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On $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-Functions

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Abstract: The purpose of this paper is to define and study the notions of new classes of functions, namely $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous, $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-closed and $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-homeomorphism functions and investigate their fundamental properties.

Keywords: Bioperation, α-open set, $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous, $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-closed and $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-homeomorphism functions.

1 Introduction

The study of topological spaces, their continuous functions and general properties make up one branch of Topology known as "General Topology". The concept of closednesss and continuity are fundamental with respect to the investigation of topological spaces. The field of mathematical science is called topology concerned with all questions directly or indirectly related to continuity. In 2013, Ibrahim [1] introduced and discussed an operation of a topology $\alpha O(X)$ into the power set $P(X)$ of a space X and also he introduced the concept of α_γ-open sets. Ibrahim [2] introduced the notion of $\alpha O(X,\tau)_{(\gamma,\gamma')}$, which is the collection of all $\alpha_{(\gamma,\gamma')}$-open sets in a topological space (X,τ) and also he defined the $\alpha_{(\gamma,\gamma')}$-$T_i$ [3] ($i = 0, \frac{1}{2}, 1, 2$) in topological spaces. In this paper, the author introduce and study the new types of functions and give some properties of these functions in topological spaces.

2 Preliminaries

Throughout this paper, by (X,τ) and (Y,σ) (or X and Y) we always mean topological spaces without any property except the mentioned in the context. The closure and the interior of a subset A of X are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a topological space (X,τ) is said to be α-open [4] if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is said to be α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$. The family of all α-open (resp. α-closed) sets in a topological space (X,τ) is denoted by $\alpha O(X,\tau)$ (resp. $\alpha C(X,\tau)$). An operation $\gamma : \alpha O(X,\tau) \to P(X)$ [1] is a mapping satisfying the condition, $V \subseteq V^\gamma$ for each $V \in \alpha O(X,\tau)$. We call the mapping γ an operation on $\alpha O(X,\tau)$. A subset A of X is called an α_γ-open set [1] if for each point $x \in A$, there exists an α-open set U of X containing x such that $U^\gamma \subseteq A$. The complement of an α_γ-open set is called α_γ-closed. The set of all α_γ-open sets of X is denote by $\alpha O(X,\tau)_\gamma$. An operation γ on $\alpha O(X,\tau)$ is said to be α-regular [1] if for every α-open sets U and V containing $x \in X$, there exists an α-open set W containing x such that $W^\gamma \subseteq U^\gamma \cap V^\gamma$. An operation γ on $\alpha O(X,\tau)$ is said to be α-open [1] if for every α-open set U containing $x \in X$, there exists an α_γ-open set V of X such that $x \in V$ and $V \subseteq U^\gamma$.

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We recall the following definitions and results from [2].

Definition 1. *A non-empty subset A of (X,τ) is said to be $\alpha_{(\gamma,\gamma')}$-open if for each $x \in A$, there exist α-open sets U and V of X containing x such that $U^{\gamma} \cup V^{\gamma'} \subseteq A$. A subset F of (X,τ) is said to be $\alpha_{(\gamma,\gamma')}$-closed if its complement $X \setminus F$ is $\alpha_{(\gamma,\gamma')}$-open. The set of all $\alpha_{(\gamma,\gamma')}$-open sets of (X,τ) is denoted by $\alpha O(X,\tau)_{(\gamma,\gamma')}$.*

Proposition 1. *If A_i is $\alpha_{(\gamma,\gamma')}$-open for every $i \in I$, then $\cup\{A_i : i \in I\}$ is $\alpha_{(\gamma,\gamma')}$-open.*

Definition 2. *A topological space (X,τ) is said to be $\alpha_{(\gamma,\gamma')}$-regular if for each point x in X and every α-open set U in X containing x, there exist α-open sets W and S in X containing x such that $W^{\gamma} \cup S^{\gamma'} \subseteq U$.*

