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Sumudu decomposition method for solving fractional Riccati equation

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Abstract: In This paper, we look forward to adapting and using Sumudu decomposition method (SDM) as a way to get approximate solutions to the fractional Riccati equation, we propose a numerical algorithm for solving fractional Riccati equation.This method is a combination of the Sumudu transform method and decomposition method. We have applied the concepts of fractional calculus too the well known population growth modled in chaotic dynamic. The fractional derivative is described in the Caputo sense.The numerical results show that the approach is easy to implement and accurate when applied to various fractional differential equations.

Keywords: Caputo derivative, Adomian polynomials, Sumudu transform method, decomposition method, Fractional Riccati equation.

1 Introduction

In recent years, fractional calculus used in many areas such as electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, optics, engineering, acoustics, viscoelasticity, material science and signal processing can be successfully modelled by linear or nonlinear fractional order differential equations [1,2,8,9,10,11,12,14,15,17,19]. As it is well known, Riccati differential equations concerned with applications in pattern formation in dynamic games, linear systems with Markovian jumps, river flows, econometric models, stochastic control, theory, diffusion problems and invariant embedding.
Inspired and motivated by the ongoing research in this area, we introduce a new method called sumudu decomposition method (SDM) [8,22,23] for solving the nonlinear equations in the present paper. It is worth mentioning that the proposed method is an elegant combination of the sumudu transform method and decomposition method which was rst introduced by Adomian [6, 7]. The proposed scheme provides the solution of the problem in a closed form while the mesh point techniques, such as Sumudu decomposition method [4,13,21] The proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. This article considers the effectiveness of the sumudu decomposition method (SDM) in solving nonlinear fractional Riccati differential equations.
The paper is organized as follows: In section 2, we provide the Basic definitions fractional calculus . Section 3, Analysis of the SDM and FRDE. Sections 4, Approximate solution of the FRDE. Sections 4, Conclusion.

2 Basic definitions of fractional calculus

In this section, we present the basic definitions and properties of the fractional calculus theory, which are used further in this paper.

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Definition 1. *[10] The Riemann-Liouville fractional integral operator of order $\alpha > 0$; for $t > 0$ is defined as*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi \tag{1}$$
$$J^0 f(t) = f(t).$$

The Riemann-liouville derivative has certain disadvantage when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by M Caputo in his work on the theory of viscoelasticity [15]

Definition 2. *[16] The Caputo fractional derivative of $f(t)$ of order $\alpha > 0$ with $t > 0$ is defined as*

$$D^{\alpha} f(t) = J^{m-\alpha} D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} f^{(m)}(\xi) d \tag{2}$$

for $m-1 < \alpha \leq m,\ m \in N,\ t > 0.$

Definition 3. *[8] The Sumudu transform is defined over the set of functions*

$$A = \left\{ f(t) \,\middle|\, \begin{matrix} \exists, T_1, T_2 > 0, |f(t)| < M\, e^{\frac{|t|}{\tau_j}} \\ if\ t \in (-1)^j \times [0,\infty) \end{matrix} \right\}, \tag{3}$$

by the following formula,

$$f'(u) = S\left[f(t)\right] = \int_0^1 f(ut)\, e^{-t}\, dt, \quad u \in (T_1, T_2). \tag{4}$$

Definition 4. *[22] The Sumudu transform of Caputo fractional derivative is defined as follows*

$$S[D^{\alpha} f(t)] = u^{\alpha} S[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k}\, f^{(k)}(0), \quad m-1 < \alpha \leq m. \tag{5}$$

