Complete lift of a tensor field of type (1,2) to semi-cotangent bundle

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Abstract: The main purpose of this paper is to define the complete lift of a projectable tensor field of type (1,2) to semi-cotangent bundle \( t^*M \). Using projectable geometric objects on \( M \), we examine lifting problem of projectable tensor field of type (1,2) to the semi-cotangent bundle. We also present the good square in the semi-cotangent bundle \( t^*M \).

Keywords: Complete lift, pull-back bundle, semi-cotangent bundle, vector field.

1 Introduction

Let \( M_n \) be a differentiable manifold of class \( C^\infty \) and finite dimension \( n \), and let \( (M_n, \pi_1, B_m) \) be a differentiable bundle over \( B_m \). We use the notation \((x') = (x^\alpha, x'^\alpha)\), where the indices \( i, j, \ldots \) run from 1 to \( n \), the indices \( a, b, \ldots \) from 1 to \( n - m \) and the indices \( \alpha, \beta, \ldots \) from \( n - m + 1 \) to \( n \). \( x^\alpha \) are coordinates in \( B_m \), \( x^i \) are fibre coordinates of the bundle \( \pi_1 : M_n \rightarrow B_m \).

Let now \((T^* (B_m), \tilde{\pi}, B_m)\) be a cotangent bundle \([1]\) over base space \( B_m \), and let \( M_n \) be differentiable bundle determined by a natural projection (submersion) \( \pi_1 : M_n \rightarrow B_m \). The semi-cotangent bundle (pull-back \([2], [3], [4], [5], [6]\)) of the cotangent bundle \((T^* (B_n), \tilde{\pi}, B_m)\) is the bundle \((t^* (B_m), \pi_2, M_n)\) over differentiable bundle \( M_n \) with a total space

\[
t^* (B_m) = \{ (x^\alpha, x'^\alpha, \bar{x}) \in M_n \times T^*_x (B_m) : \pi_1 (x^\alpha, x'^\alpha) = \tilde{\pi} (x^\alpha, x'^\alpha) = (x'^\alpha) \} \subset M_n \times T^*_x (B_m)
\]

and with the projection map \( \pi_2 : t^* (B_m) \rightarrow M_n \) defined by \( \pi_2 (x^\alpha, x'^\alpha, \bar{x}) = (x^\alpha, x'^\alpha) \), where \( T^*_x (B_m) (x = \pi_1 (\bar{x}), \bar{x} = (x^\alpha, x'^\alpha) \in M_n) \) is the cotangent space at a point \( x \) of \( B_m \), where \( \bar{x} = p_{\bar{\alpha}} \) \( \bar{x} \), \( \bar{x} \), \ldots, \( n + 1, \ldots, 2n \) are fibre coordinates of the cotangent bundle \( T^*(B_m) \).

Where the pull-back (Pontryagin \([7]\)) bundle \( t^* (B_m) \) of the differentiable bundle \( M_n \) also has the natural bundle structure over \( B_m \), its bundle projection \( \pi : t^* (B_m) \rightarrow B_m \) being defined by \( \pi : (x^\alpha, x'^\alpha, \bar{x}) \rightarrow (x'^\alpha) \), and hence \( \pi = \pi_1 \circ \pi_2 \). Thus \( (t^* (B_m), \pi_1 \circ \pi_2) \) is the composite bundle \([8], p.9\) or step-like bundle \([9]\). Consequently, we notice the semi-cotangent bundle \((t^* (B_m), \pi_2)\) is a pull-back bundle of the cotangent bundle over \( B_m \) by \( \pi_1 \) \([6]\).

If \((x') = (x^\alpha, x'^\alpha)\) is another local adapted coordinates in differentiable bundle \( M_n \), then we have

\[
\begin{aligned}
\alpha x'^\alpha &= x'^\alpha (x^\beta), \\
\beta x'^\alpha &= x'^\alpha (x^\beta).
\end{aligned}
\]
The Jacobian of (1) has components
\[
\left( A'_{ij} \right) = \left( \frac{\partial x'^i}{\partial x^j} \right) = \left( \begin{array}{cc} A'_{ib} & A'_{ib} \\ 0 & A'_{ib} \end{array} \right),
\]
where \( A'_{ib} = \frac{\partial x'^i}{\partial x^b} \), \( A'_{ib} = \frac{\partial x'^i}{\partial x^b} \), and \( A'_{ib} = \frac{\partial x'^i}{\partial x^b} \) [6].

