Categorical properties of racks

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Abstract: In this paper, we give some categorical objects of racks such as product, pullback and equalizer objects.

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1 Introduction

A rack [3] is a set with a non-associative binary operation satisfying two rack conditions. The theory of racks is connected to the group theory. This relation leads to the functor \( \text{Conj}: \text{Grp} \to \text{Rack} \) between the categories of racks and of groups which admits a left adjoint functor \( \text{As}: \text{Rack} \to \text{Grp} \); see [4], [7] for more details.

The earliest work on racks is due to Conway and Wraith [3] which is inspired by the conjugacy operation in a group and focuses in the special case of racks, called quandles; but they also were aware of the generalization. In the literature, racks are also called “automorphic sets” [2], “crystals” [8] and “(left) distributive quasigroups” [10].

In this study, we firstly recall the definitions and some examples for racks. Most of them appear in [7]. Afterwards, we give some categorical properties of racks which are the constructions of product, pullback and equalizer objects. These categorical objects are defined by the universal property diagrams in [1], [9] and examined for more specific categories such as category of crossed modules of racks and (modified) categories of interest in [5], [6].

2 Racks

We recall some notions from [7] which will be used in sequel.

Definition 1. A rack \( R \) is a set with a binary operation satisfying:

(R1) For all \( a, b \in R \), there exists a unique \( c \in R \) such that:

\[
    c \triangleright a = b,
\]

(R2) For all \( a, b, c \in R \), we have:

\[
    (a \triangleright c) \triangleleft c = (a \triangleright c) \triangleright (b \triangleleft c).
\]

A rack which additionally satisfies the idempotency condition:

\[
    r \triangleleft r = r
\]

is called a “quandle” (for all \( r \in R \)).

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Definition 2. A “pointed” rack \( R \) is a rack equipped with a fixed element \( 1 \in R \) such that (for all \( r \in R \)):

\[
1 \triangleleft r = 1 \quad \text{and} \quad r \triangleleft 1 = r.
\]

Remark. We only work with the pointed racks in the rest.

Definition 3. Let \( R \) and \( S \) be two (pointed) racks. A rack morphism is a map:

\[
f: R \rightarrow S
\]

such that:

\[
f(r \triangleleft r') = f(r) \triangleleft f(r') \quad (\text{and} \ f(1) = 1)
\]

for all \( r, r' \in R \).

Thus we get the category of (pointed) racks denoted by \( \text{Rack} \).

Some examples of racks are:

1. The trivial rack \( T_n \) of order \( n \) is the set \( \{0, 1, 2, \ldots, n-1\} \) with the rack operation (for all \( x, y \in T_n \)):

\[
x \triangleleft y = x.
\]
   
The infinitive trivial rack \( T_\infty \) is the set \( \mathbb{Z} \) equipped with the same operation.

2. The dihedral rack \( D_n \) is the set \( \{0, 1, 2, \ldots, n-1\} \) with the rack operation:

\[
x \triangleleft y = 2y - x \pmod{n}
\]
   
   for all \( x, y \in D_n \) and the infinitive dihedral rack \( D_\infty \) is the set \( \mathbb{Z} \) equipped with:

\[
x \triangleleft y = 2y - x
\]
   
   for all \( x, y \in \mathbb{Z} \).

3. The cyclic rack \( C_n \) of order \( n \) is the set \( \{0, 1, 2, \ldots, n-1\} \) with the rack operation:

\[
x \triangleleft y = x + 1 \pmod{n}
\]
   
   for all \( x, y \in C_n \), while the infinitive cyclic rack is the set \( \mathbb{Z} \) equipped with:

\[
x \triangleleft y = x + 1
\]
   
   for all \( x, y \in \mathbb{Z} \).

4. Given a group \( G \), we may define a rack structure on \( G \) by setting (for all \( g, h \in G \)):

\[
g \triangleleft h = h^{-1}gh.
\]
   
   This rack is called the “conjugation” rack of \( G \) and denoted by \( \text{ConjG} \). This construction provides a functor:

\[
\text{Conj} : \text{Grp} \rightarrow \text{Rack}.
\]

5. We may define a different rack structure on \( G \) by setting (for all \( g, h \in G \)):

\[
g \triangleleft h = hg^{-1}h.
\]
that is called “core” rack. However this construction is not functorial.

(6) Let $P$ and $R$ be two racks, then the cartesian product:

$$P \times R = \{(p, r) \mid p \in P, r \in R\}$$

has a rack structure with:

$$(p, r) \triangleright (p', r') = (p \triangleright p', r \triangleright r')$$

for all $(p, r), (p', r') \in P \times R$.