Proposition 2. *A topological space (X,τ) with operations γ and γ' on $\alpha O(X,\tau)$ is $\alpha_{(\gamma,\gamma')}$-regular if and only if $\alpha O(X,\tau) = \alpha O(X,\tau)_{(\gamma,\gamma')}$.*

Definition 3. *Let A be a subset of a topological space (X,τ). The intersection of all $\alpha_{(\gamma,\gamma')}$-closed sets containing A is called the $\alpha_{(\gamma,\gamma')}$-closure of A and denoted by $\alpha_{(\gamma,\gamma')}$-$Cl(A)$.*

Proposition 3. *Let A be subset of a topological space (X,τ). Then,*

(1) *A is $\alpha_{(\gamma,\gamma')}$-closed if and only if $\alpha_{(\gamma,\gamma')}$-$Cl(A) = A$.*
(2) *$\alpha_{(\gamma,\gamma')}$-$Cl(A)$ is $\alpha_{(\gamma,\gamma')}$-closed.*

Definition 4. *For a subset A of (X,τ), we define $\alpha Cl_{(\gamma,\gamma')}(A)$ as follows: $\alpha Cl_{(\gamma,\gamma')}(A) = \{x \in X : (U^{\gamma} \cup W^{\gamma'}) \cap A \neq \phi$ holds for every α-open sets U and W containing $x\}$.*

Theorem 1. *Let A be subset of a topological space (X,τ). Then, A is $\alpha_{(\gamma,\gamma')}$-closed if and only if $\alpha Cl_{(\gamma,\gamma')}(A) = A$.*

Definition 5. *A subset A of (X,τ) is said to be an $\alpha_{(\gamma,\gamma')}$- generalized closed (briefly, $\alpha_{(\gamma,\gamma')}$- g.closed) set if $\alpha_{(\gamma,\gamma')}$-$Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an $\alpha_{(\gamma,\gamma')}$-open set in (X,τ).*

Proposition 4. *The following statements $(1),(2)$ and (3) are equivalent for a subset A of (X,τ).*

(1) *A is $\alpha_{(\gamma,\gamma')}$-g.closed in (X,τ).*
(2) *$\alpha_{(\gamma,\gamma')}$-$Cl(\{x\}) \cap A \neq \phi$ for every $x \in \alpha_{(\gamma,\gamma')}$-$Cl(A)$.*
(3) *$\alpha_{(\gamma,\gamma')}$-$Cl(A) \setminus A$ does not contain any non-empty $\alpha_{(\gamma,\gamma')}$-closed set.*

Definition 6. [3] *A topological space (X,τ) is said to be:*

(1) *$\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$ if every $\alpha_{(\gamma,\gamma')}$-g.closed set is $\alpha_{(\gamma,\gamma')}$-closed.*
(2) *$\alpha_{(\gamma,\gamma')}$-$T_0$ if for each pair of distinct points x,y in X, there exist α-open sets U and V such that $x \in U \cap V$ and $y \notin U^{\gamma} \cup V^{\gamma'}$, or $y \in U \cap V$ and $x \notin U^{\gamma} \cup V^{\gamma'}$.*
(3) *$\alpha_{(\gamma,\gamma')}$-$T_1$ if for each pair of distinct points x,y in X, there exist α-open sets U and V containing x and α-open sets W and S containing y such that $y \notin U^{\gamma} \cup V^{\gamma'}$ and $x \notin W^{\gamma} \cup S^{\gamma'}$.*
(4) *$\alpha_{(\gamma,\gamma')}$-$T_2$ if for each pair of distinct points x,y in X, there exist α-open sets U and V containing x and α-open sets W and S containing y such that $(U^{\gamma} \cup V^{\gamma'}) \cap (W^{\gamma} \cup S^{\gamma'}) = \phi$.*

Proposition 5. [3] *A topological space (X,τ) is $\alpha_{(\gamma,\gamma')}$-$T_1$ if and only if for each $x \in X$, $\{x\}$ is $\alpha_{(\gamma,\gamma')}$-closed.*

3 $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-Continuous Functions

Throughout this section, let $f:(X,\tau)\to(Y,\sigma)$ be a function and $\gamma,\gamma':\alpha O(X,\tau)\to P(X)$ be operations on $\alpha O(X,\tau)$ and $\beta,\beta':\alpha O(Y,\sigma)\to P(Y)$ be operations on $\alpha O(Y,\sigma)$.