To a transformation (1) of local coordinates of \( M_n \), there corresponds on \( t^*(B_m) \) the change of coordinate
\[
\begin{align*}
\begin{cases}
x'^i = x'^i (x^b, x^\beta), \\
x'^i = x'^i (x^b), \\
\partial x'^i = \frac{\partial x^i}{\partial x^b} x^\beta.
\end{cases}
\end{align*}
\]

The Jacobian of coordinate system transformation (2) is:
\[
\tilde{A} = \left( A'_{ij} \right) = \left( \begin{array}{ccc} A'_{ib} & A'_{ib} & 0 \\ 0 & A'_{ib} & 0 \\ 0 & 0 & A'_{ib} \end{array} \right),
\]
where \( I = (a, \alpha, \overline{\alpha}), J = (b, \beta, \overline{\beta}), I, J, \ldots, 1, \ldots, 2n; \ A'_{ib} = \frac{\partial x'^i}{\partial x^b} \frac{\partial x^b}{\partial x^\alpha} \) [6].

Now, consider a diagram as
\[
\begin{array}{ccc}
A & \xrightarrow{\gamma} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{\pi} & D \\
\end{array}
\]

A good square of vector bundles is a diagram as above verifying
(i) \( \alpha \) and \( \beta \) are fibre bundles, but not necessarily vector bundles;
(ii) \( \gamma \) and \( \pi \) are vector bundles;
(iii) the square is commutative, i.e., \( \pi \circ \alpha = \beta \circ \gamma \);
(iv) the local expression
\[
\begin{align*}
& \begin{align*}
A & \xrightarrow{\gamma} B \\
\downarrow & \downarrow \\
C & \xrightarrow{\pi} D \\
\end{align*}
\end{align*}
\]
\[
\begin{align*}
& \begin{align*}
U^n \times R^1 \times G^i \times R^i & \rightarrow U^n \times G^i (x^i, a^\alpha, g^\lambda, \beta^\sigma) \\
\downarrow & \downarrow \\
U^n \times R^1 & \rightarrow U^n (x^i, a^\alpha) \\
\end{align*}
\end{align*}
\]

where \( G \) is a manifold and superindices denote the dimension of the manifolds [11].

By means of above definition, we have

**Theorem 1.** Let now \( \pi : t^*(B_m) \rightarrow B_m \) be a semi-cotangent bundle and \( \pi_1 : M_n \rightarrow B_m \) be a fibre bundle. Then, the following is a good square:
\[
\begin{align*}
&t^*(B_m) \xrightarrow{\tilde{\pi}_1} M_n \times T^*_(B_m) \xrightarrow{\tilde{\pi}_1} M_n (x^a, x^\alpha, x^\overline{\alpha}) \\
&\quad \downarrow \quad \downarrow \\
&t^*(B_m) \xrightarrow{\pi} B_m \times T^*_(B_m) \rightarrow B_m (x^a, x^\alpha, x^\overline{\alpha}) \\
&\quad \downarrow \quad \downarrow \\
&\end{align*}
\]

In this study, we continue to study the complete lifts of projectable tensor field of type (1,2) to semi-cotangent (pull-back) bundle (\( t^*(B_m) \), \( \pi_2 \)) initiated by F. Yildirim and A. Salimov [6].

We denote by \( \mathcal{S}_p^q(M_n) \) the set of all tensor fields of class \( C^\infty \) and of type \( (p, q) \) on \( M_n \), i.e., contravariant degree \( p \) and
covariant degree \( q \). We now put \( \mathfrak{S}(M_n) = \sum_{p,q=0}^n \bar{\mathfrak{S}}_{pq}(M_n) \), which is the set of all tensor fields on \( M_n \). Similarly, we denote by \( \bar{\mathfrak{S}}_{pq}(B_m) \) and \( \mathfrak{S}(B_m) \) respectively the corresponding sets of tensor fields in the base space \( B_m \).

Let \( \omega \) be a 1–form with local components \( \omega_{\alpha} \) on \( B_m \), so that \( \omega \) is a 1–form with local expression \( \omega = \omega_{\alpha} dx^\alpha \). On putting [6]

\[
\nu^\alpha \omega = \begin{pmatrix} 0 \\ 0 \\ \omega_{\alpha} \end{pmatrix},
\]

we have a vector field \( \nu^\alpha \omega \) on \( t^*(B_m) \). In fact, from (3) we easily see that \( (\nu^\alpha \omega)' = \bar{\omega}(\nu^\alpha) \). We call the vector field \( \nu^\alpha \omega \) the vertical lift of the 1–form \( \omega \) to \( t^*(B_m) \).