3 Analysis of the SDM and FRDE

Example 1. [18] We Consider the Fractional Riccati equation

$$D^{\alpha} y(t) = 2y(t) - y^2(t) + 1 \qquad 0 < t < 1\ , \tag{6}$$

the parameter with α refers the fractional order of time derivatve with $0 < \alpha \leq 1$ and subject to the initial condition

$$y(0) = y^0 = 0 \tag{7}$$

where$D^{\alpha} y(t)$ is the Caputo fractional derivative, $y(t)$ represents the population size, t represents the time. Taking the Sumudu transform (denoted throughout this paper by S) on both sides of we have (6)

$$S[D^{\alpha} y(t)] = S\left[2y(t) - y^2(t) + 1\right], \tag{8}$$

Using the differentiation property of the Sumudu transform and the initial conditions in (8) we have

$$S[y(t)] = y^0 + u^{\alpha} S\left[2y(t) - y^2(t) + 1\right], \tag{9}$$

Operating with the Sumudu inverse on both sides of (9) we get

$$y(t) = F(t) + S^{-1}\left[u^{\alpha} S[2y(t) - y^2(t) + 1]\right], \tag{10}$$

where $F(t)$ represent the prescribed initial conditions. Now, applying SDM. And assuming that the solution of (10) is in the form

$$y(t) = \sum_{m=0}^{\infty} y_m(t), \tag{11}$$

and the nonlinear term of (10) can be decomposed as

$$N\,y(t)=\sum_{m=0}^{\infty}A_m(t), \tag{12}$$

where A_m are He.s polynomials, which can be calculated with the formula [3,21]

$$A_m=\frac{1}{m!}\frac{d^m}{dp^m}\left[N\left(\sum_{i=0}^{\infty}p^i y_i(t)\right)\right]_{p=0}\quad m=0,1,2,... \tag{13}$$

The first five Adomian polynomials for the one variable $N\,y=f(y(t))$are given [5,6]

$$\begin{aligned}
A_0&=f(y_0),\\
A_1&=y_1 f^{'}(y_0),\\
A_2&=y_2 f^{'}(y_0)+\frac{1}{2!}y_1^2 f^{''}(y_0),\\
A_3&=y_3 f^{'}(y_0)+y_1y_2 f^{''}(y_0)+\frac{1}{3!}y_1^3 f^{'''}(y_0),
\end{aligned}$$

$$\begin{aligned}
A_0&=y_0^2,\\
A_1&=2y_0y_1,\\
A_2&=2y_0y_1+y_1y_1,
\end{aligned}$$

Substituting (11) and (12) in (10), we get

$$\sum_{m=0}^{\infty}y_m(t)=F(t)+S^{-1}\left[u^{\alpha}S\left[2\left(\sum_{m=0}^{\infty}y_m(t)\right)-\left(\sum_{m=0}^{\infty}A_m(t)\right)+1\right]\right], \tag{14}$$

On comparing both sides of(14), we get

$$\begin{aligned}
y_0(t)&=F(t),\\
y_1(t)&=S^{-1}\left[u^{\alpha}S\left[2y_0-A_0+1\right]\right]\\
y_2(t)&=S^{-1}\left[u^{\alpha}S\left[2y_1-A_1+1\right]\right]\\
&\vdots\\
y_{m+1}(t)&=S^{-1}\left[u^{\alpha}S\left[2y_m-A_m+1\right]\right],
\end{aligned} \tag{15}$$

3.1 Approximate solution of the FRDE

We start with the initial approximate $y_0(t)=0$; and by using the SDM (25), we can directly obtain the components of the solution. Consequently,the exact solution may be obtained by using(11) On comparing both sides of (25) we get

$$\begin{aligned}
y_0(t)&=0\\
y_1(t)&=\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
y_2(t)&=\frac{2t^{2\alpha}}{\Gamma(2\alpha+1)},\\
y_3(t)&=\frac{4\Gamma^2(\alpha+1)-\Gamma(2\alpha+1))}{\Gamma(3\alpha+1)\Gamma^2(\alpha+1)}t^{3\alpha},
\end{aligned}$$

Thus the solution can be obtained .For example ,when $\alpha = 0.5$, we have the solution as

$$y(t) = \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}} + 2t - \frac{16(\pi-1)}{3\pi^{\frac{3}{2}}} t^{\frac{3}{2}} - \frac{\pi-4}{\pi} t^2 + \ldots$$