**Definition 4.** Let $R$ be a rack and $X$ be a set. We say that $X$ is an $R$-set when there are bijections $(\cdot r) : X \rightarrow X$ for all $r \in R$ such that:

$$(x \cdot r) \cdot r' = (x \cdot r') \cdot (r \triangleright r'),$$

for all $x \in X$ and $r, r' \in R$.

**Definition 5.** Let $R, S$ be two racks. We say that $S$ acts on $R$ by automorphisms when there is a (right) rack action of $S$ on $R$ and:

$$(r \triangleright r') \cdot s = (r \cdot s) \triangleright (r' \cdot s)$$

for all $s \in S$ and $r, r' \in R$.

The following notion is likely to be semi-direct product of groups:

**Definition 6.** If there exists a (right) rack action of $R$ on $S$, the “hemi-semi-direct product” $S \rtimes R \subset S \times R$ is the rack defined by the rack operation:

$$(s, r) \triangleright (s', r') = (s \cdot r', r \triangleright r')$$

for all $(s, r), (s', r') \in S \times R$.

**Definition 7.** For a given rack $R$, a non empty subset $S \subseteq R$ is called a subrack if $s \triangleright s' \in S$ for all $s, s' \in S$.

### 3 Categorical properties of racks

In this section we give the constructions of product, pullback and equalizer objects for the category of racks.

**Theorem 1.** The category of racks has products.

**Proof.** Let $P$ and $R$ be two racks. Define:

$$P \times R = \{(p, r) \mid p \in P, r \in R\}.$$

We already know that $P \times R$ is a rack. Also it is easy to verify that the projection maps $p_1 : P \times R \rightarrow P$ and $p_2 : P \times R \rightarrow R$ are rack morphisms.

Now we will check the universal property. Let $T$ be any rack and $\alpha : T \rightarrow P$, $\beta : T \rightarrow R$ be two rack morphisms. Then we need to prove that there exists a unique rack morphism:

$$\varphi : T \rightarrow P \times R$$
such that makes following diagram commutes:

\[
\begin{array}{ccc}
P & \xrightarrow{p_1} & P \times R & \xrightarrow{p_2} & R \\
\downarrow{\alpha} & & \downarrow{\exists \varphi} & & \downarrow{\beta} \\
T & & & & \\
\end{array}
\]

(1)

Define:

\[\varphi : T \to P \times R\]

\[t \mapsto \varphi(t) = (\alpha(t), \beta(t)).\]

\(\varphi\) is a rack morphism since:

\[\varphi(t \triangleleft t') = (\alpha(t \triangleleft t'), \beta(t \triangleleft t'))\]

\[= (\alpha(t) \triangleleft \alpha(t'), \beta(t) \triangleleft \beta(t'))\]

\[= (\alpha(t), \beta(t)) \triangleleft (\alpha(t'), \beta(t'))\]

\[= \varphi(t) \triangleleft \varphi(t')\]

for all \(t, t' \in T\). Furthermore we get:

\[p_1 \varphi(t) = p_1(\alpha(t), \beta(t))\]

\[= \alpha(t)\]

and

\[p_2 \varphi(t) = p_2(\alpha(t), \beta(t))\]

\[= \beta(t)\]

for all \(t \in T\) that proves the commutativity of (1).

Consider \(\varphi'\) with the same property as \(\varphi\), i.e. the following conditions hold:

\[p_1 \varphi' = \alpha\]

\[p_2 \varphi' = \beta.\]

Define \((p, r) \in P \times R\) by \(\varphi'(t) = (p, r)\). We get:

\[p_1 \varphi'(t) = \alpha(t) \Rightarrow p_1(p, r) = \alpha(t)\]

\[\Rightarrow p = \alpha(t)\]

and

\[p_2 \varphi'(t) = \beta(t) \Rightarrow p_2(p, r) = \beta(t)\]

\[\Rightarrow r = \beta(t)\]
for all $t \in T$ which yields:

$$\varphi'(t) = (p, r) = (\alpha(t), \beta(t)) = \varphi(t)$$

and proves that $\varphi$ is unique.