Let \( \bar{X} \in \mathfrak{S}_1(\bar{M}_n) \) be a projectable vector field [10] with projection \( X = X^a(x^\alpha) \partial_a \).\( \bar{X} = \bar{X}^a(x^\alpha, x^\beta) \partial_a + X^a(x^\alpha) \partial_a \). Now, consider \( \bar{X} \in \mathfrak{S}_1(\bar{M}_n) \), then \( \bar{c}c \bar{X} \) (complete lift) has components on the semi-cotangent bundle \( t^*(B_m) \) [6]

\[
\bar{c}c \bar{X} = \begin{pmatrix} \bar{X}^a \\ -p_e(\partial_a x^e) \end{pmatrix}
\]

with respect to the coordinates \((x^a, x^\alpha, x^\beta)\).

2 \( \gamma \)--operators

For any \( F \in \mathfrak{S}_1(\bar{B}_m) \), if we take account of (3), we can prove that \( (\gamma F)' = \bar{A}(\gamma F) \), where \( \gamma F \) is a vector field defined by [6]:

\[
\gamma F = (\gamma F)' = \begin{pmatrix} 0 \\ 0 \\ p^b F^a_{\beta} \end{pmatrix}
\]

with respect to the coordinates \((x^a, x^\alpha, x^\beta)\) on \( t^*(B_m) \).

For any \( R \in \mathfrak{S}_1(\bar{B}_m) \), if we take account of (3), we can prove that \( \gamma R_{\alpha\beta} = A^{K}_{\alpha} A^I_{\beta} A^J_{\gamma} \gamma R_{IJ}^K \), where \( \gamma R \) has components \( R_{\alpha\beta}^K \) such that

\[
R_{\alpha\beta}^K = p^e R_{\alpha\beta}^e,
\]

all the others being zero, with respect to the induced coordinates on \( t^*(B_m) \). Where \( R_{\alpha\beta}^e \) are local components of \( R \) on \( B_m \) and also \( I = (a, \alpha, \bar{x}), J = (b, \beta, \bar{y}), K = (c, \gamma, \bar{z}) \).

**Theorem 2.** If \( \bar{X} \) and \( \bar{Y} \) be a projectable vector fields on \( M_n \) with projection \( X \in \mathfrak{S}_1(\bar{B}_m) \) and \( Y \in \mathfrak{S}_1(\bar{B}_m) \). We have

(i) \( (\gamma R)(\nu^\alpha \bar{X}, \nu^\beta \bar{Y}) = \gamma(R(X, Y)) \),
(ii) \( (\gamma R)(\nu^\alpha \omega, \nu^\beta \theta) = 0 \),
(iii) \( (\gamma R)(\nu^\alpha \omega, \nu^\beta \theta) = 0 \),
(iv) \( (\gamma R)(\nu^\alpha \omega, \gamma G) = 0 \),
(v) \( (\gamma R)(\nu^\alpha Y, \gamma G) = 0 \),
(vi) \( (\gamma R)(\nu^\alpha F, \gamma G) = 0 \)

for any \( \omega, \theta \in \mathfrak{S}_1(\bar{B}_m), F, G \in \mathfrak{S}_1(\bar{B}_m) \) and \( R \in \mathfrak{S}_1(\bar{B}_m) \).
Proof. (i) If \( R \in \mathcal{Z}_1(B_m) \), \( \tilde{X} \) and \( \tilde{Y} \) be a projectable vector fields on \( M_n \) with projection \( X, Y \in \mathcal{Z}_0(B_m) \) and

\[
\begin{pmatrix}
[(\gamma R)(cc,cc)X,ccY)]^c \\
[(\gamma R)(cc,cc)X,ccY)]^Y \\
[(\gamma R)(cc,cc)X,ccY)]^T
\end{pmatrix}
\]

are components of \([\gamma R](cc,cc)X,ccY)]^K\) with respect to the coordinates \((x^c,x^y,x^\overline{y})\) on \( t^*(B_m) \), then for \( K = \alpha \), we have

\[
[(\gamma R)(cc,cc)X,ccY)]^c = (R_{a\beta}^c)c\tilde{X}^a cc^\gamma = 0
\]

because of (5) and (7). For \( K = \gamma \), we have

\[
[(\gamma R)(cc,cc)X,ccY)]^Y = (R_{a\beta}^\gamma)cc\tilde{X}^a cc^\gamma = 0
\]

because of (5) and (7). For \( K = \overline{\gamma} \), we have

\[
[(\gamma R)(cc,cc)X,ccY)]^T = (R_{a\beta}^\overline{\gamma})cc\tilde{X}^a cc^\gamma = P_k R_{a\beta}^\overline{\gamma} cc^\gamma P_{\overline{\gamma}} = P_k (R(X,Y))^{\overline{\gamma}}
\]

because of (5) and (7). It is well known that \( \gamma(R(X,Y)) \) have components

\[
\gamma(R(X,Y)) = \begin{pmatrix} 0 \\ 0 \\ P_k (R(X,Y))^\overline{\gamma} \end{pmatrix}
\]

with respect to the coordinates \((x^c,x^y,x^\overline{y})\) on \( t^*(B_m) \). Thus, we have \( (\gamma R)(cc,cc)X,ccY) = \gamma(R(X,Y)) \). Similarly, we can easily compute another equations of Theorem 2.

