A new type of generalized quasi-Einstein manifold

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Abstract: In this paper, a new type of generalized quasi-Einstein manifold is defined. The special cases of this manifold are Einstein manifold, quasi-Einstein manifold and nearly quasi-Einstein manifold. We have shown the existence of this new type of generalised quasi-Einstein manifold by a suitable example.

Keywords: Einstein manifold, quasi-Einstein manifold, nearly quasi-Einstein manifold, conformal curvature tensor, concircular curvature tensor, Einstein field equation.

1 Introduction

Let \((\mathcal{M}^n, g)\), \((n > 2)\), be an n-dimensional Riemannian manifold. A Riemannian manifold is said to be an Einstein manifold if a non-zero Ricci tensor of the manifold satisfies relation

\[ R_{ij} = \frac{R}{n} g_{ij}, \]

where \(R_{ij}\), \(R\) and \(g_{ij}\) are Ricci tensor of type \((0, 2)\), scalar curvature and Riemannian metric respectively.

If a non-zero Ricci tensor of the manifold satisfies relation

\[ R_{ij} = \alpha g_{ij} + \beta A_i A_j, \]

then, the manifold is called a quasi-Einstein manifold, where \(A_i\) is a unit covariant vector on \(U = \{x \in M : R_{ij} \neq \frac{R}{n} g_{ij}\}\) and \(\alpha, \beta\) are scalars on \(U\). Generally an n-dimensional quasi-Einstein manifold is denoted by \((QE)_n\). The quasi-Einstein manifold is also studied by U.C.De and G.C.Ghosh[13], C. Özgür and S. Sular[6] and A. A. Shaikh , D. W. Yoon and S. K. Hui[1] and [2,3,8,9]. According to U. C. De and A. K. Gazi [12], a manifold is said to be nearly quasi-Einstein manifold, if the non-zero Ricci tensor of the manifold satisfies the relation

\[ R_{ij} = \alpha g_{ij} + \beta E_{ij}, \]

where \(E_{ij}\) is a symmetric tensor of type \((0,2)\).


\[ W_{ijkl}^h = -(n-2)b_C_{ijkl}^h + [a + (n-2)b]L_{ijkl}^h. \]
where \( a, b \) are arbitrary constants, not simultaneously zero and \( C^h_{ijk}, L^h_{ijk} \) are conformal and concircular curvature tensor respectively.

From (4) we can say that the quasi-conformal curvature tensor will be equal to conformal curvature tensor or concircular curvature tensor according as \( a = 1 \) and \( b = -\frac{1}{n-2} \) or \( a = 1 \) and \( b = 0 \) respectively.

We know that the conformal and concircular curvature tensors are defined by

\[
C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2} (R_{ij} \delta^h_k - R_{ik} \delta^h_j + R^h_k g_{ji} - R^h_j g_{ki}) + \frac{R}{(n-1)(n-2)} (g_{ij} \delta^h_k - \delta^h_j g_{ki}),
\]

and

\[
L^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)} (g_{ij} \delta^h_k - \delta^h_j g_{ki}).
\]

Putting the values of conformal and concircular curvature tensors from (5) and (6) in (4) we have an expression for quasi-conformal curvature tensor \( W \) of type \((1,3)\), given by

\[
W^h_{ijk} = aR^h_{ijk} + b(R_{ij} \delta^h_k - R_{ik} \delta^h_j + R^h_k g_{ji} - R^h_j g_{ki}) - \frac{R(n-1) + 2b}{n} (g_{ij} \delta^h_k - \delta^h_j g_{ki}).
\]

2 A new type of generalized quasi-Einstein manifold

Now we define a manifold and called it new type of generalized quasi-Einstein manifold in which a non-zero Ricci tensor satisfies a different type of relation. Generalized in sense that special cases of this manifold are an Einstein manifold, quasi-Einstein manifold and nearly quasi-Einstein manifold.

**Definition 1.** Let \((M^n, g), (n \geq 2)\), be a Riemannian manifold. If the non-zero Ricci tensor of the manifold satisfies the relation

\[
R_{ij} = \alpha g_{ij} + \beta A_i A_j + [\alpha - (n-2)\gamma]E_{ij},
\]

then, the manifold is called a new type of generalized quasi-Einstein manifold, where \( R_{ij}, E_{ij} \) and \( \alpha, \beta, \gamma \) are Ricci tensor, symmetric tensor of type \((0, 2)\) and scalars respectively.