Definition 7. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous if for each point $x\in X$ and each α-open sets W and S of (Y,σ) containing $f(x)$, there exist α-open sets U and V of (X,τ) containing x such that $f(U^{\gamma}\cup V^{\gamma'})\subseteq W^{\beta}\cup S^{\beta'}$.*

Theorem 2. *Let f be a function. Consider the following statements.*

(1) *$f:(X,\tau)\to(Y,\sigma)$ is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous.*
(2) *$f(\alpha Cl_{(\gamma,\gamma')}(A))\subseteq \alpha Cl_{(\beta,\beta')}(f(A))$ for every subset A of (X,τ).*
(3) *$\alpha Cl_{(\gamma,\gamma')}(f^{-1}(B))\subseteq f^{-1}(\alpha Cl_{(\beta,\beta')}(B))$ for every subset B of (Y,σ).*
(4) *$f^{-1}(B)$ is $\alpha_{(\gamma,\gamma')}$-closed for every $\alpha_{(\beta,\beta')}$-closed set B of (Y,σ).*
(5) *$f(\alpha_{(\gamma,\gamma')}\text{-}Cl(A))\subseteq \alpha_{(\beta,\beta')}\text{-}Cl(f(A))$ for every subset A of (X,τ).*
(6) *$f^{-1}(V)$ is $\alpha_{(\gamma,\gamma')}$-open for every $\alpha_{(\beta,\beta')}$-open set V of (Y,σ).*
(7) *For each point $x\in X$ and each $\alpha_{(\beta,\beta')}$-open W of (Y,σ) containing $f(x)$, there exist $\alpha_{(\gamma,\gamma')}$-open U of (X,τ) containing x such that $f(U)\subseteq W$.*

Then, the following implications are true: $(1)\Rightarrow(2)\Leftrightarrow(3)\Rightarrow(4)\Leftrightarrow(5)\Leftrightarrow(6)\Leftrightarrow(7)$.

Proof. $(1)\Rightarrow(2)$. Let $x\in \alpha Cl_{(\gamma,\gamma')}(A)$ and W, S be α-open sets of (Y,σ) containing $f(x)$. There exist α-open sets U and V of (X,τ) containing x such that $f(U^{\gamma}\cup V^{\gamma'})\subseteq W^{\beta}\cup S^{\beta'}$. Since $x\in \alpha Cl_{(\gamma,\gamma')}(A)$, then $(U^{\gamma}\cup V^{\gamma'})\cap A\neq\phi$, implies that $f(U^{\gamma}\cup V^{\gamma'})\cap f(A)\neq\phi$. Therefore, we have $f(A)\cap(W^{\beta}\cup S^{\beta'})\neq\phi$. Therefore $f(x)\in \alpha Cl_{(\beta,\beta')}(f(A))$, which implies that $x\in f^{-1}(\alpha Cl_{(\beta,\beta')}(f(A)))$. Hence $\alpha Cl_{(\gamma,\gamma')}(A)\subseteq f^{-1}(\alpha Cl_{(\beta,\beta')}(f(A)))$, so that $f(\alpha Cl_{(\gamma,\gamma')}(A))\subseteq \alpha Cl_{(\beta,\beta')}(f(A))$.

$(2)\Rightarrow(3)$. Let B be any subset of Y. Then $f^{-1}(B)$ is a subset of X. By (2), we have $f(\alpha Cl_{(\gamma,\gamma')}(f^{-1}(B)))\subseteq \alpha Cl_{(\beta,\beta')}(f(f^{-1}(B)))\subseteq \alpha Cl_{(\beta,\beta')}(B)$. Hence $\alpha Cl_{(\gamma,\gamma')}(f^{-1}(B))\subseteq f^{-1}(\alpha Cl_{(\beta,\beta')}(B))$.