3 Complete lift of a tensor field of type (1,2) to semi-cotangent bundle

Let \( \tilde{S} \in \mathcal{Z}_1(M_n) \) be a projectable tensor field of type (1,2) with projection \( S = S^k_{ij}(x^c,x^\alpha \partial_k \otimes dx^i \otimes dx^j) \), i.e. \( \tilde{S} \) has components such that

\[
cc^S_{a\beta} = S^c_{a\beta}
\]

with respect to the coordinates on \( M_n \). Where \( i = (a, \alpha), j = (b, \beta), k = (c, \gamma) \).

If we take account of (3), we can prove that \( cc^\tilde{S}^k_{j'i'} = A_{k'}^k A_{j'}^j cc^S_{ij} \), where \( cc^\tilde{S} \) has components \( cc^S_{ij} \) such that

\[
\begin{align*}
cc^\tilde{S}^c_{a\beta} &= S^c_{a\beta} \\
cc^\tilde{S}^y_{a\beta} &= S^y_{a\beta} \\
cc^\tilde{S}^\overline{\gamma}_{a\beta} &= -p_k (\partial_a S^c_{\gamma k} + \partial_\beta S^c_{\gamma a} + \partial_y S^c_{\gamma a} ) \\
cc^\tilde{S}^\gamma_{a\beta} &= S^\gamma_{a\beta} \\
cc^\tilde{S}^{\overline{\gamma}}_{a\beta} &= S^{\overline{\gamma}}_{a\beta}
\end{align*}
\]

all the others being zero, with respect to the induced coordinates on \( t^*(B_m) \). Where \( S_{ij}^K \) are local components of \( S \) on \( M_n \) and also \( I = (a, \alpha, \overline{\alpha}), J = (b, \beta, \overline{\beta}), K = (c, \gamma, \overline{\gamma}) \).
Proof. For convenience sake we only consider \( cc\tilde{S}_{\beta\gamma}^\gamma \). In fact,
\[
cc\tilde{S}_{\beta\gamma}^\gamma = A^\gamma_\alpha A^\alpha_\beta A^\beta_\gamma \cdot cc\tilde{S}_{\gamma\beta}^\gamma = A^\gamma_\alpha A^\alpha_\beta A^\beta_\gamma \cdot S_{\gamma\beta} = S_{\gamma\beta}'.
\]
Thus, we have \( cc\tilde{S}_{\beta\gamma}^\gamma = S_{\beta\gamma}' \). Similarly, from (3) and (8), we can easily find all other components of \( cc\tilde{S}_{ij}^k \) equal to zero, where \( I = (a,\alpha,\pi), J = (b,\beta,\bar{\beta}), K = (c,\gamma,\bar{\gamma}). \)

**Theorem 3.** Let \( \tilde{S} \in \mathcal{S}_1^1(M_n) \) be a projectable tensor field of type \((1,2)\). If \( \tilde{X},\tilde{Y} \in \mathcal{S}_0^1(M_n), \omega,\theta \in \mathcal{S}_0^1(B_m), F, G \in \mathcal{S}_1^1(B_m) \) then

(i) \( cc\tilde{S}^{(\nu\nu)}(\omega,\nu\theta) = 0 \),
(ii) \( cc\tilde{S}^{(\nu\nu)}(\omega,\nu\gamma) = 0 \),
(iii) \( cc\tilde{S}^{(\nu\nu)}(\omega,\nu\bar{\gamma}) = -\nu(\omega \circ S_Y) \),
(iv) \( cc\tilde{S}(\nu\nu,\nu\gamma) = 0 \),
(v) \( cc\tilde{S}(\nu\nu,\nu\bar{\gamma}) = -\nu(F \circ S_Y) \),
(vi) \( cc\tilde{S}(\nu\nu,\nu\bar{\gamma}) = cc(S(X,Y)) = \gamma(L_XS_Y - (L_YS)_X + S_{X,Y}) \),

where \( L_XS \) denotes the Lie derivative of \( S \) with respect to \( X \).