Transvecting (8) by \( g^{ij} \), we have

\[
R = \alpha n + \beta + [\alpha - (n-2)\gamma]E,
\]

where \( E = E_{ij} g^{ij} \) and \( A^l A_j = 1 \ (A^l = A_i g^{ij}) \). From (8) three cases arise.

**Case i.** if \( \beta = 0 \) and \( \alpha = (n-2)\gamma \) then from (8), we get

\[
R_{ij} = (n-2)\gamma g_{ij},
\]

then, manifold becomes Einstein manifold.

**Case ii.** if \( \alpha = (n-2)\gamma \) then from (8), we get

\[
R_{ij} = (n-2)\gamma g_{ij} + \beta A_i A_j,
\]

then, manifold becomes a quasi-Einstein manifold.
Definition 2. A Riemannian manifold \((M^n, g)\), \((n > 3)\) is said to be a manifold of quasi-constant curvature if it is conformally flat and its curvature tensor \(R^h_{ij}\) of type \((1,3)\) have the form

\[
R^h_{ij} = P[\delta^h_i A_j - \delta^h_j A_i + A_k A^h g_{ij} - A_j A^h g_{ik}] + Q(\delta^h_i E_j - \delta^h_j E_i) + E_k^h g_{ji} - E_j^h g_{ki} + (SE + KA + U)\delta^h_k - \delta^h_j g_{ki},
\]

where \(P = -\frac{b\beta}{a}\), \(Q = -\frac{b}{a}[\alpha - (n - 2)\gamma]\), \(S = \frac{1}{a}[\alpha - (n - 2)\gamma](\frac{a + 2(n - 1)b}{n(n - 1)})\), \(K = \frac{b}{a}\frac{a + 2(n - 1)b}{n(n - 1) + 2b}\), \(a \neq 0\) and \(U = \frac{a}{(n - 1)}\).

Now, if we take

\[
Q(\delta^h_i E_j - \delta^h_j E_i) + E_k^h g_{ji} - E_j^h g_{ki} + (SE + KA)\delta^h_k - \delta^h_j g_{ki} = 0,
\]

then from (16) and (15), we obtain

\[
R^h_{ij} = P[\delta^h_i A_j - \delta^h_j A_i + A_k A^h g_{ij} - A_j A^h g_{ik}] + U(\delta^h_k - \delta^h_j g_{ki}).
\]

In 1972 Chen and Yano [7] gave the concept of a manifold of quasi-constant curvature tensor and define.

**Theorem 1.** A quasi-conformally flat new type of generalized quasi-Einstein manifold will be a manifold of quasi-constant curvature if and only if

\[
\delta^h_i E_j - \delta^h_j E_i + E_k^h g_{ji} - E_j^h g_{ki} + (SE + KA)\delta^h_k - \delta^h_j g_{ki} = 0.
\]

Now, we propose.
Corollary 1. If a quasi-conformally flat new type of generalized quasi-Einstein manifold be a manifold of quasi-constant curvature then the symmetric tensor $E_{ij}$ satisfies the relation

$$E_{ij} = -\frac{K}{2Q + nS}A_iA_j,$$  \hspace{1cm} (19)

Proof. From Theorem 1. it is clear that if a quasi-conformally flat new type of generalized quasi-Einstein manifold be a manifold of quasi-constant curvature then

$$Q(\delta^h_k E_{ij} - \delta^h_j E_{ik} + \delta^h_i g_{kj} - \delta^h_j g_{ki}) + (SE + KA)(g_{ij} \delta^h_k - \delta^h_j g_{ki}) = 0.$$  \hspace{1cm} (20)

Contracting in $h$ and $k$, we get

$$(n-1)(2Q + nS)E_{ij} + (n-1)KA_iA_j = 0,$$  \hspace{1cm} (21)

which implies that

$$E_{ij} = -\frac{K}{2Q + nS}A_iA_j,$$  \hspace{1cm} (22)

Contraction (17) in $h$ and $k$, we get

$$R_{ij} = P[nA_iA_j - A_iA_j + \alpha A_i[g_{ij} - A_j A_i] + U(n g_{ij} - g_{ji}),$$  \hspace{1cm} (23)

which implies that

$$R_{ij} = (n-1)U g_{ij} + P(n-1)A_iA_j$$  \hspace{1cm} (24)

This is a quasi-Einstein manifold. Thus we conclude that.