**Theorem 2.** The category of racks has pullbacks.

**Proof.** Let $f : P \to T$ and $g : R \to T$ be two rack morphisms. Define:

$$P \times_T R = \{(p, r) \mid f(p) = g(r)\}$$

which is a subrack of $P \times R$; see [5]. Then we get the following commutative diagram:

Let $Q$ be any rack with two rack morphisms $\alpha : Q \to P$ and $\beta : Q \to R$ where the following diagram commutes:

Then there must be a unique rack morphism:

$$\varphi : Q \to P \times_T R$$

that makes the following diagram commutative:
namely:

\[ p_1 \varphi = \alpha \]
\[ p_2 \varphi = \beta. \]

For this aim, define:

\[ \varphi : Q \rightarrow P \times R \]
\[ q \mapsto \varphi(q) = (\alpha(q), \beta(q)). \]

Then \( \varphi \) is a rack morphism since:

\[ \varphi(q \triangleright q') = (\alpha(q \triangleright q'), \beta(q \triangleright q')) \]
\[ = (\alpha(q) \triangleright \alpha(q'), \beta(q) \triangleright \beta(q')) \]
\[ = (\alpha(q), \beta(q)) \triangleright (\alpha(q'), \beta(q')) \]
\[ = \varphi(q) \triangleright \varphi(q') \]

for all \( q, q' \in Q \). Furthermore we get:

\[ p_1 \varphi(q) = p_1 (\alpha(q), \beta(q)) \]
\[ = \alpha(q) \]
\[ p_2 \varphi(q) = p_2 (\alpha(q), \beta(q)) \]
\[ = \beta(q) \]

for all \( q \in Q \) that proves the commutativity of (2).

Consider \( \varphi' \) with the same property as \( \varphi \), i.e. the following conditions hold:

\[ p_1 \varphi' = \alpha \]
\[ p_2 \varphi' = \beta. \]

Define \((p, r) \in P \times R\) by \( \varphi'(q) = (p, r) \). We get:

\[ p_1 \varphi'(q) = \alpha(q) \Rightarrow p_1 (p, r) = \alpha(q) \]
\[ \Rightarrow p = \alpha(q) \]
\[ p_2 \varphi'(q) = \beta(q) \Rightarrow p_2 (p, r) = \beta(q) \]
\[ \Rightarrow r = \beta(q) \]

for all \( q \in Q \) which yields:

\[ \varphi'(q) = (p, r) \]
\[ = (\alpha(q), \beta(q)) \]
\[ = \varphi(q) \]

and proves that \( \varphi \) is unique.

**Theorem 3.** The category of racks has equalizers.

**Proof.** Let \( f, g : P \rightarrow R \) be two rack morphisms. Define the set:

\[ Q = \{ p \in P \mid f(p) = g(p) \}. \]
$Q$ is a subrack of $P$ since:

\[
f(p \triangleleft p') = f(p) \triangleleft f(p') \\
= g(p) \triangleleft g(p') \\
= g(p \triangleleft p')
\]

for all $p, p' \in P$.

Also the inclusion morphism $u : Q \to P$ is a rack morphism since:

\[
u(p \triangleleft p') = p \triangleleft p' \\
= u(p) \triangleleft u(p')
\]

for all $p, p' \in Q$. Furthermore for all $p \in Q$, we have:

\[
(fu)(p) = f(p) \\
= g(p) \\
= (gu)(p)
\]

and get:

\[
fu = gu.
\]

Let $T$ be any rack with a rack morphism $v : T \to P$ where:

\[
fv = gv.
\]

Then there must be a unique rack morphism:

\[
\phi : T \to Q
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
Q' & \xrightarrow{u} & P & \xrightarrow{f} & R \\
\downarrow & & \downarrow & & \downarrow \\
T & \xrightarrow{\exists \phi} & & & \\
\end{array}
\]

(3)

We can say that $v(t) \in Q$ since:

\[
f(v(t)) = g(v(t))
\]

for all $t \in T$. Define $\phi$ by $\phi(t) = v(t)$ for all $t \in T$. Then we get:

\[
u\phi(t) = uv(t) \\
= v(t)
\]

for all $t \in T$ that satisfies $u\phi = v$ and proves the commutativity of (3).
Consider \( \phi' \) with the same property as \( \phi \), i.e. \( u\phi' = v \). Define \( q \in Q \) by \( \phi'(t) = q \). We get:

\[
u \phi'(t) = v(t) \Rightarrow u(q) = v(t) \Rightarrow q = v(t)
\]

for all \( t \in T \) which yields:

\[
\phi'(t) = q = v(t) = \phi(t)
\]

and proves that \( \phi \) is unique.

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Competing interests

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Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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