$(3)\Rightarrow(2)$. Let A be any subset of X. Then $f(A)$ is a subset of Y. By (3), we have $\alpha Cl_{(\gamma,\gamma')}(f^{-1}f((A)))\subseteq f^{-1}(\alpha Cl_{(\beta,\beta')}(f(A)))$. This implies that $\alpha Cl_{(\gamma,\gamma')}(A)\subseteq f^{-1}(\alpha Cl_{(\beta,\beta')}(f(A)))$. Hence $f(\alpha Cl_{(\gamma,\gamma')}(A))\subseteq \alpha Cl_{(\beta,\beta')}(f(A))$.

$(3)\Rightarrow(4)$. Let B be an $\alpha_{(\beta,\beta')}$-closed set of (Y,σ). By (3) and Theorem 1, $\alpha Cl_{(\gamma,\gamma')}(f^{-1}(B))\subseteq f^{-1}(B)$ and hence $f^{-1}(B)$ is $\alpha_{(\gamma,\gamma')}$-closed.

$(4)\Rightarrow(5)$. Let A be any subset of X. Then $f(A)\subseteq \alpha_{(\beta,\beta')}\text{-}Cl(f(A))$ and $\alpha_{(\beta,\beta')}\text{-}Cl(f(A))$ is an $\alpha_{(\beta,\beta')}$-closed set in Y. Hence $A\subseteq f^{-1}(\alpha_{(\beta,\beta')}\text{-}Cl(f(A)))$. By (4), we have $f^{-1}(\alpha_{(\beta,\beta')}\text{-}Cl(f(A)))$ which is an $\alpha_{(\gamma,\gamma')}$-closed set in X. Therefore, $\alpha_{(\gamma,\gamma')}\text{-}Cl(A)\subseteq f^{-1}(\alpha_{(\beta,\beta')}\text{-}Cl(f(A)))$. Hence $f(\alpha_{(\gamma,\gamma')}\text{-}Cl(A))\subseteq \alpha_{(\beta,\beta')}\text{-}Cl(f(A))$.

$(5)\Rightarrow(4)$. Let B be an $\alpha_{(\beta,\beta')}$-closed set of (Y,σ). By (5), $\alpha_{(\gamma,\gamma')}\text{-}Cl(f^{-1}(B))\subseteq f^{-1}(\alpha_{(\beta,\beta')}\text{-}Cl(f(f^{-1}(B))))\subseteq f^{-1}(\alpha_{(\beta,\beta')}\text{-}Cl(B))\subseteq f^{-1}(B)$. Therefore, by Proposition 3 (1), $f^{-1}(B)$ is $\alpha_{(\gamma,\gamma')}$-closed.

$(5)\Leftrightarrow(6)$. This follows from Definition 1 and the equivalence of $(4)\Leftrightarrow(5)$.

(6) $\Rightarrow$ (7). Let W be any $\alpha_{(\beta,\beta')}$-open set in Y containing $f(x)$, so its inverse image is an $\alpha_{(\gamma,\gamma')}$-open set in X. Since $f(x) \in W$, then $x \in f^{-1}(W)$ and by hypothesis $f^{-1}(W)$ is an $\alpha_{(\gamma,\gamma')}$-open set in X containing x, so that $f(f^{-1}(W)) \subseteq W$.

(7) $\Rightarrow$ (6). Let $V \in \alpha O(Y,\sigma)_{(\beta,\beta')}$. For each $x \in f^{-1}(V)$, by (7), there exists an $\alpha_{(\gamma,\gamma')}$-open set U_x containing x such that $f(U_x) \subseteq V$. Then, we have $f^{-1}(V) = \cup\{U_x \in \alpha O(X,\tau)_{(\gamma,\gamma')} : x \in f^{-1}(V)\}$ and hence $f^{-1}(V) \in \alpha O(X,\tau)_{(\gamma,\gamma')}$ using Proposition 1.