**Proof.** (i) If \( \omega,\theta \in \mathcal{S}_0^1(B_m) \) and \( \tilde{S} \) is projectable tensor field of type \((1,2)\) on \( M_n \) with projection \( S \in \mathcal{S}_1^1(B_m) \) and

\[
\begin{pmatrix}
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\theta) \\
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\gamma) \\
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\bar{\gamma})
\end{pmatrix}^c
\]

are components of \( cc\tilde{S}^{(\nu\nu)}(\omega,\nu\theta) \) with respect to the coordinates \((x^c, x^\gamma, x^\bar{\gamma})\) on \( t^*(B_m) \), then we have

\[
\begin{pmatrix}
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\theta) \\
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\gamma) \\
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\bar{\gamma})
\end{pmatrix}^K = cc\tilde{S}_{ij}^k(\omega^{ij} \theta^J) = cc\tilde{S}_{\beta\gamma}^\gamma(\omega^{\nu\nu} \theta^\pi) = cc\tilde{S}_{\gamma\beta}^\gamma(\omega_\alpha \omega_\beta).
\]

Firstly, if \( K = c \), we have

\[
\begin{pmatrix}
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\theta) \\
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\gamma) \\
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\bar{\gamma})
\end{pmatrix}^c = cc\tilde{S}_{\gamma\beta}^\gamma(\omega_\alpha \omega_\beta) = 0
\]

by virtue of (4) and (8). Secondly, if \( K = \gamma \), we have

\[
\begin{pmatrix}
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\theta) \\
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\gamma) \\
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\bar{\gamma})
\end{pmatrix}^\gamma = cc\tilde{S}_{\gamma\beta}^\gamma(\omega_\alpha \omega_\beta) = 0
\]

by virtue of (4) and (8). Thirdly, if \( J = \bar{\beta} \), then we have

\[
\begin{pmatrix}
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\theta) \\
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\gamma) \\
cc\tilde{S}^{(\nu\nu)}(\omega,\nu\bar{\gamma})
\end{pmatrix}^\bar{\beta} = cc\tilde{S}_{\gamma\beta}^\gamma(\omega_\alpha \omega_\beta) = 0
\]

by virtue of (4) and (8). Thus (i) of Theorem 3 is proved.
(ii) If $G \in \mathcal{G}_1(B_m)$ and $\tilde{S}$ is projectable tensor field of type $(1,2)$ on $M_n$ with projection $S \in \mathcal{G}_1(B_m)$ and

$$
\begin{pmatrix}
\left(\tilde{c}\tilde{c}S^{(\nu)\omega, \gamma G}\right)^c \\
\left(\tilde{c}\tilde{c}S^{(\nu)\omega, \gamma G}\right)^\gamma \\
\left(\tilde{c}\tilde{c}S^{(\nu)\omega, \gamma G}\right)^\gamma
\end{pmatrix}
$$

are components of $\left(\tilde{c}\tilde{c}S^{(\nu)\omega, \gamma G}\right)^K$ with respect to the coordinates $(x^c, x^\gamma, x^\beta)$ on $t^*(B_m)$, then we have

$$
\left(\tilde{c}\tilde{c}S^{(\nu)\omega, \gamma G}\right)^K = \tilde{c}\tilde{S}_{IJ}^{K^\nu\omega} G^I = \tilde{c}\tilde{S}_{\alpha\beta}^{K^\nu\omega} G^\beta = \tilde{c}\tilde{S}_{\alpha\beta}^{K^\nu\omega} G^e .
$$

Firstly, if $K = c$, we have

$$
\left(\tilde{c}\tilde{S}^{(\nu)\omega, \gamma G}\right)^c = \tilde{c}\tilde{S}_{\alpha\beta}^{c^\nu\omega} G^e = 0
$$

by virtue of (4), (6) and (8). Secondly, if $K = \gamma$, we have

$$
\left(\tilde{c}\tilde{S}^{(\nu)\omega, \gamma G}\right)^\gamma = \tilde{c}\tilde{S}_{\alpha\beta}^{\gamma^\nu\omega} G^e = 0
$$

by virtue of (4), (6) and (8). Thirdly, if $J = \tilde{\beta}$, then we have

$$
\left(\tilde{c}\tilde{S}^{(\nu)\omega, \gamma G}\right)^\beta = \tilde{c}\tilde{S}_{\alpha\beta}^{\gamma^\nu\omega} G^e = 0
$$

by virtue of (4), (6) and (8). Thus (ii) of Theorem 3 is proved.