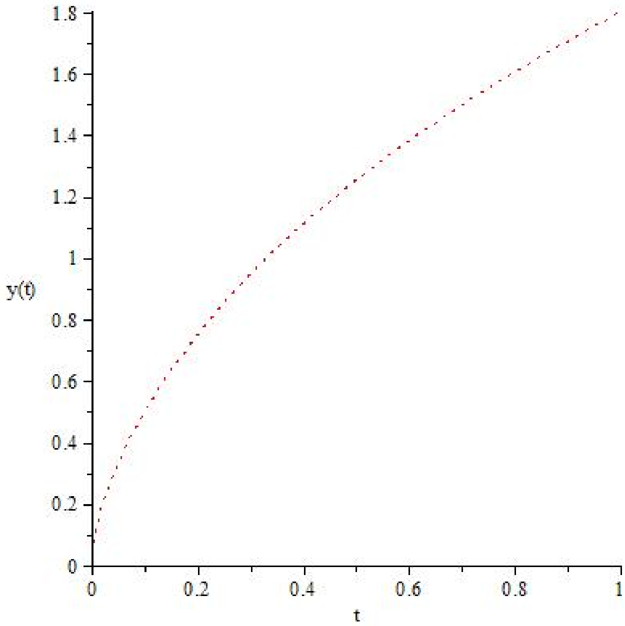

Fig. 1: The behavior of Approximate Solution of y(t) at $\alpha = 0.5$

Example 2. [18] We Consider the Fractional Riccati equation

$$D^{\alpha} y(t) = x^2 + y^2(t) \qquad 0 < t < 1 \tag{16}$$

the parameter with α refers the fractional order of time derivate with $0 < \alpha \leq 1$ and subject to the initial condition

$$y(0) = 1 \tag{17}$$

where$D^{\alpha} y(t)$ is the Caputo fractional derivative, $y(t)$ represents the population size, t represents the time. Taking the Sumudu transform (denoted throughout this paper by S) on both sides of (26), we have

$$S[D^{\alpha} y(t)] = S\left[x^2 + y^2(t)\right], \tag{18}$$

Using the differentiation property of the Sumudu transform and the initial conditions in (28), we have

$$S[y(t)] = y_0 + u^{\alpha} S\left[x^2 + y^2(t)\right], \tag{19}$$

Operating with the Sumudu inverse on both sides of (29) we get

$$y(t) = F(t) + S^{-1}\left[u^{\alpha} S[x^2 + y^2(t)]\right], \tag{20}$$

where $F(t)$ represent the prescribed initial conditions.
Now, applying SDM. And assuming that the solution of (30) is in the form

$$y(t) = \sum_{m=0}^{\infty} y_m(t), \tag{21}$$

and the nonlinear term of (30) can be decomposed as

$$N\, y(t) = \sum_{m=0}^{\infty} A_m(t), \tag{22}$$

The first five Adomian polynomials for the one variable $N\, y = f(y(t))$are given by [5,6]

$$\begin{aligned}
A_0 &= f(y_0),\\
A_1 &= y_1 f'(y_0),\\
A_2 &= y_2 f'(y_0) + \frac{1}{2!} y_1^2 f''(y_0),\\
A_3 &= y_3 f'(y_0) + y_1 y_2 f''(y_0) + \frac{1}{3!} y_1^3 f'''(y_0),
\end{aligned}$$

$$\begin{aligned}
A_0 &= y_0^2,\\
A_1 &= 2y_0 y_1,\\
A_2 &= 2y_0 y_1 + y_1 y_1,
\end{aligned}$$

Substituting (31, 32) in (30), we get

$$\sum_{m=0}^{\infty} y_m(t) = F(t) + S^{-1}\left[u^{\alpha} S\left[x^2 + \left(\sum_{m=0}^{\infty} A_m(t)\right)\right]\right], \tag{23}$$

On comparing both sides of (33), we get

$$\begin{aligned}
y_0(t) &= F(t), \\
y_1(t) &= S^{-1}\left[u^{\alpha} S\left[x^2 + A_0\right]\right],\\
y_2(t) &= S^{-1}\left[u^{\alpha} S\left[x^2 + A_1\right].\right],\\
&\vdots\\
y_{m+1}(t) &= S^{-1}\left[u^{\alpha} S\left[x^2 + A_m\right]\right].
\end{aligned} \tag{24}$$

