On initial-boundary value problem for nonlinear integro-differential equations with variable exponents of nonlinearity

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Abstract: Some nonlinear parabolic integro-differential equations with variable exponents of the nonlinearity are considered. The initial-boundary value problem for these equations is investigated and the existence theorem for the problem is proved.

Keywords: Nonlinear parabolic equation, integro-differential equation, generalized Lebesgue space, variable exponents of nonlinearity.

1 Introduction

Let \( n \in \mathbb{N} \) and \( T > 0 \) be fixed numbers, \( \Omega \subset \mathbb{R}^n \) be a bounded domain with the boundary \( \partial \Omega \), \( Q_{0,T} := \Omega \times (0,T) \), \( \Sigma_{0,T} := \partial \Omega \times (0,T) \), \( \Omega_{\tau} := \{ (x,t) \mid x \in \Omega, t = \tau \}, \tau \in \mathbb{R} \). We seek a weak solution \( u : Q_{0,T} \rightarrow \mathbb{R} \) of the problem

\[
\begin{align*}
|u|^{r-2}u_t - a \Delta u + g(x,t)|u|^{q(x)-2}u + \phi(Eu) = f(x,t), & \quad (x,t) \in Q_{0,T}, \\
\left. u \right|_{\Sigma_{0,T}} &= 0, & \left. u \right|_{t=0} = u_0(x), & \ x \in \Omega,
\end{align*}
\]

where \( a > 0 \) and \( r > 1 \) are some numbers, \( \Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \) is the Laplacian,

\[
(\tilde{E}u)(x,t) := \int_{\Omega} \delta(z) (\tilde{u}(x+z,t) - \tilde{u}(x,t)) \, dz, \quad (x,t) \in Q_{0,T},
\]

\( g, q, \phi, \delta, f, u_0 \) are some functions, and \( \tilde{u} \) is zero extension of \( u \) from \( Q_{0,T} \) into \( (\mathbb{R}^n \setminus \Omega) \times (0,T) \).

The equations of type (1) have been widely used in many applications. For example (see [4]), let us consider a circuit, which consists of a resistance, a condenser, and a generator of the jump impulses. It is well known that the supply \( U(t) \) satisfies the stochastic differential equation

\[
dU(t) = -\frac{1}{RC}U(t) \, dt + \sigma \, dW(t) + dQ(t), \quad t \in [0,T], \quad U(0) = U_0,
\]

where \( R \) is a value of the resistance, \( C \) is a capacitance of the condenser, \( \sigma \) is an intensity of the loss of the signal, \( U_0 \) is a start supply, \( T = RC \), \( \{W(t)\}_{t \in [0,T]} \) is the standard Brownian motion, and \( Q(t) \) is the compound Poisson process. Let
\( u(x, t), x \in \mathbb{R}, \) be a density of the stochastic process \( U(t), t \in [0, T]. \) If some additional conditions are satisfied, then (see [4, p. 4]) \( u \) is a solution to the Cauchy problem for the Kolmogorov-Feller equation

\[
  u_t - \frac{\sigma^2}{2} u_{xx} - \frac{1}{RC} (xu)_x - \lambda \int_{\mathbb{R}} \psi(\xi, t) (u(x + \xi, t) - u(x, t)) \, d\xi = 0, \quad (x, t) \in \mathbb{R} \times (0, T), \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R},
\]

(5)

where \( u_0 \) is a density of \( U_0, \) \( \lambda \) is an intensity of the jump creation, \( \psi \) is a density of the jump size.

If \( n = 1, \) then equation (1) is a nonlinear modification of (5) in bounded domain \( Q_{0, T} \) instead of the unbounded strip \( \mathbb{R} \times (0, T). \) Equation (5) also arises in the Merton and Kou models of the option pricing (see [15], [31], and [34]). The Cauchy problem for the equations of type (5) is considered in [9], [16], [21], and [41].

The number \( r \) and the function \( q \) (see (1)) are called exponents of the nonlinearity of double nonlinear parabolic equation (1). Since \( q \) is a function, here we have a variable exponent of the nonlinearity. The double nonlinear parabolic equations with variable exponents of the nonlinearity without the integral terms are considered in [2], [7]. In [6], [17], [18], [19], [39], and [40], the authors investigate the problems for nonlinear parabolic equations with constant exponents of the nonlinearity and with the integral terms which differ from term (3).

In [38], J.P. Pinoso proved the existence of the local solution and the nonexistence of the global solution to the initial-boundary value problem for the equation

\[
  u_t - \Delta u - |u|^{\alpha(x)} = \int_{\Omega} |u(z, t)|^{\beta(z)} \, dz,
\]

(6)

where \( \alpha(x) > 1 \) and \( \beta(y) > 1. \)

In this paper we extend our investigations which is started in [13] for the double nonlinear equation

\[
  u_t - a \Delta (|u|^r - 2u) + g(x, t)|u|^{q(x) - 2} u + \phi(Eu) = f(x, t), \quad (x, t) \in Q_{0, T}.
\]

(7)

We recall that if \( r \in [2, 3], \) then the existence of the weak solution to problem (7), (2) is proved in [13]. The Cauchy problem for the equations of type (7) is considered in [20] if \( q(x) \equiv 2, \) \( \phi(s) = s, \) and \( r = 3. \)

### 2 Notation and statement of main result

Let \((\cdot, \cdot)_H\) be a scalar product of some Hilbert space \( H, \) \( \| \cdot \|_H \equiv \| \cdot \|; \) a norm of some Banach space \( B, B^* \) a dual space, and \((\cdot, \cdot)_B\) a scalar product between \( B^* \) and \( B. \) For the Banach spaces \( X \) and \( Y \) the notation \( X \odot Y \) means the continuous embedding; the notation \( X \overset{\kappa}{\subset} Y \) means the continuous and densely embedding; the notation \( X \subset Y \) means the compact embedding.

Suppose that \( m, N \in \mathbb{N}, p \in [1, \infty], Q = \Omega \) or \( Q = Q_{0, T}, \) \( \mathcal{M}(Q) \) is a set of all measurable functions \( v : Q \to \mathbb{R} \) (see [24, p. 120]), Lip\((Q) \) is a set of all Lipschitz-continuous functions \( v : Q \to \mathbb{R} \) (see [32, p. 29]), \( C^m(Q) \) and \( C^m_0(Q) \) are determined from [1, p. 9], \( L^p(Q) \) is the Lebesgue space (see [1, p. 22, 24]), \( W^{m,p}(Q) \) and \( W_0^{m,p}(Q) \) are Sobolev spaces (see [1, p. 45]), \( H^m(Q) := W^{m+2}_0(Q), H^m_0(Q) := W_0^{m+2}(Q), C([0, T]; X) \) and \( C^m([0, T]; X) \) are determined from [27, p. 147], \( L^p(0, T; X) \) is determined from [27, p. 155], \( W^{m,p}(0, T; X) \) is determined from [25, p. 286],
\(H^m(0,T;X) := W^{m,2}(0,T;X)\), and
\[\mathcal{B}^+(Q) := \{ \delta \in L^\infty(Q) \mid \text{ess inf}_{y \in Q} \delta(y) > 0 \}.\]

For every \(\delta \in \mathcal{B}^+(Q)\), by definition, put
\[
\delta_0 := \text{ess inf}_{y \in Q} \delta(y), \quad \delta^0 := \text{ess sup}_{y \in Q} \delta(y), \quad S_\delta(s) := \max\{s^{\delta_0}, s^{\delta^0}\}, \quad s \geq 0,
\]
and
\[
\delta'(y) := \frac{\delta(y)}{\delta(y) - 1} \quad \text{for a.e.} \quad y \in Q \quad \left(\text{note that} \quad \frac{1}{\delta(y)} + 1 = 1 \quad \text{and} \quad \delta' \in \mathcal{B}^+(Q)\right).
\]

\[
\rho_\delta(v; Q) := \int_Q |v(y)|^{\delta'(y)} \, dy, \quad v \in \mathcal{M}(Q).
\]

For every function \(u \in L^1(Q_{0,T}) = L^1(0,T;L^1(\Omega))\) we have \(u(\cdot, t) \in L^1(\Omega), t \in (0,T)\). For the sake of convenience we shall write \(u(t)\) instead of 
\(u(\cdot, t)\).

Assume that \(\delta \in \mathcal{B}^+(Q)\) and \(\delta_0 > 1\). The set
\[
L^{\delta(\cdot)}(Q) := \{ v \in \mathcal{M}(Q) \mid \rho_\delta(v; Q) < +\infty \}
\]
with the Luxemburg norm
\[
||v; L^{\delta(\cdot)}(Q)|| := \inf\{\lambda > 0 \mid \rho_\delta(v/\lambda; Q) \leq 1\}
\]
is called a generalized Lebesgue space. It is well known that \(L^{\delta(\cdot)}(Q)\) is the Banach space which is reflexive and separable (see [30, p. 599, 600, 604]). The generalized Lebesgue space was introduced in [36]. Its properties were widely studied in [2], [10], [22], [26], and [30].

We shall need the following assumptions:
(A): \(a > 0, \quad \frac{1}{2} < r \leq 2;\)
(Q): \(q \in \mathcal{B}^+(\Omega), 1 < q_0 \leq q^0 \leq 2;\)
(G): \(g \in \mathcal{B}^+(Q_{0,T});\)
(E): \(s \in L^\infty(Q_{0,T} \times \Omega);\)
(\Phi): \(\phi \in \text{Lip}(\mathbb{R}), \quad |\phi(\xi)| = \phi^*(|\xi|)\) for every \(\xi \in \mathbb{R}, \quad \phi^* \in [0, +\infty);\)
(F): \(f \in L^r(Q_{0,T}), \quad \text{where} \quad \frac{1}{r} + \frac{1}{2} = 1;\)
(U): \(u_0 \in H^2(\Omega) \cap H^1_0(\Omega).\)

By definition, put
\[
\gamma := \frac{4}{4 - r} \quad \left(\text{note that} \quad \gamma \in \left[\frac{8}{3}, 2\right] \quad \text{iff} \quad r \in \left(\frac{3}{2}, 2\right)\right),
\]
\[
(Gu)(x,t) := g(x,t)|u(x,t)|^{(r-2)/r}u(x,t), \quad (x,t) \in Q_{0,T}, \quad u \in \mathcal{M}(Q_{0,T}),
\]
\[
(\mathcal{B}u)(x,t) := \frac{1}{r - 1}|u(x,t)|^{r-2}u(x,t), \quad (x,t) \in Q_{0,T}, \quad u \in \mathcal{M}(Q_{0,T}).
\]

Notice that if \(u \in C^1(Q_{0,T})\) and \(u \neq 0\), then
\[
(\mathcal{B}u)_t = |u|^{r-2}u_t.
\]

Then we formally rewrite equation (1) as
\[
(\mathcal{B}u)_t - a\Delta u + Gu + \phi(Eu) = f(x,t), \quad (x,t) \in Q_{0,T}.
\]
Definition 1. A real-valued function $u \in L^2(\{0,T\}; H^1_0(\Omega)) \cap C([0,T]; L^2(\Omega))$ is called a weak solution of problem (1)-(2) if $\mathcal{A} u, G u, E u \in L^2(Q_{0,T})$, $u$ satisfies (2), and for every $v \in H^1_0(Q_{0,T})$ we have

$$\int_{Q_{0,T}} \left[ -\mathcal{A} u \nabla v + a(\nabla u, \nabla v) + G u v + \Phi(E u) v \right] dx dt = \int_{Q_{0,T}} f v dx dt. \quad (15)$$

Here $\nabla v := (v_{x_1}, \ldots, v_{x_n})$, $(\cdot, \cdot)$ is a scalar product in $\mathbb{R}^n$.