(iii) If $\tilde{Y} \in \mathcal{G}_1(M_n)$ and $\tilde{S}$ is projectable tensor field of type $(1,2)$ on $M_n$ with projection $S \in \mathcal{G}_1(B_m)$ and

$$
\begin{pmatrix}
\left(\tilde{c}\tilde{S}^{(\nu)\omega, \gamma \tilde{Y}}\right)^c \\
\left(\tilde{c}\tilde{S}^{(\nu)\omega, \gamma \tilde{Y}}\right)^\gamma \\
\left(\tilde{c}\tilde{S}^{(\nu)\omega, \gamma \tilde{Y}}\right)^\gamma
\end{pmatrix}
$$

are components of $\left(\tilde{c}\tilde{S}^{(\nu)\omega, \gamma \tilde{Y}}\right)^K$ with respect to the coordinates $(x^c, x^\gamma, x^\beta)$ on $t^*(B_m)$, then we have

$$
\left(\tilde{c}\tilde{S}^{(\nu)\omega, \gamma \tilde{Y}}\right)^K = \tilde{c}\tilde{S}_{IJ}^{K^\nu\omega} \tilde{Y}^I = \tilde{c}\tilde{S}_{\alpha\beta}^{K^\nu\omega} \tilde{Y}^\beta + \tilde{c}\tilde{S}_{\alpha\beta}^{K^\nu\omega} \tilde{Y}^e .
$$

Firstly, if $K = c$, we have

$$
\left(\tilde{c}\tilde{S}^{(\nu)\omega, \gamma \tilde{Y}}\right)^c = \tilde{c}\tilde{S}_{\alpha\beta}^{c^\nu\omega} \tilde{Y}^e = 0
$$
by virtue of (4), (5) and (8). Secondly, if $K = \gamma$, we have
\[
\left( c \tilde{S}^{(\gamma)}(\gamma, \gamma) \right)^{\gamma} = \frac{c \tilde{\alpha}_{\gamma}^{\gamma}}{0} (\gamma, \gamma) + \frac{c \tilde{\beta}_{\gamma}^{\gamma}}{0} (\gamma, \gamma) + \frac{c \tilde{\gamma}_{\gamma}^{\gamma}}{0} (\gamma, \gamma) = 0
\]
by virtue of (4), (5) and (8). Thirdly, if $K = \gamma$, then we have
\[
\left( c \tilde{S}^{(\gamma)}(\gamma, \gamma) \right)^{\gamma} = \frac{c \tilde{\alpha}_{\gamma}^{\gamma}}{0} (\gamma, \gamma) + \frac{c \tilde{\beta}_{\gamma}^{\gamma}}{0} (\gamma, \gamma) + \frac{c \tilde{\gamma}_{\gamma}^{\gamma}}{0} (\gamma, \gamma) = 0
\]
by virtue of (4), (5) and (8). On the other hand, we know that $\gamma^\gamma(\omega \circ S)$ have components
\[
\gamma^\gamma(\omega \circ S) = \begin{pmatrix} 0 \\ 0 \\ (\omega \circ S)_{\gamma} \end{pmatrix}
\]
with respect to the coordinates $(x^c, x^\gamma, x^\beta)$ on $t^*(B_m)$. Thus, we have $c \tilde{S}^{(\gamma)}(\gamma, \gamma) = -\gamma^\gamma(\omega \circ S)$.

(iv) If $F, G \in \mathcal{S}_1(B_m)$ and $\tilde{S}$ is projectable tensor field of type $(1, 2)$ on $M_\omega$ with projection $S \in \mathcal{S}_2(B_m)$ and
\[
\left( c \tilde{S}(\gamma F, \gamma G) \right)^c = \left( c \tilde{S}(\gamma F, \gamma G) \right)^\gamma = \left( c \tilde{S}(\gamma F, \gamma G) \right)^\beta
\]
are components of $\left( c \tilde{S}(\gamma F, \gamma G) \right)^K$ with respect to the coordinates $(x^c, x^\gamma, x^\beta)$ on $t^*(B_m)$, then we have
\[
\left( c \tilde{S}(\gamma F, \gamma G) \right)^K = c \tilde{S}_{\gamma}^{K \gamma} F^{\gamma} G^{\gamma} = c \tilde{S}_{\gamma}^{K \gamma} (\gamma F)(\gamma G) = c \tilde{S}_{\gamma}^{K \gamma} (p_c F^c_a) (p_c G^c_b).
\]
Firstly, if $K = c$, we have
\[
\left( c \tilde{S}(\gamma F, \gamma G) \right)^c = c \tilde{S}_{\gamma}^{K \gamma} (p_c F^c_a) (p_c G^c_b) = 0
\]
by virtue of (6) and (8). Secondly, if $K = \gamma$, we have
\[
\left( c \tilde{S}(\gamma F, \gamma G) \right)^\gamma = c \tilde{S}_{\gamma}^{K \gamma} (p_c F^c_a) (p_c G^c_b) = 0
\]
by virtue of (6) and (8). Thirdly, if $J = \beta$, then we have
\[
\left( c \tilde{S}(\gamma F, \gamma G) \right)^\beta = c \tilde{S}_{\gamma}^{K \gamma} (p_c F^c_a) (p_c G^c_b) = 0
\]
by virtue of (6) and (8). Thus (iv) of Theorem 3 is proved.