Corollary 1. *If (Y,σ) is an $\alpha_{(\beta,\beta')}$-regular space, then all statements of Theorem 2 are equivalent.*

Proof. By Theorem 2, it is sufficient to prove the implication (6) $\Rightarrow$ (1), where (1) and (6) are the properties of Theorem 2. Let $x \in X$ and W, S be α-open sets of (Y,σ) containing $f(x)$. By Proposition 2, $W \cup S$ is $\alpha_{(\beta,\beta')}$-open. Then, $f^{-1}(W \cup S)$ is $\alpha_{(\gamma,\gamma')}$-open set of (X,τ) containing x by (6). Therefore, there exist α-open sets U and V of (X,τ) containing x such that $U^{\gamma} \cup V^{\gamma'} \subseteq f^{-1}(W \cup S)$ and so $f(U^{\gamma} \cup V^{\gamma'}) \subseteq W^{\beta} \cup S^{\beta'}$. This implies that f is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous.

Remark. The converse of implication (1) $\Rightarrow$ (6) in Theorem 2 is not true in general as shown by the following example.

Example 1. Let $X = \{a,b,c\}$, $Y = \{1,2,3\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma = \{\phi, Y, \{1\}, \{2\}, \{1,2\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be a function defined by

$$f(x) = \begin{cases} 2, & \text{if } x = a, \\ 3, & \text{if } x = b, \\ 1, & \text{if } x = c. \end{cases}$$

For each $A \in \alpha O(X)$, we define two operations γ and γ', respectively, by $A^{\gamma} = Cl(A)$ and $A^{\gamma'} = A$.
For each $K \in \alpha O(Y)$, we define two operations β and β', respectively, by $K^{\beta} = K$ and $K^{\beta'} = Cl(K)$. Then, the condition (6) in Theorem 2 is true. It is shown that f is not $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous.

Theorem 2 suggests the following.

Remark. If $f : (X,\tau) \to (Y,\sigma)$ is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous, then the induced function $f : (X, \alpha O(X,\tau)_{(\gamma,\gamma')}) \to (Y, \alpha O(Y,\sigma)_{(\beta,\beta')})$ is continuous.

Let (X,τ), (Y,σ) and (Z,η) be spaces and $\gamma, \gamma' : \alpha O(X) \to P(X)$, $\beta, \beta' : \alpha O(Y) \to P(Y)$ and $\delta, \delta' : \alpha O(Z) \to P(Z)$, be operations on $\alpha O(X,\tau)$, $\alpha O(Y,\sigma)$ and $\alpha O(Z,\eta)$, respectively.

Theorem 3. *If $f : (X,\tau) \to (Y,\sigma)$ is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous and $g : (Y,\sigma) \to (Z,\eta)$ is $(\alpha_{(\beta,\beta')}, \alpha_{(\delta,\delta')})$-continuous, then its composition $gof : (X,\tau) \to (Z,\eta)$ is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\delta,\delta')})$-continuous.*

Proof. Let $x \in X$, K and L be α-open sets of Z containing $g(f(x))$. Since g is $(\alpha_{(\beta,\beta')}, \alpha_{(\delta,\delta')})$-continuous, then there exist α-open sets W and S of Y containing $f(x)$ such that $g(W^{\beta} \cup S^{\beta'}) \subseteq K^{\delta} \cup L^{\delta'}$. Also, since f is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-continuous, then there exist α-open sets U and V of X containing x such that $f(U^{\gamma} \cup V^{\gamma'}) \subseteq W^{\beta} \cup S^{\beta'}$. This implies that $f(U^{\gamma} \cup V^{\gamma'}) \subseteq W^{\beta} \cup S^{\beta'} \subseteq g^{-1}(K^{\delta} \cup L^{\delta'})$. Then, we obtain $(gof)(U^{\gamma} \cup V^{\gamma'}) \subseteq K^{\delta} \cup L^{\delta'}$. Therefore, gof is $(\alpha_{(\gamma,\gamma')}, \alpha_{(\delta,\delta')})$-continuous.