Note that the integral operator $E : L^2(Q_{0,T}) \to L^2(Q_{0,T})$ (see (3)) is a linear bounded operator and for every $u \in L^2(Q_{0,T})$ we have

$$||E u; L^2(Q_{0,T})|| \leq E^* ||u; L^2(Q_{0,T})||, \quad (16)$$

where $E^* > 0$ is independent of $u$. Indeed, using the Cauchy-Bunyakowski-Schwarz inequality (see Lemma 6.1 [27, Chapter 1, §6]), we get

$$\int_{Q_{0,T}} |(E u)(x,t)|^2 dx dt = \int_{Q_{0,T}} \int_\Omega |\mathcal{A}(x,t,z)(u(x+z,t) - u(x,t))|^2 dz dx dt \leq \int_{Q_{0,T}} \left( \int_\Omega |\mathcal{A}(x,t,z)|^2 dz \right) \left( \int_\Omega |u(x+z,t) - u(x,t)|^2 dz \right) dx dt$$

$$\leq C_1 \int_{Q_{0,T}} \int_\Omega (|u(x+z,t)|^2 + |u(x,t)|^2) dz dx dt \leq C_2 \int_{Q_{0,T}} |u(x,t)|^2 dx dt$$

and so (16) holds.

Assume that the following condition is satisfied:

$(Z)$: $a > \Phi^* E^* M_\Omega$, where $a$ is defined by (1), $\Phi^*$ is determined from $(\Phi)$, $E^*$ is defined by (16), $M_\Omega$ is determined from the Friedrichs inequality (see Lemma 1.26 [27, Chapter 2, §1])

$$\int_\Omega |w|^2 dx \leq M_\Omega \sum_{i=1}^n |w_{x_i}|^2 dx, \quad w \in H^1_0(\Omega). \quad (17)$$

Notice that $M_\Omega$ depends on $n$ and does not depend on $u$.

Theorem 1. Suppose that $\partial \Omega \in C^4$, conditions $(A)$-$(U)$ hold, and $(Z)$ is satisfied. If $g_L \in L^\infty(Q_{0,T})$ and if $\Phi \in L^\infty(\mathbb{R})$, then problem (1)-(2) has a weak solution $u$ such that $u \in W^{1,\gamma}(Q_{0,T})$ and $u^{\frac{2}{\gamma}-1} u \in H^1(0,T; L^2(\Omega))$.

3 Auxiliary facts

3.1 Functional spaces and some operators

First we recall some properties of the generalized Lebesgue spaces.

Proposition 1. ([37, p. 31]). If $\delta \in \mathcal{D}_+(\Omega)$ and $\delta_0 > 1$, then for every $\eta > 0$ there exists a number $Y_\delta(\eta) > 0$ such that for every $a, b \geq 0$ and for a.e. $y \in Q$ the generalized Young inequality (the Young inequality, if $\delta \equiv \text{const}$)

$$ab \leq \eta a^\delta(y) + Y_\delta(\eta) b^{\delta(y)} \quad (18)$$
holds. In addition, \( Y_3(\eta) \) depends on \( \delta_0, \delta^0 \) and it is independent of \( y \), \( Y_2(\eta) = \frac{1}{\eta}, Y_2(\frac{1}{\eta}) = \frac{1}{\eta}, \) \( Y_3(+0) = +\infty \), and \( Y_3(+\infty) = 0 \).

**Proposition 2.** Assume that \( \delta \in B_+(Q) \) and \( \delta_0 > 1 \). Then the following statements are fulfilled:

(i) ([30, p. 600]) if \( \delta(y) \geq r(y) \geq 1 \) for a.e. \( y \in Q \), then \( L^{\delta(y)}(Q) \cap L^{r(y)}(Q) \) and

\[
\|v; L^{\delta(y)}(Q)\| \leq (1 + \text{mes} Q)\|v; L^{r(y)}(Q)\|, \quad v \in L^{\delta(y)}(Q);
\]

(ii) ([26, p. 431]) for every \( u \in L^{\delta(y)}(Q) \) and \( v \in L^{\delta(y)}(Q) \) we get \( uv \in L^1(Q) \) and the following generalized Hölder inequality is true

\[
\int_Q |u(y)v(y)| \, dy \leq 2||v; L^{\delta(y)}(Q)|| \cdot \|v; L^{\delta(y)}(Q)\|.
\]  

(19)

**Proposition 3.** ([11, p. 168]). Suppose that \( \delta \in B_+(Q) \), \( \delta_0 > 1 \), \( S_\delta \) is defined by (8), and \( \rho_\delta \) is defined by (10). Then for every \( v \in \mathcal{M}(Q) \) the following statements are fulfilled:

(i) \( \|v; L^{\delta(y)}(Q)\| \leq S_{1/\delta}(\rho_\delta(v; Q)) \) if \( \rho_\delta(v; Q) < +\infty \);

(ii) \( \rho_\delta(v; Q) \leq S_\delta(\|v; L^{\delta(y)}(Q)\|) \) if \( \|v; L^{\delta(y)}(Q)\| < +\infty \).

**Proposition 4.** (Lemma 1 [7, p. 714]). Suppose that \( \delta \in B_+(Q) \), \( \delta_0 > 1 \), and \( \{u^m\}_{m \in \mathbb{N}} \subset L^{\delta(y)}(Q) \). If \( u^m \rightharpoonup u \) weakly in \( L^{\delta(y)}(Q) \) and if \( u^m \rightharpoonup v \) almost everywhere in \( Q \), then \( u = v \).

Let \( \Delta^0 v := v, \Delta^1 v := \Delta v, \Delta^r v := \Delta(\Delta^{r-1} v) \), and

\[
H^{2r}_{\Delta}(\Omega) := \{ v \in H^{2r}(\Omega) \mid v|_{\partial\Omega} = \Delta v|_{\partial\Omega} = \ldots = \Delta^{r-1} v|_{\partial\Omega} = 0 \}, \quad r \in \mathbb{N}.
\]  

(20)

Take a number \( r \in \mathbb{N} \). It is easy to verify that \( H^{2r}_{\Delta}(\Omega) \) is the reflexive Hilbert space with respect to the scalar product

\[
(u, v)_{H^{2r}_{\Delta}(\Omega)} := \int_{\Omega} \Delta^r u(x) \Delta^r v(x) \, dx, \quad u, v \in H^{2r}_{\Delta}(\Omega).
\]

If \( \partial\Omega \subset C^1 \), then the following integration by parts formulae is true

\[
\int_{\Omega} v \Delta^r u \, dx = \int_{\Omega} u \Delta^r v \, dx, \quad u, v \in H^{2r}_{\Delta}(\Omega).
\]  

(21)

**Proposition 5.** (Lemma 3 [35, p. 229]). If \( \partial\Omega \subset C^{2r} \), then there exists a constant \( C_3 > 0 \) such that

\[
\|v; H^{2r}_{\Delta}(\Omega)\| \leq C_3 \|\Delta^r v; L^2(\Omega)\|
\]  

(22)

holds for all \( v \in H^{2r}_{\Delta}(\Omega) \).

Proposition 5 and definition (20) imply that

\[
H^{2r}_{\Delta}(\Omega) \cap H^{2r}(\Omega), \quad H^{2r}_{\Delta}(\Omega) \cap L^2(\Omega) \cap [H^{2r}_{\Delta}(\Omega)]^*.
\]  

(23)

Let \( \{w^j\}_{j \in \mathbb{N}} \) is an orthonormal basis for the space \( L^2(\Omega) \) that consists of all eigenfunctions of the problem

\[
-\Delta w = \lambda w \text{ in } \Omega, \quad w|_{\partial\Omega} = 0, \quad w \in H^1_0(\Omega),
\]  

(24)
and \( \{ \lambda_j \}_{j \in \mathbb{N}} \subset \mathbb{R}_+ \) is the set of the corresponding eigenvalues. It is easy to verify that the functions \( \{ w^j \}_{j \in \mathbb{N}} \) satisfy the equalities
\[
(-1)^j \Delta^j w^j = \lambda_j^w w^j \quad \text{in} \quad \Omega, \quad w^j \big|_{\partial \Omega} = \Delta w^j \big|_{\partial \Omega} = \ldots = \Delta^{r-1} w^j \big|_{\partial \Omega} = 0.
\] (25)

The following statements are needed for the sequel.

**Proposition 6.** (Theorem 8 [35, p. 230]). If \( \partial \Omega \subset C^{2r} \), then the set \( \{ w^j \}_{j \in \mathbb{N}} \) is a basis for the space \( H^2_{\Delta} (\Omega) \).

**Remark 1** Let us consider the eigenvalue problem
\[
-\lambda_j^w v_{xxxx} - \lambda_j^w v_{xxt} - \lambda_j^w v_{xx} = \lambda_j^w v \quad \text{in} \quad Q_0, T, \quad v|_{\partial \Omega \times (0, T)} = 0, \quad v|_{t=0} = 0, \quad v|_{t=T} = 0.
\] (26)

Separating of variables \( v(x, t) = w(x) \theta(t) \) yields
\[
-\theta''(t) - w_{xx} \theta(t) - w_{x} \theta(t) = \lambda_j^w \theta(t),
\]
and so \( \lambda_j^w = \sigma + \lambda, \quad \theta''(t) + \sigma \theta(t) = 0 \) for \( t \in (0, T) \), and \( \Delta w(x) + \hat{\lambda} w(x) = 0 \) for \( x \in \Omega \). Hence, instead of (26) we obtain the uncouple system of the following problem:

(i) problem (27);
(ii) the problem
\[
\theta''(t) + \sigma \theta(t) = 0 \quad \text{for} \quad t \in (0, T), \quad \theta(0) = 0 = \theta'(T).
\] (27)

The eigenvalues and the corresponding eigenfunctions of problem (27) have the following form:
\[
\hat{\lambda}_m = \left( \frac{\pi (2k - 1)}{2T} \right)^2, \quad k \in \mathbb{N}, \quad \theta^k(t) = \sin \left( \frac{\pi (2k - 1)}{2T} t \right), \quad t \in (0, T), \quad k \in \mathbb{N}.
\]

Whence, the eigenvalues and the corresponding eigenfunctions of problem (26) have the following form:
\[
\lambda_j^w = \sigma^k, \quad v(x, t) = w^j(x) \sin \left( \frac{\pi (2k - 1)}{2T} t \right), \quad (x, t) \in Q_0, T, \quad j, k \in \mathbb{N}.
\]

If we renumber it, then we get the set \( \{ v^m \}_{m \in \mathbb{N}} \) of the eigenfunctions and the set \( \{ \lambda_m^w \}_{m \in \mathbb{N}} \) of the corresponding eigenvalues \( \lambda_m^w = \lambda_m + \sigma_m \) \( (m \in \mathbb{N}) \) of problem (26). Similarly to [42, Section 11.1] we prove that \( \{ v^m \}_{m \in \mathbb{N}} \) is a complete set in \( L^2 \quad (Q_0, T) \).

For problem (24) we have Proposition 6. In particular, if \( r \in \mathbb{N} \) and if the condition \( \partial \Omega \subset C^{2r} \) holds, we obtain \( \{ w^j \}_{j \in \mathbb{N}} \subset H^2_{\Delta} (\Omega) \subset H^{2r} (\Omega) \) and so \( \{ v^m \}_{m \in \mathbb{N}} \subset C^n ([0, T]; H^{2r} (\Omega)) \subset H^{2r} (Q_0, T) \). Moreover, the positive numbers \( \{ \lambda_m \}_{m \in \mathbb{N}} \) and \( \{ \sigma_m \}_{m \in \mathbb{N}} \) satisfy the equalities
\[
\hat{\lambda}_m v^m = -\Delta v^m \quad \text{in} \quad Q_0, T, \quad \sigma_m v^m = -v^m_{xx} \quad \text{in} \quad Q_0, T, \quad m \in \mathbb{N}.
\] (28)

**Lemma 1.** If \( p \in [1, +\infty) \) and if \( z, z_t \in L^p (Q_0, T) \), then the following inequalities are true:
\[
\int_\Omega |z(x, \tau)|^p \, dx \leq C_\delta \left( \int_\Omega |z(x, 0)|^p \, dx + \int_0^T \int_\Omega |z_t(x, \tau)|^p \, dx \, d\tau \right), \quad \tau \in [0, T],
\] (29)
\[
\begin{align*}
\int_\Omega |z(x, \tau)|^p \, dx & \leq C_4 \left( \int_\Omega |z(x, T)|^p \, dx + \int_{Q_{0,T}} |z(x, t)|^p \, dx \right), \quad \tau \in [0, T], \\
\int_\Omega |z(x, \tau)|^p \, dx & \leq C_4 \int_{Q_{0,T}} \left[ |z(x, t)|^p + |z(x, t)|^p \right] \, dx, \quad \tau \in [0, T],
\end{align*}
\] (30)

where \( C_4 > 0 \) is independent of \( z, \tau \).