(v) If \( \tilde{\gamma} \in \mathcal{S}_0^1(M_n) \) and \( \tilde{S} \) is projectable tensor field of type \((1,2)\) on \( M_n \) with projection \( S \in \mathcal{S}_1^1(B_m) \) and

\[
\left( \begin{array}{c}
\left( \begin{array}{c}
(\varepsilon \tilde{S}(\gamma F, \varepsilon \tilde{Y})^c) \\
(\varepsilon \tilde{S}(\gamma F, \varepsilon \tilde{Y})^\gamma) \\
(\varepsilon \tilde{S}(\gamma F, \varepsilon \tilde{Y})^\tau)
\end{array} \right)
\end{array} \right)
\]

are components of \( \left( \varepsilon \tilde{S}(\gamma F, \varepsilon \tilde{Y})^K \right)^c \) with respect to the coordinates \((x^c, x^\gamma, x^\tau)\) on \( t^*(B_m) \), then we have

\[
\left( \varepsilon \tilde{S}(\gamma F, \varepsilon \tilde{Y})^K \right)^c = \varepsilon \tilde{S}^{cK}_{\varepsilon F}(\gamma F)^c \left( \varepsilon \tilde{Y} \right)^b + \varepsilon \tilde{S}^{\gamma K}_{\varepsilon F}(\gamma F)^\gamma \left( \varepsilon \tilde{Y} \right)^\beta + \varepsilon \tilde{S}^{\tau K}_{\varepsilon F}(\gamma F)^\tau \left( \varepsilon \tilde{Y} \right)^\bar{\beta}.
\]

Firstly, if \( K = \varepsilon \), we have

\[
\left( \varepsilon \tilde{S}(\gamma F, \varepsilon \tilde{Y})^\varepsilon \right)^c = \varepsilon \tilde{S}^{\varepsilon \gamma}_{\varepsilon F}(\gamma F)^c \left( \varepsilon \tilde{Y} \right)^b = 0
\]

by virtue of (5), (6) and (8). Secondly, if \( K = \gamma \), we have

\[
\left( \varepsilon \tilde{S}(\gamma F, \varepsilon \tilde{Y})^\gamma \right)^\gamma = \varepsilon \tilde{S}^{\varepsilon \gamma}_{\varepsilon F}(\gamma F)^\gamma \left( \varepsilon \tilde{Y} \right)^b + \varepsilon \tilde{S}^{\gamma \gamma}_{\varepsilon F}(\gamma F)^\gamma \left( \varepsilon \tilde{Y} \right)^\beta + \varepsilon \tilde{S}^{\tau \gamma}_{\varepsilon F}(\gamma F)^\gamma \left( \varepsilon \tilde{Y} \right)^\bar{\beta} = 0
\]

by virtue of (5), (6) and (8). Thirdly, if \( K = \tau \), then we have

\[
\left( \varepsilon \tilde{S}(\gamma F, \varepsilon \tilde{Y})^\tau \right)^\gamma = \varepsilon \tilde{S}^{\varepsilon \gamma}_{\varepsilon F}(\gamma F)^\tau \left( \varepsilon \tilde{Y} \right)^b + \varepsilon \tilde{S}^{\gamma \gamma}_{\varepsilon F}(\gamma F)^\tau \left( \varepsilon \tilde{Y} \right)^\beta + \varepsilon \tilde{S}^{\tau \gamma}_{\varepsilon F}(\gamma F)^\tau \left( \varepsilon \tilde{Y} \right)^\bar{\beta}
\]

\[
= -\varepsilon \tilde{S}^{\varepsilon \gamma}_{\varepsilon F} F_{\alpha}^\varepsilon Y^\alpha - p_{\varepsilon}(F \circ S_{\gamma})^\varepsilon_{\gamma}
\]

by virtue of (5), (6) and (8). On the other hand, we know that \( \gamma(F \circ S_{\gamma}) \) have components

\[
\gamma(F \circ S_{\gamma}) = \begin{pmatrix}
0 \\
p_{\varepsilon}(F \circ S_{\gamma})^\varepsilon_{\gamma}
\end{pmatrix}
\]

with respect to the coordinates \((x^c, x^\gamma, x^\tau)\) on \( t^*(B_m) \). Thus, we have \( \varepsilon \tilde{S}(\gamma F, \varepsilon \tilde{Y}) = -\gamma(F \circ S_{\gamma}) \).