3.1.1 Approximate solution of the FRDE

We start with the initial approximate $y_0(t) = 1$; and by using the SDM (34), we can directly obtain the components of the solution. Consequently, the exact solution may be obtained by using (31). On comparing both sides of (34) we get

$$\begin{aligned}
y_0(t) &= F(t),\\
y_1(t) &= S^{-1}\left[u^{\alpha} S[2y_0 - A_0 + 1]\right],\\
y_2(t) &= S^{-1}\left[u^{\alpha} S[2y_1 - A_1 + 1]\right],\\
&\vdots\\
y_{m+1}(t) &= S^{-1}\left[u^{\alpha} S[2y_m - A_m + 1]\right].
\end{aligned} \tag{25}$$

Thus the solution can be obtained .For example ,when $\alpha = 0.5$, we have the solution as

$$y(t) = \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} + 2t - \frac{16(\pi-1)}{3\pi^{\frac{3}{2}}}t^{\frac{3}{2}} - \frac{\pi-4}{\pi}t^2 + \ldots$$

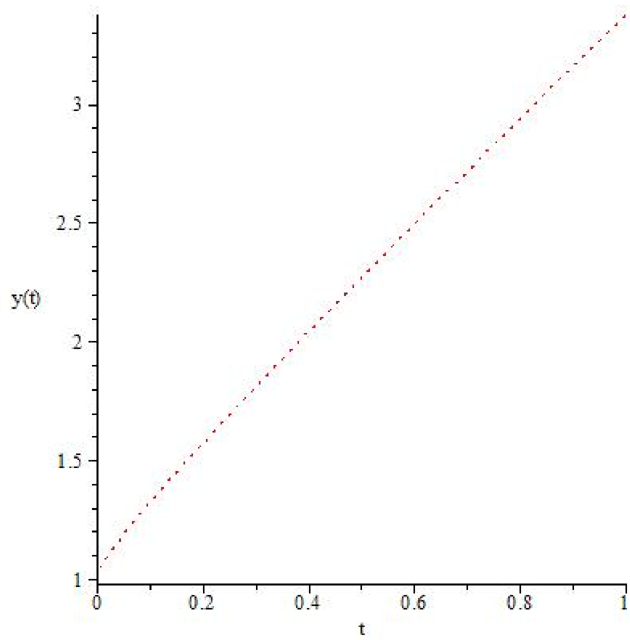

Fig. 2: The behavior of Approximate Solution of y(t) at $\alpha = 0.5$

Example 3. [18] We Consider the Fractional Riccati equation

$$D^{\alpha}y(t) = x^2 + y^2(t) 0 < t < 1, \tag{26}$$

the parameter with α refers the fractional order of time derivate with $0 < \alpha \leq 1$ and subject to the initial condition

$$y(0) = 1 \tag{27}$$

where$D^{\alpha}y(t)$ is the Caputo fractional derivative, $y(t)$ represents the population size, t represents the time.
Taking the Sumudu transform (denoted throughout this paper by S) on both sides of (26), we have

$$S[D^{\alpha}y(t)] = S\left[x^2 + y^2(t)\right]. \tag{28}$$

Using the differentiation property of the Sumudu transform and the initial conditions in (28), we have

$$S[y(t)] = y_0 + u^{\alpha}S\left[x^2 + y^2(t)\right]. \tag{29}$$

Operating with the Sumudu inverse on both sides of (29) we get

$$y(t) = F(t) + S^{-1}\left[u^{\alpha}S[x^2 + y^2(t)]\right], \tag{30}$$

where $F(t)$ represent the prescribed initial conditions.
Now, applying SDM. And assuming that the solution of (30) is in the form

$$y(t) = \sum_{m=0}^{\infty} y_m(t), \tag{31}$$

and the nonlinear term of (30) can be decomposed as

$$N\,y(t) = \sum_{m=0}^{\infty} A_m(t), \tag{32}$$

The first five Adomian polynomials for the one variable $N\,y = f(y(t))$are given by[5,6].