**Proof.** The proof of estimate (31) is found in Lemma 6 [13]. In the same way we prove (29) and (30). \( \square \)

**Lemma 2.** Suppose that the Nemyskij operators \( G \) and \( R \) are defined by (12) and (13) respectively. If \( q, r \in E(\Omega) \), \( q_0 > 1, r_0 > 1, \) and \( g \in L^{\infty}(Q_{0,T}) \), then the operators \( G : L^q(\Omega) \to L^r(\Omega) \) and \( R : L^{r_1}(\Omega) \to L^{r_2}(\Omega) \) are bounded and continuous.

We shall omit the proof because it is analogous to the proof which was given in [20, p. 613].

**Proposition 7.** (the Aubin theorem, see [3] and [5, p. 393]). If \( s, h > 1 \) are fixed numbers, \( \mathcal{W}, \mathcal{L}, \mathcal{R} \) are the Banach spaces, and \( \mathcal{W} \subset \mathcal{L} \subset \mathcal{R} \), then \( \{ u \in L^1(0, T; \mathcal{W}) \mid u_t \in L^h(0, T; \mathcal{R}) \} \subset L^s([0, T], \mathcal{R}) \cap C([0, T]; \mathcal{R}) \).

**Proposition 8.** (the Vishyk lemma, see Lemma 4.3 [33, p. 66] and [23, p. 60]). Suppose that \( m \in \mathbb{N} \), the vector valued function \( P := (P_1, \ldots, P_m) : \mathbb{R}^m \to \mathbb{R}^m \) is continuous, and there exists a number \( \rho > 0 \) such that \( (P(z), z)_{\mathbb{R}^m} \geq 0 \) holds for all \( z \in \mathbb{R}^m \) such that \( |z| = \rho \). Then there exists a vector \( z^m \in \mathbb{R}^m \) such that \( |z^m| \leq \rho \) and \( P(z^m) = 0 \).

### 3.2 Differentiability of nonlinear expressions

The following statements are needed for the sequel.

**Theorem 2.** Suppose that \( r \in \mathcal{R}_+(\Omega) \). Then the following statements are fulfilled:

1. If \( r_0 > 1 \), then the equality
\[
\left( |u|^r \right)_t = r(x)|u|^r - 2u \tag{32}
\]
is true if one of the following alternatives hold:
   (i) \( u \in C^1(\overline{Q_{0,T}}) \) (here we have \( |u|^r \), \( \left( |u|^r \right)_t \in L^\infty(\Omega) \));
   (ii) \( u, u_t \in L^p(\Omega) \) and \( p(x) \geq r(x) \) for a.e. \( x \in \Omega \) (here we have \( |u|^r \), \( \left( |u|^r \right)_t \in L^\frac{p(x)}{r(x)}(\Omega) \)).

2. If \( r_0 > 2 \), then the equality
\[
\left( |u|^r - 2u \right)_t = (r(x) - 1)|u|^r - 2u \tag{33}
\]
is true if one of the following alternatives hold:
   (i) \( u \in C^1(\overline{Q_{0,T}}) \) (here we have \( |u|^r - 2u \), \( \left( |u|^r - 2u \right)_t \in L^\infty(\Omega) \));
   (ii) \( u, u_t \in L^p(\Omega) \) and \( p(x) \geq r(x) - 1 \) for a.e. \( x \in \Omega \) (here \( |u|^r - 2u \), \( \left( |u|^r - 2u \right)_t \in L^\frac{p(x)}{r(x)-1}(\Omega) \)).

This Theorem 2 coincides with Theorem 3 [13] (see also Remark 1 [13]). Then the proof is omitted.

**Proposition 9.** (Theorem 2 [25, p. 286]). If \( X \) is the Banach space and if \( p \in [1, \infty) \), then \( W^{1,p}(0, T; X) \subset C([0, T]; X) \) and the following integration by parts formula is true
\[
\int_s^\tau u_t(t) \, dt = u(\tau) - u(s), \quad 0 \leq s < \tau \leq T, \quad u \in W^{1,p}(0, T; X). \tag{34}
\]
Lemma 3. Suppose that \( \Omega \subset \mathbb{R}^n \) is a bounded \( C^{0,1} \)-domain (see [27, p. 48]). Then the integration by parts formulae

\[
\int_{Q,\tau} w_z \, dx \, dt = \int_\Omega w_z \, dx \bigg|_{t=s}^{t=\tau} - \int_{Q,\tau} w_{z_t} \, dx \, dt, \quad 0 \leq s < \tau \leq T, \tag{35}
\]

holds if one of the following alternatives hold:

(i) \( w \in L^{q(x)}(Q_{0,T}) \), where \( q \in \Gamma_0(\Omega) \), \( q_0 > 1 \), \( w_t \in L^1(Q_{0,T}) \), \( z \in L^\infty(Q_{0,T}) \), and \( z_t \in L^{q(x)}(Q_{0,T}) \);

(ii) \( w, w_t \in L^1(Q_{0,T}) \) and \( z, z_t \in L^\infty(Q_{0,T}) \).

Proof. (i). Take \( W := \{ w \in L^{q(x)}(Q_{0,T}) \mid w_t \in L^1(Q_{0,T}) \} \) and \( Z := \{ z \in L^\infty(Q_{0,T}) \mid z_t \in L^{q(x)}(Q_{0,T}) \} \). If \( \varphi \in C^1([0,T]) \) and if \( z \in Z \), then \( \varphi z \in W^{1,1}(0, T; L^{q(x)}(\Omega)) \). By (34) with \( u = \varphi(t)z(x,t) \), we get

\[
\int_{S}^\tau \varphi(t)z(x,t) \, dt = \varphi(\tau)z(x,\tau) - \varphi(s)z(x,s) - \int_{S}^\tau \varphi(t)z_t(x,t) \, dt, \quad x \in \Omega. \tag{36}
\]

Take a function \( v \in C^1(\overline{\Omega}) \). By (36), we obtain

\[
\int_{Q,\tau} \varphi v z \, dx \, dt = \int_\Omega \varphi v z \, dx \bigg|_{t=s}^{t=\tau} - \int_{Q,\tau} \varphi v z_t \, dx \, dt. \tag{37}
\]

Clearly, \( C^1([0,T]; C^1(\overline{\Omega})) \supseteq W \supseteq W^{1,1}(0, T; L^1(\Omega)) \). Then the set

\[
\left\{ \sum_{i=1}^m \varphi_i(t)v_i(x) \mid m \in \mathbb{N}, \varphi_1, \ldots, \varphi_m \in C^1([0,T]), v_1, \ldots, v_m \in C^1(\overline{\Omega}) \right\}
\]

is dense in \( W \) and (37) yields (35).

We shall omit the proof of (ii) because it is analogous to the previous one. \( \square \)

Lemma 4. Suppose that \( \alpha \in \Gamma_0(Q) \),

\[
\psi_{\alpha(y)}(s) := \begin{cases} s^{\alpha(y)} & \text{if } s > 0, \\ 0 & \text{if } s \leq 0, \end{cases} \quad y \in Q, \tag{38}
\]

\( p, \delta \in \Gamma_0(Q), p_0, \delta_0 > 1, p(y) \geq \alpha(y) \) for a.e. \( y \in Q \), and \( \delta(y) \leq \frac{\delta_0(y)}{\alpha(y)} \) for a.e. \( y \in Q \). Then for every \( u \in L^{p(y)}(Q) \) we have that \( \psi_{\alpha(y)}(u) \in L^{\alpha(y)}(Q) \),

\[
\rho_{p/\alpha}(\psi_{\alpha(y)}(u); Q) \leq \rho_{p}(u; Q), \tag{39}
\]

\[
\| \psi_{\alpha(y)}(u); L^{\delta(y)}(Q) \| \leq C_{5}\| u; L^{\alpha(y)}(Q) \|, \tag{40}
\]

where \( C_{5} > 0 \) is independent of \( u \).

Proof. Clearly, \( \frac{\delta_0(y)}{p(y)} \geq 1 \) for a.e. \( y \in Q \), \( |\psi_{\alpha(y)}(u)|^{\frac{\delta(y)}{p(y)}} \leq |u|^{\frac{\delta_0(y)}{p(y)}} \in L^{1}(Q) \). Then by [29, p. 297], we obtain

\[
\psi_{\alpha(y)}(u) \in L^{\alpha(y)}(Q). \text{ Moreover, (39) and}
\]

\[
\| \psi_{\alpha(y)}(u); L^{\delta(y)}(Q) \| \leq C_{6}\| u; L^{p(y)/\alpha(y)}(Q) \| \leq C_{6}\| \rho_{p/\alpha}(\psi_{\alpha(y)}(u); Q) \|
\]
hold. This inequality and (39) imply (40). □

**Lemma 5.** Suppose that \( \beta \in \mathcal{B}_+(\Omega) \), \( \psi_{\beta(x)} \) is determined from (38) if we replace \( \alpha(y) \) by \( \beta(x) \), and

\[
\chi_k(s) := \begin{cases} 
1 & \text{if } s > \frac{1}{k}, \\
k & \text{if } s \leq \frac{1}{k},
\end{cases}
\]

(41)

If \( u \in C^1(Q_{0,T}) \) and if \( v, v_i \in L^1(Q_{0,T}) \), then

\[
\lim_{k \to +\infty} \int_{Q_{0,T}} \chi_k(u) \beta(x) \psi_{\beta(x)-1}(u) v \, dx dt = \int_{\Omega} \psi_{\beta(x)}(u) v \, dx \bigg|_{t=0}^{t=T} - \int_{Q_{0,T}} \psi_{\beta(x)}(u) v_i \, dx dt.
\]

(42)

**Proof.** By definition, put

\[
\psi_{\beta(x),k}(s) := \begin{cases} 
k^{\beta(s)} & \text{if } s \geq k, \\
k^{\beta(s)} - 1 & \text{if } 1 < s < k, \\
\frac{1}{k} & \text{if } s \leq 1,
\end{cases}
\]

\[
\tilde{\psi}_{\beta(x),k}(s) := \begin{cases} 
\beta(x) k^{\beta(s)-1} & \text{if } 1 < s < k, \\
0 & \text{if } s \leq 1 \text{ and } s \geq k,
\end{cases}
\]

Clearly, \( \psi_{\beta(x),k}(s) \to \psi_{\beta(x)}(s) \) if \( s \in \mathbb{R} \) and \( x \in \Omega \). In addition, for \( k \in \mathbb{N} \) \( (k \geq 2) \) and \( x \in \Omega \) the function \( s \to \psi_{\beta(x),k}(s) \) has the Lipschitz property in \( \mathbb{R} \) and it is not differentiable only in the point \( s = \frac{1}{k} \) and \( s = k \). Moreover, \( \frac{d}{ds} \psi_{\beta(x),k}(s) = \tilde{\psi}_{\beta(x),k}(s) \) if \( s \neq \frac{1}{k} \) and \( s \neq k \). Whence, by Lemma 4 [13], we obtain

\[
(\psi_{\beta(x),k}(u))_t = \tilde{\psi}_{\beta(x),k}(u) u_t \quad \text{almost everywhere in } Q_{0,T}.
\]

(43)

Thus, \( \psi_{\beta(x),k}(u), (\psi_{\beta(x),k}(u))_t \in L^\infty(Q_{0,T}) \). Using case (ii) of Lemma 3 with \( z = \psi_{\beta(x),k}(u) \) and \( w = v \), we get (see (35))

\[
\int_{Q_{0,T}} (\psi_{\beta(x),k}(u))_t v \, dx dt = \int_{\Omega} \psi_{\beta(x),k}(u) v \, dx \bigg|_{t=0}^{t=T} - \int_{Q_{0,T}} \psi_{\beta(x),k}(u) v_i \, dx dt.
\]

(44)