Definition 8. *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-closed if for $\alpha_{(\gamma,\gamma')}$-closed set A of X, $f(A)$ is $\alpha_{(\beta,\beta')}$-closed in Y.*

Proposition 6. *Let $f : (X,\tau) \to (Y,\sigma)$ be an $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-closed function. Then, for each subset B of (Y,σ) and each $\alpha_{(\gamma,\gamma')}$-open set U containing $f^{-1}(B)$, there exists an $\alpha_{(\beta,\beta')}$-open set V such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Let $V = Y \setminus f(X \setminus U)$. Then V is $\alpha_{(\beta,\beta')}$-open. Thus $f^{-1}(B) \subseteq U$ implies $B \subseteq V$ and $f^{-1}(V) = f^{-1}(Y \setminus f(X \setminus U)) = X \setminus f^{-1}(f(X \setminus U)) \subseteq X \setminus (X \setminus U) = U$, or $f^{-1}(V) \subseteq U$.

Proposition 7. *Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then, f is $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-closed if and only if $\alpha_{(\beta,\beta')}$-$Cl(f(A))\subseteq f(\alpha_{(\gamma,\gamma')}$-$Cl(A))$ for every subset A of X.*

Proof. Obvious.

Corollary 2. *If $f:(X,\tau)\to(Y,\sigma)$ is bijective and $f^{-1}:(Y,\sigma)\to(X,\tau)$ is $(\alpha_{(\beta,\beta')},\ \alpha_{(\gamma,\gamma')})$-continuous, then f is $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-closed.*

Proof. Follows from Theorem 2.

Proposition 8. *If $f:(X,\tau)\to(Y,\sigma)$ is $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-continuous and $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-closed, then*

(1) *$f(A)$ is $\alpha_{(\beta,\beta')}$-g.closed for every $\alpha_{(\gamma,\gamma')}$-g.closed set A of (X,τ).*
(2) *$f^{-1}(B)$ is $\alpha_{(\gamma,\gamma')}$-g.closed for every $\alpha_{(\beta,\beta')}$-g.closed set B of (Y,σ).*

Proof. (1) Let V be an $\alpha_{(\beta,\beta')}$-open set containing $f(A)$. Then, $f^{-1}(V)$ is an $\alpha_{(\gamma,\gamma')}$-open set containing A by Theorem 2 and so $\alpha_{(\gamma,\gamma')}$-$Cl(A)\subseteq f^{-1}(V)$. It follows that $f(\alpha_{(\gamma,\gamma')}$-$Cl(A))$ is an $\alpha_{(\beta,\beta')}$-closed set and hence $\alpha_{(\beta,\beta')}$-$Cl(f(A))\subseteq \alpha_{(\beta,\beta')}$-$Cl(f(\alpha_{(\gamma,\gamma')}$-$Cl(A)))=f(\alpha_{(\gamma,\gamma')}$-$Cl(A))\subseteq V$. This implies that $f(A)$ is $\alpha_{(\beta,\beta')}$-g.closed.

(2) Let U be any $\alpha_{(\gamma,\gamma')}$-open set such that $f^{-1}(B)\subseteq U$. Let $F=\alpha_{(\gamma,\gamma')}$-$Cl(f^{-1}(B))\cap(X\setminus U)$, then F is $\alpha_{(\gamma,\gamma')}$-closed in (X,τ). This implies $f(F)$ is $\alpha_{(\beta,\beta')}$-closed set in (Y,σ). Since $f(F)=f(\alpha_{(\gamma,\gamma')}$-$Cl(f^{-1}(B))\cap(X\setminus U))\subseteq\alpha_{(\beta,\beta')}$-$Cl(B)\cap f(X\setminus U)\subseteq\alpha_{(\beta,\beta')}$-$Cl(B)\cap(Y\setminus B)$ by Theorem 2, it is shown that $\alpha_{(\beta,\beta')}$-$Cl(B)\setminus B$ contains an $\alpha_{(\beta,\beta')}$-closed set $f(F)$. It follows from Proposition 4 that $f(F)=\phi$ and hence $F=\phi$. Therefore $\alpha_{(\gamma,\gamma')}$-$Cl(f^{-1}(B))\subseteq U$. This shows that $f^{-1}(B)$ is $\alpha_{(\gamma,\gamma')}$-g.closed.