Theorem 2. If a quasi-conformally flat new type of generalized quasi-Einstein manifold be a manifold of quasi-constant curvature then this becomes a quasi-Einstein manifold.

3 Einstein field equation in a new type of generalized quasi-Einstein manifold

The Einstein field equation with a cosmological term is given by [8]

$$R_{ij} - R \frac{1}{2} g_{ij} + \alpha g_{ij} = kT_{ij},$$  \hspace{1cm} (25)

where $\wedge$, $k$ and $T_{ij}$ are cosmological constant, Gravitational constant and energy momentum tensor respectively.

Using (8) and (9) in (25), we have

$$\alpha g_{ij} + \beta A_iA_j + \alpha - (n-2) \gamma E_{ij} - \frac{1}{2}(\alpha n + \beta A + [\alpha - (n-2) \gamma] E) g_{ij} + \alpha g_{ij} = kT_{ij},$$  \hspace{1cm} (26)

equation (26) which implies that

$$\alpha (1 - \frac{n}{2}) + \wedge g_{ij} + \beta 1 (1 - \frac{n}{2}) |\alpha - (n-2) \gamma| E_{ij} = kT_{ij},$$  \hspace{1cm} (27)
which is required Einstein field equation in a new type of generalized quasi-Einstein manifold. Taking covariant derivative of (27) and suppose $\nabla_j A_i = 0$, we have

$$\left(\alpha \left(1 - \frac{n}{2}\right) + \gamma\right) \nabla_h g_{ij} + \left(1 - \frac{n}{2}\right) \alpha \nabla_h E_{ij} = k \nabla_h T_{ij},$$

(28)
equation (28) which implies that

$$\left(1 - \frac{n}{2}\right) \alpha \nabla_h E_{ij} = k \nabla_h T_{ij}.$$  

(29)

Thus, we conclude.

**Theorem 3.** In a new type of generalized quasi-Einstein manifold, if $A_i$ be covariant constant then

(i) Symmetric tensor $E_{ij}$ is covariant constant if the energy momentum tensor is covariant constant,

(ii) Symmetric tensor $E_{ij}$ is recurrent if the energy momentum tensor is recurrent.

Taking covariant derivative of (27) and $A_i$ be a covariant constant, we get

$$\left(\alpha \left(1 - \frac{n}{2}\right) + \gamma\right) g_{ij,k} + \left(1 - \frac{n}{2}\right) \alpha \nabla_k E_{ij} = k T_{ij,k},$$

(30)

Interchanging $i, j$ and $k$ in cyclic order in (30), we have

$$\left(\alpha \left(1 - \frac{n}{2}\right) + \gamma\right) g_{jk,i} + \left(1 - \frac{n}{2}\right) \alpha \nabla_i E_{jk} = k T_{jk,i},$$

(31)

and

$$\left(\alpha \left(1 - \frac{n}{2}\right) + \gamma\right) g_{ki,j} + \left(1 - \frac{n}{2}\right) \alpha \nabla_j E_{ki} = k T_{ki,j},$$

(32)

Adding (30), (31) and (32), we get

$$\left(1 - \frac{n}{2}\right) \alpha \nabla E_{ij} = k \left(T_{ij,k} + T_{jk,i} + T_{ki,j}\right).$$

(33)

Now, if symmetric tensor $E_{ij}$ satisfies the Bianchi second identity, then

$$E_{ij,k} + E_{jk,i} + E_{ki,j} = 0,$$

(34)

therefore from (33), we get

$$T_{ij,k} + T_{jk,i} + T_{ki,j} = 0.$$  

(35)

i.e. the energy momentum tensor satisfies the Bianchi second identity. Thus, we conclude that.

**Theorem 4.** In a new type of generalized quasi-Einstein manifold if $A_i$ be covariant constant then the symmetric tensor $E_{ij}$ satisfies the Bianchi second identity if and only if energy momentum tensor satisfies the Bianchi second identity.