Let \( M := \max\{|u(x,t)| : (x,t) \in \bar{Q}_{0,T}\}, k_0 \in \mathbb{N}, k_0 \geq \max\{2,M\} \). Since \( |u| \leq M \leq k_0 \leq k \), from (43) we have

\[
(\psi_{\beta(x),k}(u))_t = \tilde{\psi}_{\beta(x),k}(u) u_t = \chi_k(u) \beta(x) \psi_{\beta(x)-1}(u) u_t,
\]

where \( k \geq k_0 \). By the inequality \( |\psi_{\beta(x),k}(u(x,t))| \leq M^{\beta(x)} \chi_k(x,t) \in \bar{Q}_{0,T} \) and the Lebesgue Dominate Convergence Theorem, we obtain

\[
\lim_{k \to +\infty} \int_{Q_{0,T}} \psi_{\beta(x),k}(u) v \, dx = \int_{\Omega} \psi_{\beta(x)}(u) v \, dx \quad \text{if } t = 0 \quad \text{and } t = T,
\]

\[
\lim_{k \to +\infty} \int_{Q_{0,T}} \psi_{\beta(x),k}(u) v_i \, dx dt = \int_{Q_{0,T}} \psi_{\beta(x)}(u) v_i \, dx dt.
\]

Therefore, (42) follows from (44). □

**Theorem 3.** Suppose that \( \alpha \in \mathcal{B}_+(\Omega) \), \( \frac{1}{2} < \alpha_0 \leq \alpha(x) \leq \alpha^0 \) for a.e. \( x \in \Omega \), and \( \psi_{\alpha(x)} \) is defined by (38) if we replace \( \alpha(y) \) by \( \alpha(x) \). Then the following statements are fulfilled:
(1) if \( u \in C^2(Q_{0,T}) \), then \( \psi_{\alpha(x)}(u) \in L^\infty(Q_{0,T}) \), \( (\psi_{\alpha(x)}(u))_t \in L^2(Q_{0,T}) \), and
\[
(\psi_{\alpha(x)}(u))_t = \alpha(x) \psi_{\alpha(x)-1}(u) u_t; \quad (45)
\]

(2) if \( u, u_t, u_{tt} \in L^p(Q_{0,T}) \), \( p \in L^\infty(\Omega) \), and \( p(x) \geq 2\alpha(x) \) for a.e. \( x \in \Omega \), then \( \psi_{\alpha(x)}(u) \in W^{1,2}(0,T;L^2(\Omega)) \), formulae (45) holds, and
\[
||\psi_{\alpha(x)}(u)||_{L^2(Q_{0,T})} \leq C_7 S_{\alpha/p} (p(u;Q_{0,T})), \quad (46)
\]
\[
||((\psi_{\alpha(x)}(u))_t ||_{L^2(Q_{0,T})} \leq C_8 S_{\alpha/p} (p(u;Q_{0,T}) + p(u;Q_{0,T}) + p(u_t;Q_{0,T})), \quad (47)
\]
where \( C_7, C_8 > 0 \) are independent of \( u \).

**Proof.** First let us prove Case 1. Take a function \( u \in C^2(Q_{0,T}) \). Since \( u_t, u_{tt} \in L^1(Q_{0,T}) \), using (42) with \( \beta = 2\alpha - 1 > 0 \) and \( v = u_t \), we obtain
\[
\lim_{k \to \infty} \int_{Q_{0,T}} f_k \, dxdt = \int_{Q_{0,T}} \psi_{2\alpha(x)-1}(u) u_t \, dxdt \bigg|_{t=0}^{t=T} - \int_{Q_{0,T}} (\psi_{\alpha(x)-1}(u) u_t) \, dxdt, \quad (48)
\]
where \( \chi_k \) is defined by (41), \( f_k = \chi_k(u)(2\alpha(x)-1)\psi_{2\alpha(x)-2}(u)|u_t|^2, \quad k \in \mathbb{N} \). Clearly,
\[
f_k(x,t) \xrightarrow{k \to \infty} f(x,t) \quad \text{if} \quad (x,t) \in Q_{0,T}, \quad (49)
\]
where \( f = (2\alpha(x)-1)\psi_{2\alpha(x)-2}(u)|u_t|^2 = (2\alpha(x)-1)|\psi_{\alpha(x)-1}(u) u_t|^2 \). The existence of the limit in (48) implies that there exists a constant \( C_9 > 0 \) such that for every \( k \in \mathbb{N} \) we have the estimate
\[
\int_{Q_{0,T}} f_k \, dxdt \leq C_9.
\]
Clearly, \( f_k \leq f_{k_2} \) if \( k_1 \leq k_2 \). Then the Levi Monotone Convergence Theorem (see Theorem 4.1 [8, p. 90]) and (49) yield that \( f \in L^1(Q_{0,T}) \) and so \( \psi_{\alpha(x)-1}(u) u_t \in L^2(Q_{0,T}) \). In addition,
\[
\int_{Q_{0,T}} f_k \, dxdt \xrightarrow{k \to \infty} \int_{Q_{0,T}} f \, dxdt. \quad (50)
\]
Hence, by (48), we get
\[
\int_{Q_{0,T}} (2\alpha(x)-1)|\psi_{\alpha(x)-1}(u) u_t|^2 \, dxdt = \int_{Q_{0,T}} (\psi_{2\alpha(x)-1}(u) u_t) \, dxdt \bigg|_{t=0}^{t=T} - \int_{Q_{0,T}} (\psi_{\alpha(x)-1}(u) u_t) \, dxdt. \quad (51)
\]
By (50), we obtain
\[
||\chi_k(u) \sqrt{2\alpha - 1}\psi_{\alpha(x)-1}(u) u_t;L^2(Q_{0,T})|| \longrightarrow ||\sqrt{2\alpha - 1}\psi_{\alpha(x)-1}(u) u_t;L^2(Q_{0,T})||. \quad (52)
\]
Whence, using (49), we prove the following convergence
\[
\chi_k(u)\psi_{\alpha(x)-1}(u) u_t \xrightarrow{k \to \infty} \psi_{\alpha(x)-1}(u) u_t \quad \text{strongly in} \quad L^2(Q_{0,T}). \quad (53)
\]
Take \( v \in L^2(Q_{0,T}) \) such that \( v_t \in L^1(Q_{0,T}) \). Using (53) and (42) with \( \beta = \alpha \), we obtain

\[
\int_{Q_{0,T}} \alpha(x) \psi_{\alpha(x)}^{-1} (u_t) v \ dx dt = \int_{\Omega_0} \psi_{\alpha(x)}(u) v \ dx \bigg|_t^{t=T} - \int_{Q_{0,T}} \psi_{\alpha(x)}(u) v_t \ dx dt.
\]

(54)

Taking in (54) function \( v \in C^0_{\infty}(Q_{0,T}) \), we get

\[
\int_{Q_{0,T}} \alpha(x) \psi_{\alpha(x)}^{-1} (u_t) v \ dx dt = - \int_{Q_{0,T}} \psi_{\alpha(x)}(u) v_t \ dx dt.
\]

Therefore, (45) holds. Thus, \( \psi_{\alpha(x)}(u) v \in L^2(Q_{0,T}) \). Moreover, \( \psi_{\alpha(x)}(u) \in L^\infty(Q_{0,T}) \).

Further let us prove Case 2. We start from proof of (46) and (47) if \( u \in C^2(\overline{Q_{0,T}}) \). Since \( p \geq 2\alpha \) and so \( 2 \leq \frac{p}{\alpha} \), estimate (40) with \( Q = Q_{0,T} \) and \( q \equiv 2 \) yields that (46) holds. Using (45) (we already proved it), by (51), we obtain

\[
\int_{Q_{0,T}} \left( \psi_{\alpha(x)}(u) \right)_t^2 \ dx dt \leq \frac{[\alpha^0]^2}{2\alpha - 1} \int_{\Omega_0} \psi_{2\alpha(x)-1}(u)|u_t| \ dx + \int_{\Omega_T} \psi_{2\alpha(x)-1}(u)|u_t| \ dx + \int_{Q_{0,T}} \psi_{2\alpha(x)-1}(u)|u_t| \ dx dt.
\]

(55)

On the other hand, for every \( r \in \mathcal{R}_+(\Omega) \) \((r_0 > 1)\) the estimate

\[
J_1 := \int_{\Omega_0} \psi_{2\alpha(x)-1}(u)|u_t| \ dx \leq \| \psi_{2\alpha(x)-1}(u(0)); \mathcal{L}^{r'(x)}(\Omega) \| \cdot \| u_t(0); \mathcal{L}^{r(x)}(\Omega) \|
\]

holds. Since \( p \geq 2\alpha > 1 \), for \( r = 2\alpha \) the following inequalities are true: \( 1 < r \leq p, 2\alpha - 1 < p \), and \( 1 < r' = \frac{p}{\alpha} = \frac{2\alpha - 1}{\alpha - 1} \leq \frac{p}{2\alpha - 1} \). Then estimate (40) with \( Q = \Omega \) and \( q = r' \) implies that

\[
\| \psi_{2\alpha(x)-1}(u(0)); \mathcal{L}^{r'(x)}(\Omega) \| \leq C_{10} S_{2\alpha-1/p}(\rho_p(u(0); \Omega))
\]

holds. In addition, by Proposition 3, we obtain

\[
\| u_t(0); \mathcal{L}^{r(x)}(\Omega) \| \leq C_{11} \| u_t(0); \mathcal{L}^{p(x)}(\Omega) \| \leq C_{11} S_{1/p}(\rho_p(u(0); \Omega)).
\]

Therefore, by (31) with \( p = p(x) \) and by the equality \( S_a(z)S_b(z) = S_{a+b}(z) \ (a, b, z \geq 0) \), we get

\[
J_1 \leq C_{12} S_{2\alpha-1/p}(\int_{\Omega} |u(0)|^{p(x)} \ dx) S_{1/p}(\int_{\Omega} |u_t(0)|^{p(x)} \ dx)
\]

\[
\leq C_{13} S_{2\alpha/p}(\int_{Q_{0,T}} \left[ |u(x,t)|^{p(x)} + |u_t(x,t)|^{p(x)} + |u_{tt}(x,t)|^{p(x)} \right] \ dx dt)
\]

where \( C_{13} > 0 \) is independent of \( u \). Continuing in the same way, by (55), we obtain (47).

Finally let us take \( u, u_t, u_{tt} \in L^{p(x)}(Q_{0,T}), p \geq 2\alpha \), and \( \{ u^\ell \}_{\ell \in \mathbb{N}} \subset C^2(\overline{Q_{0,T}}) \), where

\[
u^\ell \longrightarrow u, u_t^\ell \longrightarrow u_t, u_{tt}^\ell \longrightarrow u_{tt} \text{ strongly in } L^{p(x)}(Q_{0,T}).
\]
Estimates (46) and (47) imply that the sequence \( \{ \psi_{\alpha(x)}(u^f) \} \) is bounded in \( W^{1,2}(0,T;L^2(\Omega)) \). Then there exists a subsequence \( \{ u^{f_k} \} \subseteq \{ u^f \} \) such that
\[
\psi_{\alpha(x)}(u^{f_k}) \rightharpoonup \chi \quad \text{weakly in} \quad W^{1,2}(0,T;L^2(\Omega))
\]
and so \( \alpha \psi_{\alpha(x)-1}(u^{f_k})u^{f_k} = \psi_{\alpha(x)}(u^{f_k}) \rightharpoonup \zeta \) weakly in \( L^2(Q_{0,T}) \). By choosing the sequence \( \{ u^f \} \subseteq \mathbb{N} \), there exists a subsequence (we call it \( \{ u^{f_k} \} \)) such that
\[
{u^{f_k}}_{k\to\infty} \to u, \quad {u^{f_k}}_{k\to\infty} \to u \quad \text{almost everywhere in} \quad Q_{0,T},
\]
\[
\alpha(x){u^{f_k}}_{k\to\infty} \to \alpha(x)u \quad \text{almost everywhere in} \quad Q_{0,T}.
\]
Therefore \( \chi = \psi_{\alpha(x)}(u) \) and \( \zeta = \alpha \psi_{\alpha(x)-1}(u)u \). Thus, (54) with \( v \in W^{1,2}(0,T;L^2(\Omega)) \) holds and so (45) is true.