Theorem 4. *Suppose that there exists an $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-continuous and $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-closed function, say $f:(X,\tau)\to(Y,\sigma)$.*

(1) *If f is injective and (Y,σ) is $\alpha_{(\beta,\beta')}$-$T_{\frac{1}{2}}$, then (X,τ) is $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$.*
(2) *If f is surjective and (X,τ) is $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$, then (Y,σ) is $\alpha_{(\beta,\beta')}$-$T_{\frac{1}{2}}$.*

Proof. (1) Let A be an $\alpha_{(\gamma,\gamma')}$-g.closed set of (X,τ). We claim that A is $\alpha_{(\gamma,\gamma')}$-closed in (X,τ). By Proposition 8 (1), $f(A)$ is $\alpha_{(\beta,\beta')}$-g.closed. Since (Y,σ) is $\alpha_{(\beta,\beta')}$-$T_{\frac{1}{2}}$, this implies that $f(A)$ is $\alpha_{(\beta,\beta')}$-closed. Since f is $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-continuous and injective, then, we have $A=f^{-1}(f(A))$ is $\alpha_{(\gamma,\gamma')}$-closed by Theorem 2. Hence (X,τ) is $\alpha_{(\gamma,\gamma')}$-$T_{\frac{1}{2}}$.

(2) Let B be an $\alpha_{(\beta,\beta')}$-g.closed set in (Y,σ). By Proposition 8 (2) and assumption, it is shown that $B=f(f^{-1}(B))$ is $\alpha_{(\beta,\beta')}$-closed and hence (Y,σ) is $\alpha_{(\beta,\beta')}$-$T_{\frac{1}{2}}$.

Theorem 5. *Suppose that there exists an $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-continuous injection. If (Y,σ) is $\alpha_{(\beta,\beta')}$-T_i, then (X,τ) is $\alpha_{(\gamma,\gamma')}$-$T_i$, where $i=0,1,2$.*

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be the $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-continuous injection. The proof for $i=1$ is as follows:
Let $x\in X$. Then, by Proposition 5, $\{f(x)\}$ is $\alpha_{(\beta,\beta')}$-closed in (Y,σ). By Theorem 2 and Proposition 5, $\{x\}$ is $\alpha_{(\gamma,\gamma')}$-closed and hence (X,τ) is $\alpha_{(\gamma,\gamma')}$-$T_1$. The proofs for $i=0,2$ follow from Definition 6 (2), (4) and by Theorem 2.

Definition 9. *A function $f:(X,\tau)\to(Y,\sigma)$ is called an $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-homeomorphism if f is an $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-continuous bijection and $f^{-1}:(Y,\sigma)\to(X,\tau)$ is $(\alpha_{(\beta,\beta')},\ \alpha_{(\gamma,\gamma')})$-continuous. The collection of all $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\gamma,\gamma')})$-homeomorphisms from (X,τ) onto itself is denoted by $\alpha_{(\gamma,\gamma')}$-$h(X,\tau)$.*

Theorem 6. *Suppose that $f:(X,\tau)\to(Y,\sigma)$ is an $(\alpha_{(\gamma,\gamma')},\ \alpha_{(\beta,\beta')})$-homeomorphism function. Then, (X,τ) is $\alpha_{(\gamma,\gamma')}$-$T_i$ if and only if (Y,σ) is $\alpha_{(\beta,\beta')}$-T_i, where $i=0,\frac{1}{2},1,2$.*

