( \alpha(\beta, \beta') \)-semiopen set containing \( f(x) \) and \( A \) be an \( \alpha(\gamma', \gamma) \)-open set of \( X \) containing \( x \). By Lemma 1 and Theorem 6, \( x \in \alpha(\gamma, \gamma') \)-Int\( f^{-1}(\alpha(\beta, \beta')-sCl(B)) \), then \( A \cap \alpha(\gamma', \gamma) \)-Int\( f^{-1}(\alpha(\beta, \beta')-sCl(B)) \neq \emptyset \). Take \( U = A \cap \alpha(\gamma, \gamma') \)-Int\( f^{-1}(\alpha(\beta, \beta')-sCl(B)) \). Thus, \( U \) is a nonempty \( \alpha(\gamma', \gamma) \)-open set by Proposition 1, and hence \( U \subseteq A \) and \( U \subseteq \alpha(\gamma', \gamma) \)-Int\( f^{-1}(\alpha(\beta, \beta')-sCl(B)) \) \( \subseteq \alpha(\gamma, \gamma') \)-Cl\( f^{-1}(\alpha(\beta, \beta')-sCl(B)) \).

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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