Since \( \{ u^f \} \subseteq C^2(\overline{Q_{0,T}}) \) inequality (46) and (47) with \( u = u^f \) are true, i.e.
\[
\| \psi_{\alpha(x)}(u^{f_k});L^2(\Omega) \rangle \| \leq C_7S_{\alpha/p}(p_p(u^{f_k});Q_{0,T}),
\]
\[
\| (\psi_{\alpha(x)}(u^{f_k}))_t;L^2(Q_{0,T}) \rangle \| \leq C_8S_{\alpha/p}(p_p(u^{f_k});Q_{0,T}) + p_p(u^{f_k};Q_{0,T}) + p_p(u^{f_k};Q_{0,T}),
\]
where \( C_7, C_8 > 0 \) are independent of \( u^{f_k}, \ell_k \). Thus, using Lemma 5.3 [27, p. 20], we obtain (46) and (47). □

Notice that the case \( \alpha(x) \in (0, \frac{1}{2}] \) (see Theorem 3) is considered in [12].

**Theorem 4.** Assume that \( r \in \mathcal{P}_+(\Omega) \). Then the following statements are fulfilled:

1. If \( \frac{1}{2} < r_0 \leq r^0 \leq 1 \), then the equality (32) is true if one of the following alternatives hold:
   
   (i) \( u \in C^2(\overline{Q_{0,T}}) \) (here we have \( |u|^{r(x)} \in L^\infty(Q_{0,T}) \) and \( \{ |u|^{r(x)} \}_x \in L^2(Q_{0,T}) \));
   
   (ii) \( u, u, u_0 \in L^{r(x)}(Q_{0,T}) \) and \( p(x) \geq 2r(x) \) for a.e. \( x \in \Omega \) (here we have \( |u|^{r(x)} \in H^1_0(0,T;L^2(\Omega)) \)) and
   \[
   \| |u|^{r(x)};H^1(0,T;L^2(\Omega)) \| \leq C_{14}S_{r/p}(p_p(u;Q_{0,T}) + p_p(u_0;Q_{0,T}) + p_p(u_0;Q_{0,T})),
   \]
   where \( C_{14} > 0 \) is independent of \( u \).

2. If \( \frac{1}{2} < r_0 \leq r^0 \leq 2 \), then the equality (33) is true if one of the following alternatives hold:
   
   (i) \( u \in C^2(\overline{Q_{0,T}}) \) (here we have \( |u|^{r(x)}-2u \in L^\infty(Q_{0,T}) \) and \( \{ |u|^{r(x)}-2u \}_x \in L^2(Q_{0,T}) \));
   
   (ii) \( u, u, u_0 \in L^{r(x)}(Q_{0,T}) \) and \( p(x) \geq 2(r(x) - 1) \) for a.e. \( x \in \Omega \) (here we have \( |u|^{r(x)}-2u \in H^1_0(0,T;L^2(\Omega)) \)) and
   \[
   \| |u|^{r(x)}-2u;H^1(0,T;L^2(\Omega)) \| \leq C_{15}S_{r-1/p}(p_p(u;Q_{0,T}) + p_p(u_0;Q_{0,T}) + p_p(u_0;Q_{0,T})),
   \]
   where \( C_{15} > 0 \) is independent of \( u \).

**Proof.** The proof of Theorem 4 follows from Theorem 3 if we recall that
\[
|s|^{r(x)} = \psi_{r(x)}(s) + \psi_{r(x)}(-s), \quad |s|^{r(x)}-2s = \psi_{r(x)-1}(s) - \psi_{r(x)-1}(-s), \quad x \in \Omega, \quad s \in \mathbb{R}.
\]

Notice that we omit the notation \( \psi \) in the right-hand sides of (32) and (33). Thus the right-hand sides of (32) and (33) equal zero on \( \{ (x,t) \in Q_{0,T} \mid u(x,t) = 0 \} \). □

If we take \( p(x) \equiv \text{const} \) and \( r(x) \equiv \text{const} \), we rewrite Theorem 4 as follows.
Theorem 5. Assume that $r > 0$. Then the following statements are fulfilled:

(1) If $\frac{1}{2} < r \leq 1$, then the equality

$$\langle |u|^r \rangle_t = r|u|^{r-2}u_t$$

is true if one of the following alternatives hold:

(i) $u \in C^2(Q_{0,T})$ (here we have $|u|^r \in C(Q_{0,T})$ and $\langle |u|^r \rangle_t \in L^2(Q_{0,T})$);
(ii) $u \in W^{2,p}(0,T;L^p(\Omega))$ and $p \geq 2r$ (here we have $|u|^r \in H^1(0,T;L^2(\Omega))$ and

$$\| |u|^r; H^1(0,T;L^2(\Omega)) \| \leq C_{16} (\| u; L^p(Q_{0,T}) \|^r + \| u_t; L^p(Q_{0,T}) \|^r)$$

where $C_{16} > 0$ is independent of $u$.

(2) If $\frac{1}{2} < r \leq 2$, then the equality

$$\langle |u|^{r-2}u \rangle_t = (r-1)|u|^{r-2}u_t$$

is true if one of the following alternatives hold:

(i) $u \in C^2(Q_{0,T})$ (here we have $|u|^{r-2}u \in C(Q_{0,T})$ and $\langle |u|^{r-2}u \rangle_t \in L^2(Q_{0,T})$);
(ii) $u \in W^{2,p}(0,T;L^p(\Omega))$ and $p \geq 2(r-1)$ (here we have $|u|^{r-2}u \in H^1(0,T;L^2(\Omega))$ and

$$\| |u|^{r-2}u; H^1(0,T;L^2(\Omega)) \| \leq C_{17} (\| u; L^p(Q_{0,T}) \|^{r-1} + \| u_t; L^p(Q_{0,T}) \|^{r-1} + \| u_t; L^p(Q_{0,T}) \|^{r-1})$$

where $C_{17} > 0$ is independent of $u$.

3.3 Boundary value problem for some elliptic equation

The following Dirichlet-Neumann boundary value problem is needed for the sequel

$$-\varepsilon u_{tt} + (\partial u^\varepsilon)_{t} - a \Delta u^\varepsilon + G u^\varepsilon + \phi (E u^\varepsilon) = f(x,t), \quad (x,t) \in Q_{0,T},$$

$$u^\varepsilon|_{t=0} = 0, \quad u^\varepsilon|_{t=\tau} = 0,$$

where $\varepsilon > 0$. By definition, put $U_0(Q_{0,T}) := \{ v \in H^1(Q_{0,T}) \mid v|_{t=0} = 0, \ v|_{t=\tau} = 0 \}$,

$$D(x,t) := \left( 1 - \frac{t}{T} \right)^2 u_0(x), \quad (x,t) \in Q_{0,T}.$$  

(64)

Clearly, (64) and condition (U) yield that $D \in H^2(Q_{0,T})$.

$$D|_{t=0} = 0, \quad D|_{t=\tau} = 0, \quad D|_{t=\tau} = 0.$$  

(65)

Definition 2. A real-valued function $u^\varepsilon \in H^1(Q_{0,T})$ is called a weak solution of problem (62)-(63) if $u$ satisfies (63)1, (63)2 (i.e. $u^\varepsilon - D \in U_0(Q_{0,T})$, where $D$ is defined by (64)), $\partial u^\varepsilon, Gu^\varepsilon, Eu^\varepsilon \in L^2(Q_{0,T})$, and for every $v \in U_0(Q_{0,T})$ we have

$$\int_{Q_{0,T}} \left[ \varepsilon u^\varepsilon_t v_t - \partial u^\varepsilon v_t + a(\nabla u^\varepsilon, \nabla v) + Gu^\varepsilon v + \phi (E u^\varepsilon) v \right] dt dx + \int_{\Omega} \partial u^\varepsilon v(T) dx = \int \varepsilon v dx dt.$$  

(66)
For functions \( w, v : Q_{0,T} \rightarrow \mathbb{R}^1 \) by definition, put
\[
\mathcal{L}_\varepsilon(w,v) := \int_{Q_{0,T}} \left[ \varepsilon w_t v_t + |w|^{r-2}w_t v + a(\nabla w, \nabla v) + Gw v + \phi(Ew)v \right] \, dxdt. \tag{67}
\]

**Lemma 6.** If \( r \in (1, 2) \), conditions \((G)-(\Phi)\) hold, and \((Z)\) is satisfied, then there exists a constant \( \alpha_0 > 0 \) such that for every \( u \in H^1(Q_{0,T}) \) and \( \varepsilon > 0 \) the estimate
\[
\mathcal{L}_\varepsilon(u,u) \geq \frac{1}{r} \int_{Q_{0,T}} |u(x,t)|^r \, dx \bigg|_{t=0}^{t=T} + \int_{Q_{0,T}} \left[ |u_t|^2 + \alpha_0 (|\nabla u|^2 + |u|^2 + |u|^r) + g_0|u|^{q(x)} \right] \, dxdt - C_{18} \tag{68}
\]
holds, where \( C_{18} > 0 \) is independent of \( u, \varepsilon \).

**Proof.** By (67), it follows that
\[
\mathcal{L}_\varepsilon(u,u) = \int_{Q_{0,T}} \left[ |u_t|^2 + |u_t|^2 u_t + a(\nabla u, \nabla u) + g|u|^{q(x)} + \phi(Eu)u \right] \, dxdt. \tag{69}
\]

Since \( u \in H^1(Q_{0,T}) \) and \( 2 \geq r > 1 \), Case 1.ii of Theorem 2 with \( p(x) \equiv 2 \) and \( r(x) \equiv r \) (see also Proposition 9) implies that
\[
|u|^r \in W^{1,2}(0,T;L^2(\Omega)), \quad (|u|^r)_t = r|u|^{r-2}u_t u, \quad \int_{Q_{0,T}} (|u|^r)_t \, dxdt = \int_{\Omega} |u(x,t)|^r \, dx \bigg|_{t=0}^{t=T}.
\tag{70}
\]
Taking into account condition \((\Phi)\), the Cauchy-Bunyakowski-Schwarz inequality, (16), and (17), we obtain
\[
\int_{Q_{0,T}} \phi(Eu)u \, dxdt \leq \Phi^* ||Eu; L^2(Q_{0,T})|| \cdot ||u; L^2(Q_{0,T})|| \leq \Phi^* E^* ||u; L^2(Q_{0,T})||^2 \leq \Phi^* E^* M_\Omega \int_{Q_{0,T}} |\nabla u|^2 \, dxdt.
\]

Using the Young inequality with the exponents \( \frac{2}{r}, \frac{2}{2-r} \geq 1 \) and inequality (17), we get
\[
\int_{\Omega} |u_t|^r \, dx \leq \int_{\Omega} |u(t)|^2 \, dx + C_{19} \leq M_\Omega \int_{\Omega} |\nabla u(t)|^2 \, dx + C_{19}, \quad t \in [0,T],
\]
where \( C_{19} > 0 \) is independent of \( u, t \). Then for \( \alpha := a - \Phi^* E^* M_\Omega > 0 \) we have
\[
\int_{Q_{0,T}} \left[ a|\nabla u|^2 + \phi(Eu)u \right] \, dxdt \geq \alpha \int_{Q_{0,T}} |\nabla u|^2 \, dxdt = \int_{Q_{0,T}} \left( \frac{\alpha}{3} + \frac{\alpha}{3} \right)|\nabla u|^2 \, dxdt \geq \int_{Q_{0,T}} \left[ \frac{\alpha}{3} |\nabla u|^2 + \frac{\alpha}{3M_\Omega} |u|^2 + \frac{\alpha}{3M_\Omega} |u|^r \right] \, dxdt - C_{20}.
\]
Taking into account (69) and the transformations above, we obtain (68). \( \square \)

**Theorem 6.** Suppose that \( \partial \Omega \in C^4 \) and the following conditions are satisfied:
\begin{itemize}
  \item[(A):] \( a > 0, \ \frac{2}{3} < r \leq 2; \)
  \item[(Q):] \( q \in \mathcal{B}_+(\Omega), \ 1 < q_0 \leq q \leq 2; \)
  \item[(G):] \( g \in \mathcal{B}_+(Q_{0,T}); \)
  \item[(E):] \( \bar{\xi} \in L^\infty(Q_{0,T} \times \Omega); \)
  \item[(\Phi):] \( \phi \in \text{Lip}(\mathbb{R}), \ |\phi(\xi)| \leq \phi^*(|\xi|) \text{ for every } \xi \in \mathbb{R}, \text{ where } \phi^* \in [0, +\infty); \)
  \item[(F):] \( f \in L^\rho(Q_{0,T}), \text{ where } \frac{1}{r} + \frac{\rho}{r} = 1; \)
\end{itemize}
\( (U) \): \( u_0 \in H^2(\Omega) \cap H_0^1(\Omega) \);

\( (Z) \): \( a > \phi^* E M_\Omega \), where \( E^* \) is defined by \( (16) \), \( M_\Omega \) is determined from \( (17) \).