$$\begin{aligned}
A_0 &= f(y_0),\\
A_1 &= y_1 f^{'}(y_0),\\
A_2 &= y_2 f^{'}(y_0) + \frac{1}{2!} y_1^2 f^{''}(y_0),\\
A_3 &= y_3 f^{'}(y_0) + y_1 y_2 f^{''}(y_0) + \frac{1}{3!} y_1^3 f^{'''}(y_0).
\end{aligned}$$

$$\begin{aligned}
A_0 &= y_0^2,\\
A_1 &= 2y_0 y_1,\\
A_2 &= 2y_0 y_1 + y_1 y_1,\\
&\vdots
\end{aligned}$$

Substituting (31) and (32) in (30), we get

$$\sum_{m=0}^{\infty} y_m(t) = F(t) + S^{-1}\left[u^{\alpha} S\left[x^2 + \left(\sum_{m=0}^{\infty} A_m(t)\right)\right]\right]. \tag{33}$$

On comparing both sides of (33), we get

$$\begin{aligned}
y_0(t) &= F(t),\\
y_1(t) &= S^{-1}\left[u^{\alpha} S\left[x^2 + A_0\right]\right],\\
y_2(t) &= S^{-1}\left[u^{\alpha} S\left[x^2 + A_1\right]\right]\\
&\vdots\\
y_{m+1}(t) &= S^{-1}\left[u^{\alpha} S\left[x^2 + A_m\right]\right].
\end{aligned} \tag{34}$$

3.1.2 Approximate solution of the FRDE

We start with the initial approximate $y_0(t) = 1$; and by using the SDM (34), we can directly obtain the components of the solution. Consequently,the exact solution may be obtained by using (31). On comparing both sides of (34) we get

$$y_0(t) = 1 + x^2$$

$$\begin{aligned}
y_1(t) &= S^{-1}\left[u^{\alpha} S\left[x^2 + A_0\right]\right]\\
& S^{-1}\left[u^{\alpha} S\left[x^2 + 1\right]\right]
\end{aligned}$$

$$y_1(t) = \left(x^2+1\right)^2 \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$y_2(t) = S^{-1}\left[u^{\alpha} S\left[x^2 + A_1\right]\right]$$
$$y_2(t) = S^{-1}\left[u^{\alpha} S\left[x^2 + 2\left[x^2+1\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]\right]$$
$$y_2(t) = 2\left(x^2+1\right)^3 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Hence

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots$$
$$y(t) = \left(1+x^2\right)[1 + \left(x^2+1\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2\left(x^2+1\right)^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots]$$

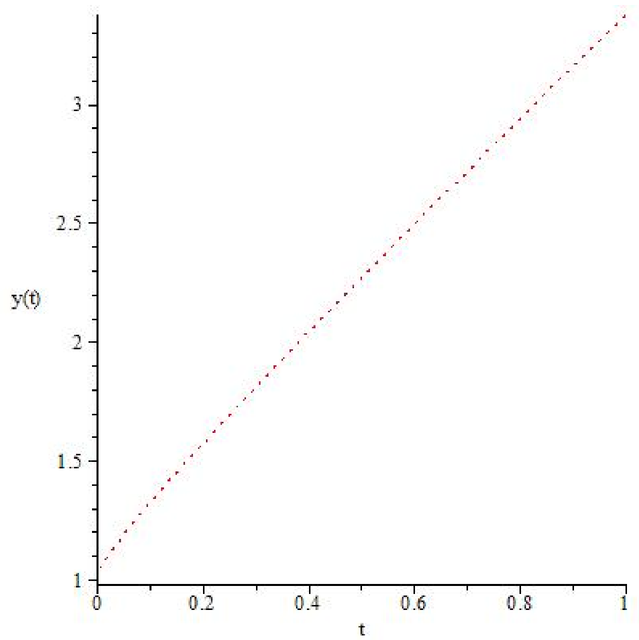

Fig. 3: The behavior of Approximate Solution of y(t) at $\alpha = 0.5$

4 Conclusions

This present analysis exhibits the applicability of the Sumudu decomposition method to solve fractional Riccati equation. The work emphasized our belief that the method is a reliable technique to handle linear and nonlinear fractional differential equations. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, restrictive assumptions. The numerical results obtained with the proposed techniques are in an excellent agreement with the exact solution. All numerical results are obtained using Maple 16.

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