Proof. The proof follows from Theorems 4, 5 and Definition 9

Theorem 7. *Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a function. Then,*

(1) *For each topological space (X,τ), the collection $\alpha_{(\gamma,\gamma')}$-$h(X,\tau)$ forms a group under the composition of functions.*
(2) *For an $(\alpha_{(\gamma,\gamma')}, \alpha_{(\beta,\beta')})$-homeomorphism $f:(X,\tau)\rightarrow(Y,\sigma)$, there exists a group isomorphism, say $f_*:\alpha_{(\gamma,\gamma')}$-$h(X,\tau)\rightarrow\alpha_{(\beta,\beta')}$-$h(Y,\sigma)$.*

Proof. Putting $H_X=\alpha_{(\gamma,\gamma')}$-$h(X,\tau)$.

(1) First we prove that: if $a\in H_X$ and $b\in H_X$, then $b\circ a\in H_X$. Indeed, since a and b (resp. a^{-1} and b^{-1}) are $(\alpha_{(\gamma,\gamma')}, \alpha_{(\gamma,\gamma')})$-continuous bijectons, $b\circ a$ (resp. $a^{-1}\circ b^{-1}=(b\circ a)^{-1}$) is also an $(\alpha_{(\gamma,\gamma')}, \alpha_{(\gamma,\gamma')})$-continuous bijection by Theorem 3 and so $b\circ a\in H_X$, where $b\circ a:X\rightarrow X$ is the composite functions of $a:X\rightarrow X$ and $b:X\rightarrow X$ such that $(b\circ a)(x)=b(a(x))$ for every point $x\in X$. Thus, the following binary operation $\eta_X:H_X\times H_X\rightarrow H_X$ is well defined by $\eta_X(a,b)=b\circ a$. Putting $a\cdot b=\eta_X(a,b)$, we have the following properties:
   (a) $(a\cdot b)\cdot c=a\cdot(b\cdot c)$ holds for every elements $a,b,c\in H_X$;
   (b) for all element $a\in H_X$, there exists an element $e\in H_X$ such that $a\cdot e=e\cdot a=a$ hold in H_X;
   (c) for each element $a\in H_X$, there exists an element $a_1\in H_X$ such that $a\cdot a_1=a_1\cdot a=e$ hold in H_X.
   Indeed, (a) is obtained obviously;
   (b) is obtained by taking $e=1_X$ and using the fact that $1_X\in H_X$, where $1_X:X\rightarrow X$ is the identity function;
   (c) is obtained by taking $a_1=a^{-1}$ for each $a\in H_X$. Then, by definition of groups, the pair (H_X,η_X) forms a group under the composition of functions.
(2) The required group isomorphism $f_*:H_X\rightarrow H_Y$ is well defined by $f_*(a)=f\circ(a\circ f^{-1})$ for every element $a\in H_X$. Indeed, $f_*(a)\in H_Y$ holds for every $a\in H_X$ by Theorem 3, $f_*(a\cdot b)=f\circ(b\circ a)\circ f^{-1}=f\circ b\circ f^{-1}\circ f\circ a\circ f^{-1}=(f_*(b))\circ(f_*(a))=f_*(a)\cdot f_*(b)$ hold for every elements $a,b\in H_X$ and so $f_*:H_X\rightarrow H_Y$ is a homomorphism. Hence $f_*:H_X\rightarrow H_Y$ is the required isomorphism.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Ibrahim, H. Z. (2013). On a class of α_γ-open sets in a topological space, *Acta Scientiarum. Technology*, 35 (3), 539-545.
[2] Ibrahim, H. Z. (2017). On $\alpha_{(\gamma,\gamma')}$-open Sets in Topological Spaces, *(submitted)*.
[3] Ibrahim, H. Z. (2017). On $\alpha_{(\gamma,\gamma')}$-Separation Axioms, *(submitted)*.
[4] Njastad, O. (1965). On some classes of nearly open sets, *Pacific J. Math.* 15, 961-970.