Then problem \((62)-(63)\) has a weak solution \( u^\varepsilon \) such that \( u^\varepsilon \in H^2(Q_{0,T}) \cap H^1(0,T;H_0^1(\Omega)) \) and \( \mathcal{B} u^\varepsilon \in H^1(0,T;L^2(\Omega)) \).

**Proof.** The Case \( r = 2 \) is trivial. Let us assume that \( r \in (\frac{1}{2},2) \). The solution will be constructed via Galerkin’s method.

**Step 1.** Let \( \{v^m\}_{m \in \mathbb{N}} \) be determined from Remark 1, \( D \) defined by \( (64) \), \( \varepsilon > 0 \) a fixed number, \( \mathcal{L}_\varepsilon \) defined by \( (67) \),

\[
  u^{\varepsilon,m}(x,t) := D(x,t) + \varepsilon^{m}(x,t), \quad \varepsilon^m(x,t) := \sum_{\mu = 1}^{m} \phi^{m,\mu} \phi^{m,\mu}(x,t), \quad (x,t) \in Q_{0,T}, \quad m \in \mathbb{N},
\]

and \( \phi := (\phi^{1,\varepsilon}, \ldots, \phi^{m,\varepsilon}) \in \mathbb{R}^m \). For the sake of convenience we have omitted index \( \varepsilon \) in \( u^{\varepsilon,m}, \phi^{1,\varepsilon}, \ldots, \phi^{m,\varepsilon} \), and \( \varepsilon^{m} \). Notice also that

\[
  \varepsilon^m|_{\Sigma_{0,T}} = 0, \quad \varepsilon^m|_{t=0} = 0, \quad \varepsilon^m|_{t=T} = 0, \quad (72)
\]

\[
  u^m|_{\Sigma_{0,T}} = 0, \quad u^m|_{t=0} = u_0, \quad u^m|_{t=T} = 0. \quad (73)
\]

Since \( r < 2 \), we have \( 2 \geq 2(r-1) \). In addition, the inequality \( r > \frac{3}{2} \) holds. Since \( \partial \Omega \in C^2 \), Remark 1 implies that \( \{v^m\}_{m \in \mathbb{N}} \subset H^2(Q_{0,T}) \). Then \( \{v^m\}_{m \in \mathbb{N}}, \{u^m\}_{m \in \mathbb{N}} \subset H^2(Q_{0,T}) \) and Case 2 ii of Theorem 5 with \( p = 2 \) yields that

\[
  \mathcal{B} u^m \in H^1(0,T;L^2(\Omega)), \quad (\mathcal{B} u^m)_r = |u^m|^{r-2}u^m. \quad (74)
\]

Assume that the unknown vector \( \phi \) (see \( (71) \)) satisfies the following equalities:

\[
  \int_{Q_{0,T}} \left[ E u^\mu \phi^\mu + |u^m|^{r-2}u^\mu \phi^\mu + a(\nabla u^m, \nabla \phi^\mu) + G u^m \phi^\mu + \phi(Ed^m)\phi^\mu \right] dx \, dt = \int_{Q_{0,T}} f \phi^\mu dx \, dt, \quad (75)
\]

where \( \mu = 1, m \). Let us prove that this \( \phi \) exists. Equality \( (75) \) we rewrite to read

\[
  P(\phi) = 0, \quad (76)
\]

where \( P := (P_1, \ldots, P_m), P_\mu(\phi) := \mathcal{L}_\varepsilon(u^\mu, \phi^\mu) - \int_{Q_{0,T}} f \phi^\mu dx \, dt, \quad \mu = 1, m \). Clearly,

\[
  (P(\phi), \phi)|_{\mathbb{R}^m} = \sum_{\mu = 1}^{m} \left( \mathcal{L}_\varepsilon(u^\mu, \phi^\mu) - \int_{Q_{0,T}} f \phi^\mu dx \, dt \right) \phi^\mu = \mathcal{L}_\varepsilon(u^m, \phi^m) - \int_{Q_{0,T}} f \phi^m dx \, dt
\]

\[
  = \mathcal{L}_\varepsilon(u^m, u^m - D) - \int_{Q_{0,T}} f(u^m - D) dx \, dt = \int_{Q_{0,T}} f D dx \, dt - \int_{Q_{0,T}} f u^m dx \, dt + \mathcal{L}_\varepsilon(u^m, u^m) - \mathcal{L}_\varepsilon(u^m, D). \quad (77)
\]

Using Young’s inequality, we obtain

\[
  |fD| \leq \frac{1}{2} |f|^2 + \frac{1}{2} |D|^2, \quad |fu^m| \leq \varkappa_1 |u^m|^2 + \frac{1}{2\varkappa_1} |f|^2, \quad \varkappa_1 > 0.
\]

By \( (68) \) and \( (73_2) \), we get

\[
  \mathcal{L}_\varepsilon(u^m, u^m) \geq \frac{1}{r} \int_{Q_{0,T}} |u^m(T)|^r \, dx + \int_{Q_{0,T}} \left[ E |u^\mu|^2 + a_0 |\nabla u^m|^2 + a_0 |\nabla \phi^\mu|^2 + \frac{a_0}{r} |u^m|^r + g_0 |u^m|^r \phi^\mu \right] dx \, dt - \frac{1}{r} \int_{Q_{0,T}} |u_0|^r \, dx - C_{21}.
\]

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where $C_{21} > 0$ is independent of $m, \varepsilon$.

Now let us consider the expression

$$\mathcal{L}_n(u^m, D) = \int_{\Omega \times T} \left[ Eu^m_t D_t + |u^m|^{r-2} u^m D + a(\nabla u^m, \nabla D) + Gu^m D + \phi(Eu^m) D \right] \, dx dt.$$

Using Young’s inequality, we obtain

$$|u^m D_t| \leq \frac{1}{2} |u^m|^2 + \frac{1}{2} |D_t|^2, \quad |a(\nabla u^m, \nabla D)| \leq a \varkappa_2 |\nabla u^m|^2 + \frac{a}{4 \varkappa_2} |\nabla D|^2, \quad \varkappa_2 > 0.$$

Taking into account (74), integration by parts formulae, (65), (73), and the Young inequality with $r', r > 1$, we obtain

$$\int_{\Omega \times T} |u^m|^{r-1} |D_t| \, dx \, dt \leq \int_{\Omega \times T} |u^m|^{r-1} |D_t| \, dx \, dt \leq \int_{\Omega \times T} \left[ \varkappa_3 |u^m|^r + C_{22}(\varkappa_3) |D_t|^r \right] \, dx \, dt, \quad \varkappa_3 > 0.$$

By generalized Young’s inequality with $q'(x), q(x) > 1$, it follows that

$$|Gu^m D| \leq \varepsilon_0 |u^m|^{q'(x)} |D| \leq \varepsilon_0 |u^m|^{q(x)} + C_{23}(\varepsilon_4) |D|^{q(x)}, \quad \varepsilon_4 > 0.$$

Taking into account the Cauchy-Bunyakowski-Schwarz inequality, the Young inequality with $r', r > 1$, and (16), we get

$$\int_{\Omega \times T} \phi(Eu^m) D \, dx \, dt \leq \phi^*(|Eu^m|; L^2(\Omega \times T)) \cdot |D; L^2(\Omega \times T)|$$

$$\leq \phi^*(|u^m|; L^2(\Omega \times T)) \cdot |D; L^2(\Omega \times T)| \leq \varepsilon_5 \int_{\Omega \times T} |u^m|^2 \, dx \, dt + C_{24}(\varepsilon_5) \int_{\Omega \times T} |D|^2 \, dx \, dt, \quad \varepsilon_5 > 0.$$

Here $C_{22}, \ldots, C_{24} > 0$ are independent of $m$.

By (U) and (F), it follows that $\int_{\Omega} |u_0|^r \, dx + \int_{\Omega} |f|^2 + |D|^2 + |D_t|^r + |D_t|^2 + |\nabla D|^2 \, dx \, dt \leq C_{25}$. Therefore, (77) yields that

$$\mathcal{L}_n(u^m, \varphi^m) - \int_{\Omega \times T} f u^m \, dx \, dt \geq \frac{1}{7} \int_{\Omega \times T} |u^m(T)|^r \, dx \, dt + \int_{\Omega \times T} \left[ \left( \varepsilon - \frac{\varepsilon}{7} \right) |u^m|^2 + (\alpha_0 - \alpha_2 |\nabla u^m|^2$$

$$+ (\alpha_0 - \alpha_1 - \alpha_5) |u^m|^2 + (\alpha_0 - \alpha_3) |u^m|^r + (\alpha_0 - \alpha_4) |u^m|^{q(x)} \right] \, dx \, dt - C_{26}(\alpha_1, \ldots, \alpha_5),$$

where $C_{26} > 0$ is independent of $m$. Choosing $\alpha_1, \ldots, \alpha_5 > 0$ small enough, we obtain the existence of the nonnegative constants $C_{27}$ and $C_{28}$ such that (see (77))

$$(P(\varphi, \varphi))_{\Omega \times T} \geq C_{27} \int_{\Omega \times T} \left[ |\nabla u^m|^2 + |u^m|^2 \right] \, dx \, dt - C_{28} \rightarrow +\infty \quad \text{as} \quad |\varphi| \rightarrow +\infty.$$

Thus, the Vishny Lemma (see Proposition 8) implies that there exist a solution $\varphi^m_1, \ldots, \varphi^m_m$ of system (76), i.e. (75).
Step 2. Multiplying (75) by \( \varphi_m^\mu \) and summing in \( \mu = 1, m \), we obtain

\[
\mathcal{L}_\varepsilon(u^m, \varepsilon^m) = \int_{Q_0,T} f \varepsilon^m \, dxdt.
\]

Then 0 = \( \mathcal{L}_\varepsilon(u^m, \varepsilon^m) - \int_{Q_0,T} f \varepsilon^m \, dxdt \) and, using (78), we get

\[
\varepsilon \int_{Q_0,T} |u^m(t)|^2 \, dxdt + \int_{\Omega} |u^m(T)|^t \, dx + \int_{Q_0,T} \left[ |\nabla u^m|^2 + |u^m|^2 + |u^m|^\prime + |u^m|^q(s) \right] \, dxdt \leq C_{29}.
\]  (79)

By (79) and Proposition 3, we have

\[
||u^m; L^q(s)(Q_{0,T})|| \leq C_{30}.
\]  (80)

Taking into account Lemma 2, (79), and (80), we obtain

\[
||Gu^m; L^q(s)(Q_{0,T})|| \leq C_{31}, \quad ||\mathcal{R}_u u^m; L^q(t)(Q_{0,T})|| \leq C_{32},
\]  (81)

\[
||\mathcal{R}_u u^m(T); L^q(t)(\Omega)|| \leq C_{33}.
\]  (82)

Here \( C_{29}, \ldots, C_{33} \geq 0 \) are independent of \( m, \varepsilon \).