4 An example of a new type of generalized quasi-Einstein manifold

Now, we take a manifold $(M, g)$ such that $M = R^4$ and the metric $g$ in $R^4$ is given by

$$ds^2 = g_{ij} dx^i dx^j = f(x^4)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2,$$

(36)
the only non-vanishing components of Christoffel symbols and the curvature tensor are given by

\[ \Gamma^a_{11} = \Gamma^a_{33} = \Gamma^a_{22} = -\frac{1}{2} f'(x^4), \quad \Gamma^a_{14} = \Gamma^a_{34} = \Gamma^a_{24} = \frac{1}{2} \left( \frac{f'(x^4)}{f(x^4)} \right), \]  

(37)

and

\[ R_{1441} = R_{2442} = R_{4334} = \frac{1}{2} f''(x^4) - \frac{1}{4} \left( \frac{f'(x^4)}{f(x^4)} \right)^2, \quad R_{3112} = R_{3312} = \frac{1}{4} \left( \frac{f'(x^4)}{f(x^4)} \right)^2. \]  

(38)

The only non-zero Ricci tensors are given by

\[ R_{11} = R_{22} = R_{33} = -\frac{1}{2} f''(x^4) - \frac{1}{2} \left( \frac{f'(x^4)}{f(x^4)} \right)^2, \quad R_{44} = -\frac{3}{2} f''(x^4) + \frac{3}{4} \left( \frac{f'(x^4)}{f(x^4)} \right)^2, \]  

(39)

assuming

\[ \alpha = -\frac{f''(x^4)}{2f(x^4)} - \frac{1}{2} \left( \frac{f'(x^4)}{f(x^4)} \right)^2, \quad \beta = \frac{3}{2f(x^4)} \left\{ f'(x^4) \left( \frac{f'(x^4)}{f(x^4)} \right)^2 \right\}, \quad \gamma = -\left( \frac{f'(x^4)}{f(x^4)} \right)^2 + \frac{1}{2} \frac{f''(x^4)}{f(x^4)}, \]  

and

\[ A_i = \begin{cases} \sqrt{f(x^4)}, & i = 1 \\ 0, & i = 2, 3, 4, \end{cases} \]  

\[ E_{ij} = \begin{cases} 1, & i = j = 1 \\ 0, & i \neq j \text{ and } i = j = 2, 3 \\ \frac{1}{2} \left( \frac{4f''(x^4) f(x^4) - 5f'(x^4)^2}{f(x^4)^2} \right), & i = j = 4. \end{cases} \]  

Now, using above value, we have

\[ \alpha g_{11} + \beta A_1 A_1 + (\alpha - 2\gamma) E_{11} = -\frac{1}{2} f''(x^4) - \frac{1}{2} \left( \frac{f'(x^4)}{f(x^4)} \right)^2, \]  

\[ \alpha g_{22} + \beta A_2 A_2 + (\alpha - 2\gamma) E_{22} = -\frac{1}{2} f''(x^4) - \frac{1}{2} \left( \frac{f'(x^4)}{f(x^4)} \right)^2, \]  

\[ \alpha g_{33} + \beta A_3 A_3 + (\alpha - 2\gamma) E_{33} = -\frac{1}{2} f''(x^4) - \frac{1}{2} \left( \frac{f'(x^4)}{f(x^4)} \right)^2, \]  

\[ \alpha g_{44} + \beta A_4 A_4 + (\alpha - 2\gamma) E_{44} = -\frac{3}{2} f''(x^4) + \frac{3}{4} \left( \frac{f'(x^4)}{f(x^4)} \right)^2. \]  

(40)

Now, from equation (39) and (40), we have

1. \( R_{11} = \alpha g_{11} + \beta A_1 A_1 + (\alpha - 2\gamma) E_{11}, \)
2. \( R_{22} = \alpha g_{22} + \beta A_2 A_2 + (\alpha - 2\gamma) E_{22}, \)
3. \( R_{33} = \alpha g_{33} + \beta A_3 A_3 + (\alpha - 2\gamma) E_{33}, \)
4. \( R_{44} = \alpha g_{44} + \beta A_4 A_4 + (\alpha - 2\gamma) E_{44}. \)

We shall now show that the 1-forms are unit

\[ g^{ij} A_i A_j = g^{11} A_1 A_1 + g^{22} A_2 A_2 + g^{33} A_3 A_3 + g^{44} A_4 A_4 = 1, \]  

(41)
this shows that \((\mathbb{R}^4,g)\) is a new type of generalized quasi-Einstein manifold.

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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