(vi) If \( \tilde{X}, \tilde{Y} \in \mathcal{S}_0^1(M_n) \) and \( \tilde{S} \) is projectable tensor field of type \((1,2)\) on \( M_n \) with projection \( S \in \mathcal{S}_1^1(B_m) \) and

\[
\left( \begin{array}{c}
\left( \begin{array}{c}
(\varepsilon \tilde{S}(\varepsilon \tilde{X}, \varepsilon \tilde{Y})^c) \\
(\varepsilon \tilde{S}(\varepsilon \tilde{X}, \varepsilon \tilde{Y})^\gamma) \\
(\varepsilon \tilde{S}(\varepsilon \tilde{X}, \varepsilon \tilde{Y})^\tau)
\end{array} \right)
\end{array} \right)
\]

are components of \( (\alpha \tilde{S}(\alpha X, \alpha Y))^K \) with respect to the coordinates \((x^\alpha, x^\beta, x^\gamma)\) on \( t^*(B_m) \), then we have

\[
(\alpha \tilde{S}(\alpha X, \alpha Y))^K = \alpha \tilde{S}_J^K \left( \alpha X^J \right) \left( \alpha Y \right)^K + \alpha \tilde{S}_{\alpha \beta}^K \left( \alpha X^\alpha \right)^\beta + \alpha \tilde{S}_{\alpha \gamma}^K \left( \alpha X^\alpha \right)^\gamma.
\]

Firstly, if \( K = \alpha \), we have

\[
\left( \alpha \tilde{S}(\alpha X, \alpha Y) \right)^\alpha = \alpha \tilde{S}_{\alpha \beta} \left( \alpha X^\alpha \right)^\alpha \left( \alpha Y \right)^\alpha + \alpha \tilde{S}_{\alpha \gamma} \left( \alpha X^\alpha \right)^\gamma \left( \alpha Y \right)^\gamma.
\]

by virtue of (5) and (8). Secondly, if \( K = \gamma \), we have

\[
\left( \alpha \tilde{S}(\alpha X, \alpha Y) \right)^\gamma = \alpha \tilde{S}_{\alpha \beta} \left( \alpha X^\gamma \right)^\beta \left( \alpha Y \right)^\gamma + \alpha \tilde{S}_{\alpha \beta} \left( \alpha X^\gamma \right)^\beta \left( \alpha Y \right)^\gamma.
\]

by virtue of (5) and (8). Thirdly, if \( K = \beta \), then we have

\[
\left( \alpha \tilde{S}(\alpha X, \alpha Y) \right)^\beta = -p_e (\partial_{\alpha \beta} \alpha X^\beta Y^e) - p_e (\partial_{\alpha \gamma} \alpha X^\gamma) - p_e (\partial_{\alpha \beta} \alpha X^\beta Y^e) - p_e (\partial_{\alpha \gamma} \alpha X^\gamma) - p_e (\partial_{\alpha \beta} \alpha X^\beta Y^e) - p_e (\partial_{\alpha \gamma} \alpha X^\gamma) - p_e (\partial_{\alpha \beta} \alpha X^\beta Y^e) - p_e (\partial_{\alpha \gamma} \alpha X^\gamma).
\]

by virtue of (5) and (8). We know that \( \alpha \tilde{S}(\alpha X, \alpha Y)^\gamma, \alpha \tilde{S}(\alpha X, \alpha Y)^\alpha, \alpha \tilde{S}(\alpha X, \alpha Y)^\beta \) have respectively, components on \( t^*(B_m) \).
with respect to the coordinates \((x^c, x^γ, x^γ)\). Where the same equations are denoted by \(A_1, A_2, \ldots, A_9\). On the other hand, we know that 

\[
\gamma((L_X S)_Y - (L_Y S)_X + S_{[X,Y]})
\]

have respectively, components

\[
\begin{pmatrix}
(S(X,Y))^c \\
(S(X,Y))^γ \\
-p_ε∂_γ(S(X,Y))^c
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 \\
0 \\
p_α((L_X S)_Y - (L_Y S)_X + S_{[X,Y]})^γ
\end{pmatrix}
\]

with respect to the coordinates \((x^c, x^γ, x^γ)\) on \(t^*(B_m)\). Thus, we have

\[
\gamma((L_X S)_Y - (L_Y S)_X + S_{[X,Y]}) = \gamma((L_X S)_Y - (L_Y S)_X + S_{[X,Y]})^c
\]

by the necessary simplifications made in equalities.

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

**References**