Estimates (79)-(82) yield that there exists a subsequence \( \{u^{m_j}\}_{j \in \mathbb{N}} \subset \{u^m\}_{m \in \mathbb{N}} \) such that

\[
u^{m_j} \rightarrow u \quad \text{weakly in} \quad H^1(Q_{0,T}) \cap L^t(Q_{0,T}) \cap L^q(s)(Q_{0,T}),
\]  (83)

\[Gu^{m_j} \rightarrow \tilde{\chi}_1 \quad \text{weakly in} \quad L^q(s)(Q_{0,T}),
\]  (84)

\[\mathcal{R}_u u^{m_j} \rightarrow \tilde{\chi}_2 \quad \text{weakly in} \quad L^t(Q_{0,T}),
\]  (85)

\[\mathcal{R}_u u^{m_j}(T) \rightarrow \tilde{\chi}_3 \quad \text{weakly in} \quad L^t(\Omega).
\]  (86)

By (83), the Rellich-Kondrachov theorem (see Lemma 1.28 [27, p. 47]), and Lemma 1.18 [27, p. 39], it follows that there exists a subsequence (we call it \( \{u^{m_j}\}_{j \in \mathbb{N}} \) again) such that

\[
u^{m_j} \rightarrow u \quad \text{strongly in} \quad L^2(Q_{0,T}) \quad \text{and almost everywhere in} \quad Q_{0,T}.
\]  (87)

By the Aubin theorem (see Proposition 7), we get

\[
u^{m_j} \rightarrow u \quad \text{in} \quad C([0,T];L^2(\Omega)).
\]  (88)

Therefore, since \( E \) is continuous and \( \phi \) satisfies the Lipschitz condition, we obtain

\[
\phi(Eu^{m_j}) \rightarrow \phi(Eu) \quad \text{strongly in} \quad L^2(Q_{0,T}).
\]  (89)

In addition,

\[
u^{m_j}(T) \rightarrow u(T) \quad \text{strongly in} \quad L^2(\Omega) \quad \text{and almost everywhere in} \quad \Omega.
\]  (90)
By (87), it follows that $Gu^{m_j} \rightarrow Gu$ almost everywhere in $Q_{0,T}$. Hence, Proposition 4 yields that $\chi_1 = Gu$. Continuing in the same way, we see that $\chi_2 = Gu$ and $\chi_3 = Gu(T)$.

Using (75) with $m = m_j$, (74), and the condition $\nu^\mu|_{t=0} = 0$, we obtain

$$\int_{Q_{0,T}} \left[ Eu^{m_j} v^\mu_t - \overline{R} u^{m_j} v^\mu_t + a(\nabla u^{m_j}, \nabla v^\mu) + Gu^{m_j} v^\mu + \phi(Eu^{m_j}) v^\mu \right] dxdt + \int_\Omega \overline{R} u^{m_j}(T) v^\mu(T) dx = \int f v^\mu dxdt,$$  

(91)

where $\mu = \sum m_j$. Letting $j \rightarrow \infty$, taking into account the properties of $\{\nu^\mu\}_{\mu \in \mathbb{N}}$ and (91), we get (66). Inequalities $r' > 2$, $q'(x) \geq 2$ for a.e. $x \in \Omega$, (16), and (81) imply that $\overline{R} u, Gu, Eu \in L^2(Q_{0,T})$. Thus, $u$ is a weak solution to problem (62)-(63).

**Step 3.** Now we shall prove the additional estimates. By definition, put

$$N'w := Gw + \phi(Ew), \quad w \in L^2(Q_{0,T}).$$  

(92)

By inequalities $r' > 2$, $q'(x) \geq 2$ for a.e. $x \in \Omega$, (16), and (81), it follows that

$$\int_{Q_{0,T}} \left[ |\overline{R} u|^2 + |N^\nu|^2 \right] dxdt \leq C_{34},$$  

(93)

where $C_{34} > 0$ is independent of $m, \varepsilon$.

We recall that (74) holds, $\frac{1}{2} < r \leq 2$, and $u^m \in H^2(Q_{0,T})$. Whence, using (61) with $p = 2$ (note that $p \geq 2(r-1)$), we obtain

$$\left| (|u^m|^{r-2}u^m)_t;L^2(Q_{0,T}) \right| \leq C_{35}(||u^m;L^2(Q_{0,T})||^{r-1} + ||u^m_t;L^2(Q_{0,T})||^{r-1} + ||u^m_t;L^2(Q_{0,T})||^{r-1}),$$  

(94)

where $C_{35} > 0$ is independent of $m$. By (30) and (73), it follows that

$$||u^m_t;L^2(Q_{0,T})||^2 \leq C_{36} \int_\Omega \left[ \left| u^m_t(x,T) \right|^2 dx + \left| u^m_t;L^2(Q_{0,T}) \right|^2 \right] dt = C_{37} ||u^m_t;L^2(Q_{0,T})||^2.$$

Thus, taking into account (79), the Young inequality with $\frac{1}{\rho+1} = \frac{1}{2} > 1$, and (94), we get

$$\left| (\overline{R} u^m)_t;L^2(Q_{0,T}) \right| \leq \varepsilon ||u^m_t;L^2(Q_{0,T})|| + C_{38}(\varepsilon_1), \quad \varepsilon_1 > 0,$$  

(95)

where $C_{38}(\varepsilon_1) > 0$ is independent of $m, \varepsilon$.

**Step 4.** Assume that the positive numbers $\{\sigma_m\}_{m \in \mathbb{N}}$ are determined from Remark 1. Hence, in particular, (28) hold. Using (73) and the condition $\nu^\mu|_{t=0} = 0$, we have

$$\int_{Q_{0,T}} Eu^{m_j} v^\mu_t dxdt = \int_{\Omega} Eu^{m_j} v^\mu dx \bigg|_{t=0} - \int_{Q_{0,T}} Eu^{m_j} v^\mu dxdt = - \int_{Q_{0,T}} Eu^{m_j} v^\mu dxdt.$$
Then we multiply (75) by $\varphi_\mu m^n$, use (28), and sum in $\mu = 1, m$. Taking into account (74) and (71), we obtain
\[
\int_\Omega \left[ -\varepsilon m^n(D_H - u_H^n) + (\mathcal{R}u^n)\xi_t(D_H - u_H^n) + a(\nabla u^n, \nabla (-u_H^n)) + \mathcal{N}u^n(D_H - u_H^n) \right] \, dxdt = \int_\Omega f(D_H - u_H^n) \, dxdt. \tag{96}
\]

The Young inequality and condition (Φ) imply that
\[
|u_H^n D_H| \leq \frac{1}{2} |u_H^n|^2 + \frac{1}{2} |D_H|^2, \quad |f D_H| \leq \frac{1}{2} |f|^2 + \frac{1}{2} |D_H|^2, \quad |f u_H^n| \leq \kappa_2 |u_H^n|^2 + \frac{1}{2} \kappa_2 |f|^2, \quad \kappa_2 > 0,
\]
\[
|(\mathcal{R} u^n)_t(D_H - u_H^n)| \leq \frac{\kappa_3}{2} |u_H^n|^2 + C_{39} (\kappa_3) \left( |(\mathcal{R} u^n)|_2^2 + |D_H|^2 \right), \quad \kappa_3 > 0,
\]
\[
|\mathcal{N} u^n(D_H - u_H^n)| \leq \kappa_4 |u_H^n|^2 + C_{40} (\kappa_4) (|\mathcal{N} u^n|^2 + |D_H|^2), \quad \kappa_4 > 0.
\tag{97}
\]

By (97) and (95) with $\kappa_1 > 0$ small enough, it follows that
\[
\int_\Omega \left( (\mathcal{R} u^n)_t(D_H - u_H^n) \right) \, dxdt \leq \kappa_3 \int_\Omega |u_H^n|^2 \, dxdt + C_{41} (\kappa_3) \left( 1 + \int_\Omega |D_H|^2 \, dxdt \right).
\]

Since $\partial \Omega \in C^4$, Remark 1 yields that $\{v^m\}_{m \in \mathbb{N}} \subset H^4(\Omega, \Omega)$. Integrating by parts and using (72a, b), we get
\[
\int_\Omega a(\nabla u^n, -\nabla z^m) \, dxdt = - \int_\Omega a(\nabla D, \nabla z^m, \nabla z^m) \, dxdt = I_1 - \int_\Omega a(\nabla z^m, \nabla z^m) \, dxdt
\]
\[
= I_1 - \int_\Omega a(\nabla z^m, \nabla z^m) \, dx|_{t=0}^{t=T} + \int_\Omega a(\nabla z^m) \, dxdt = I_1 + \int_\Omega a(\nabla u^m - \nabla D)^2 \, dxdt
\]
\[
\geq I_1 + \frac{\alpha}{2} \int_\Omega |\nabla u^m|^2 \, dx + C_{42} \int_\Omega |D_D|^2 \, dxdt, \tag{98}
\]
where
\[
I_1 := - \int_\Omega a(\nabla D, \nabla z^m) \, dxdt.
\]

Integrating by parts in $t$, using (72a), and using (73a), we obtain
\[
I_1 = - \int_\Omega a(\nabla D, \nabla z^m) \, dx|_{t=0}^{t=T} + \int_\Omega a(\nabla D, \nabla z^m) \, dxdt = I_2 + \int_\Omega a(\nabla D, \nabla z^m) \, dxdt.
\]
where
\[
I_2 := \int_\Omega a(\nabla D, \nabla z^m(0)) \, dx.
\]

Integrating by parts in $x$ and using condition (U), we get
\[
I_2 = \int_{\partial \Omega} a(\nabla D(0), \nu) z^m(0) \, dS - \int_\Omega a(\Delta D(0), z^m(0)) \, dx = - \int_\Omega a z^m(0) \Delta D(0) \, dx.
\]
Moreover, \(|a(Du, Dz^m)| = |a(Du, Du^m - Du)| \leq \kappa_5|Du|^2 + C_{43}(\kappa_5)|Du|^2|Dz|^2\). Whence,
\[
|I_1| \leq \int_\Omega \left[ |z^m_0|^2 + C_{44}(\kappa_6)|\Delta D(0)|^2 \right] dx + \int_{Q_0,T} \left[ \kappa_5|Du|^2 + C_{45}(\kappa_5)|DV|^2 \right] dxdt, \quad \kappa_5, \kappa_6 > 0.
\]

By the estimate of type (30), (72), and (71), it follows that
\[
\int \int_\Omega |z^m(0)|^2 \leq C_{46} \left( \int \int_\Omega |z^m(T)|^2 dx + \int \int_{Q_0,T} |z^m_{\nu}|^2 dxdt \right) \leq C_{46} \int \int_{Q_0,T} |u^m_{\nu} - Du|^2 dxdt \leq C_{47} \int \int_{Q_0,T} |u^m|^2 + |Du|^2 dxdt,
\]
where \(C_{47} > 0\) is independent of \(m\).

Finally, from (98) we get
\[
\int_{Q_0,T} a(Du^m, Dz^m) \, dxdt \geq \int_{Q_0,T} \left[ \left( \frac{a}{2} - \kappa_5 \right)|Du|^2 - C_{47}(\kappa_6)|Du|^2 \right] dxdt
- C_{48}(\kappa_5, \kappa_6) \left( \int \int_\Omega |\Delta D(0)|^2 dx + \int \int_{Q_0,T} |DV|^2 \right) dxdt \right).
\]

Since \(\int_{\Omega} |\Delta D(0)|^2 dx \leq 0\), by (93), (95), and (96), it follows that
\[
\int_{Q_0,T} \left[ \left( \frac{a}{2} - \kappa_2 - \kappa_3 - \kappa_4 - C_{47}(\kappa_6) \right)|u^m_{\nu}|^2 + \left( \frac{a}{2} - \kappa_5 \right)|Du|^2 \right] dxdt \leq C_{50}(\kappa_2, \kappa_3), \kappa_4, \kappa_5, \kappa_6).
\]

Hence, choosing \(\kappa_2, \ldots, \kappa_6 > 0\) small enough, we have
\[
\int_{Q_0,T} \left[ |u^m_{\nu}|^2 + |DV|^2 \right] dxdt \leq C_{51}.
\]

Thus, (95) yields that
\[
||\hat{\Delta}u^m_{\nu}||L^2(Q_0,T) || \leq C_{52}.
\]

Here \(C_{50}, \ldots, C_{52} > 0\) are independent of \(m\).

**Step 5.** Assume that the positive numbers \(\tilde{\lambda}_{\mu} \in \mathbb{N}\) are determined from Remark 1. Hence, in particular, (28) hold. Clearly (here and below \(v\) denotes the outward unit vector field on \(\partial \Omega\)),
\[
\int_{Q_0,T} a(Du^m, Dv^m) \, dxdt = \int_{\Sigma_0,T} a(Du^m, v^m) \, dSdt - \int_{Q_0,T} a\Delta u^m \, v^m \, dxdt = - \int_{Q_0,T} a\Delta u^m \, v^m \, dxdt.
\]

Multiply (75) by \(\tilde{\lambda}_{\mu} Q_{\mu}^m\), use (28) and (74), and sum in \(\mu = 1, m\). Then we get
\[
\int_{Q_0,T} \left[ \epsilon_1 u^m_{\nu}(-\Delta z^m) + (\hat{\Delta} u^m, (\Delta D - Du^m) - a\Delta u^m(DT - Du^m)
+ \mathcal{N} u^m(DT - Du^m) \right] \, dxdt = \int_{Q_0,T} f(DT - Du^m) \, dxdt.
\]
Clearly (we recall that \( \{z^m\}_{m \in \mathbb{N}} \subset H^2(Q_{0,T}) \)),
\[
- \int_{Q_{0,T}} \varepsilon u_m^m \Delta z^m \, dxdt = - \int_{Q_{0,T}} \varepsilon u_m^m (\nabla z^m, \nu) \, dSdt + \int_{Q_{0,T}} \varepsilon (\nabla u_m^m, \nabla z^m) \, dxdt
\]
\[
= \int_{Q_{0,T}} \varepsilon (\nabla u_m^m, \nabla u_m^m - \nabla D_t) \, dxdt \geq \frac{\varepsilon}{2} \int_{Q_{0,T}} \left| \nabla u_t^m \right|^2 \, dxdt.
\]

The Young inequalities yields that
\[
|f| \Delta D \leq \frac{1}{2} |f|^2 + \frac{1}{2} |\Delta D|^2, \quad |a \Delta u^m \Delta D| \leq \varkappa_1 |\Delta u^m|^2 + \frac{a^2}{4\varkappa_1} |\Delta D|^2, \quad \varkappa_1 > 0,
\]
\[
|f| \Delta u^m \leq \varkappa_2 |\Delta u^m|^2 + 1 \frac{|f|^2}{4\varkappa_2}, \quad \varkappa_2 > 0,
\]
\[
| (\tilde{R} u^m, (\Delta D - \Delta u^m) ) | \leq \varkappa_3 |\Delta u^m|^2 + C_{53}(\varkappa_3) (| (\tilde{R} u^m)|^2 + |\Delta D|^2) , \quad \varkappa_3 > 0,
\]
\[
| \partial \partial u^m (\Delta D - \Delta u^m) | \leq \varkappa_4 |\Delta u^m|^2 + C_{54}(\varkappa_4) (| \partial \partial u^m|^2 + |\Delta D|^2) , \quad \varkappa_4 > 0.
\]

Since \( \int_{Q_{0,T}} [ |f|^2 + |\Delta D|^2 + |\nabla D_t|^2 ] \, dxdt \leq C_{55} \), by (93), (95), and (101), it follows that
\[
\int_{Q_{0,T}} \left( \frac{\varepsilon}{2} |\nabla u_t^m|^2 + \left( a - \varkappa_1 - \varkappa_2 - \varkappa_3 - \varkappa_4 \right) |\Delta u^m|^2 \right) \, dxdt \leq C_{56}(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4),
\]
where \( C_{56} > 0 \) is independent of \( m \). Therefore, choosing \( \varkappa_1, \ldots, \varkappa_4 > 0 \) small enough, we obtain
\[
\int_{Q_{0,T}} |\Delta u^m|^2 \, dxdt \leq C_{57},
\]
where \( C_{57} > 0 \) is independent of \( m \).

By (100) and (87), it follows that
\[
(\tilde{R} u^m)_{j \to \infty} \to (\tilde{R} u), \quad \text{weakly in } L^2(Q_{0,T}).
\]

By (99) and (102), we get
\[
u^m_{j \to \infty} \to u \quad \text{weakly in } H^2(Q_{0,T})
\]
and so \( z^m_{j \to \infty} \to z \) weakly in \( H^2(Q_{0,T}) \). Since every function \( z^m \) satisfies (72), the convergence above implies that \( z \in U_0(Q_{0,T}) \). Thus, \( u - D = z \in U_0(Q_{0,T}) \) and Theorem 6 is proved. \( \Box \)

4 Proof of main Theorem

The Case \( r = 2 \) is trivial. Assume that \( r \in (1, 2) \). The solution will be constructed via the elliptic regularization method.

Step 1. For every \( \varepsilon > 0 \) let us denote by \( u^\varepsilon \) a weak solution to problem (62)-(63) (see Theorem 6). Then (66) holds, \( u^\varepsilon \in H^2(Q_{0,T}) \cap H^1(0, T; H^1(\Omega)) \), \( \tilde{R} u^\varepsilon \in H^1(0, T; L^2(\Omega)) \), and \( Gu^\varepsilon, \phi(Eu^\varepsilon) \in L^2(Q_{0,T}) \). By (79) and (80), we get
\[
\varepsilon \int_{Q_{0,T}} |u_{\varepsilon t}^\varepsilon|^2 \, dxdt + \int_{Q_{0,T}} \left[ |\nabla u^\varepsilon|^2 + |u^\varepsilon|^2 + |u^\varepsilon|^r + |u^\varepsilon|^q(a) \right] \, dxdt \leq C_{58},
\]
\[
\|u^\varepsilon; L^g(t)(Q, T)\| \leq C_{59}.
\] (105)

Whence, similarly to (81), we obtain
\[
\|Gu^\varepsilon; L^g(t)(Q, T)\| \leq C_{60}, \quad \|\Re u^\varepsilon; L'(Q, T)\| \leq C_{61}.
\] (106)

In addition, by (16), (\Phi), and (104), it follows that
\[
\|\phi(Eu^\varepsilon); L^2(Q, T)\| \leq C_{62}.
\] (107)

Here \( C_{58}, \ldots, C_{62} > 0 \) are independent of \( \varepsilon \).

Estimates (104)-(107) implies that there exists a sequence \( \{\varepsilon_j\}_j \subset \mathbb{R}_+ \) such that \( \varepsilon_j \to 0 \) and
\[
u^\varepsilon_j \to u \quad \text{weakly in} \quad L^2(0, T; H^1(Q, \Omega)) \cap L'(Q, T) \cap L^g(t)(Q, T),
\] (108)
\[
Gu^\varepsilon_j \to \chi_1 \quad \text{weakly in} \quad L^g(t)(Q, T),
\] (109)
\[
\Re u^\varepsilon_j \to \chi_2 \quad \text{weakly in} \quad L'(Q, T),
\] (110)
\[
\phi(Eu^\varepsilon_j) \to \chi_3 \quad \text{weakly in} \quad L^2(Q, T),
\] (111)
\[
\sqrt{\varepsilon_j} u^\varepsilon_j \to \chi_4 \quad \text{weakly in} \quad L^2(Q, T).
\] (112)

Step 2. Notice that the constants in (99), (100), and (102) depend on \( \varepsilon \) and so we can not use these estimates here. We shall prove the additional estimates.

Since \( u^\varepsilon \in H^2(Q, T) \) and \( 2 \geq 2(\frac{r}{2} + 1 - 1) \) holds, Case 2.ii of Theorem 5 with \( p = 2 \) and \( \frac{r}{2} + 1 \) instead of \( r \) yields that
\[
|\nabla u^\varepsilon|^2 \leq 2 |u^\varepsilon|^2 \quad \text{in} \quad H^1(0, T; \Omega),
\] (113)
\[
\left( |u^\varepsilon|^2 \right)_t = \frac{r}{2} |u^\varepsilon|^2 - 2 u^\varepsilon_t.
\]

Since \( u^\varepsilon \in H^2(Q, T) \) and \( \frac{r}{2} < r \leq 2 \) holds, similarly to (74), we have
\[
\Re u^\varepsilon \in H^1(0, T; L^2(\Omega)), \quad (\Re u^\varepsilon)_t = |u^\varepsilon|^2 - 2 u^\varepsilon_t.
\] (114)

Moreover, \( u^\varepsilon \in H^1(Q, T), 2 \geq r \), and so similarly to (70), we obtain
\[
|u^\varepsilon|^r \in W^{1, \frac{r}{2}}(0, T; L^2(\Omega)), \quad \left( |u^\varepsilon|^r \right)_t = r |u^\varepsilon|^r - 2 u^\varepsilon_t.
\] (115)

Continuing in the same way (we recall that \( u^\varepsilon, u^\varepsilon_t, u^\varepsilon_{t_t}, \ldots, u^\varepsilon_{n} \in H^1(Q, T) \)), using Case 1.ii of Theorem 2 with \( p(x) \equiv 2 \) and \( r(x) \equiv 2 \), we see that
\[
|u^\varepsilon|^2 \in W^{1, 1}(0, T; L^1(\Omega)), \quad \left( |u^\varepsilon|^2 \right)_t = 2 u^\varepsilon u^\varepsilon_t,
\] (116)
\[
|u^\varepsilon|^2 \in W^{1, 1}(0, T; L^1(\Omega)), \quad \left( |u^\varepsilon|^2 \right)_t = 2 u^\varepsilon u^\varepsilon_t,
\] (117)
\[
|\nabla u^\varepsilon|^2 \in W^{1, 1}(0, T; L^1(\Omega)), \quad \left( |\nabla u^\varepsilon|^2 \right)_t = 2 (\nabla u^\varepsilon, \nabla u^\varepsilon).
\] (118)
Finally, since \( u^\epsilon \in H^1(Q_{0,T}) \) and \( q^0 \leq 2 \) holds, Case 1.ii of Theorem 2 with \( r(x) = q(x) \) and \( p(x) \equiv 2 \) yields that

\[
|u^\epsilon|^q(\cdot) \in L^{\frac{2}{q}}(Q_{0,T}), \quad (|u^\epsilon|^q(\cdot))_r = q(x)|u^\epsilon|^q(\cdot) - 2u^\epsilon u^\epsilon_r.
\] (119)

Integrating by parts (see (114)) and using (63), we obtain

\[
\int_{Q_{0,T}} \left[ \epsilon u^\epsilon_r v - \mathcal{A} u^\epsilon v \right] \, dx \, dt = \int_{Q_{0,T}} \left[ \epsilon u^\epsilon_r v - \mathcal{A} u^\epsilon v \right] \, dx \bigg|_{t=0}^{T} - \int_{Q_{0,T}} \left[ \epsilon u^\epsilon_r v - (\mathcal{A} u^\epsilon)_r v \right] \, dx \, dt
\]

\[
= - \int_{\Omega} \mathcal{A} u^\epsilon(T) v(T) \, dx + \int_{Q_{0,T}} \left[ -\epsilon u^\epsilon_r v + (\mathcal{A} u^\epsilon)_r v \right] \, dx \, dt, \quad v \in U_0(Q_{0,T}).
\]

Then from (66) it follows that

\[
\int_{Q_{0,T}} \left[ -\epsilon u^\epsilon_r v + (\mathcal{A} u^\epsilon)_r v + a(\nabla u^\epsilon, \nabla v) + G u^\epsilon v + \phi(E u^\epsilon)v \right] \, dx \, dt = \int_{Q_{0,T}} f v \, dx \, dt, \quad v \in U_0(Q_{0,T}).
\] (120)

Assume that \( w \in H^1_0(\Omega), \varphi \in C^1([0,T]), \) and \( \varphi(0) = 0. \) Substituting \( v(x,t) = w(x)\varphi(t) \) in (120), we get

\[
\int_{0}^{T} \left( \int_{\Omega} \left[ -\epsilon u^\epsilon_r w + (\mathcal{A} u^\epsilon)_r w + a(\nabla u^\epsilon, \nabla w) + G u^\epsilon w + \phi(E u^\epsilon)w - f w \right] \, dx \right) \varphi(t) \, dt = 0.
\] (121)

Take a function \( \psi \in C^\infty_0((0,T)). \) Substituting \( \varphi = \psi \) in (121), we have the following equality in \( H^{-1}(\Omega) := [H^1_0(\Omega)]^*: \)

\[
-\epsilon u^\epsilon_r(t) + (\mathcal{A} u^\epsilon(t))_r + Au^\epsilon(t) + Gu^\epsilon(t) + \phi(E u^\epsilon(t)) = f(t), \quad t \in (0,T),
\] (122)

where the operator \( A : H^1_0(\Omega) \to H^{-1}(\Omega) \) is defined by the rule

\[
\langle Av, w \rangle_{H^1_0(\Omega)} = \int_{\Omega} a(\nabla v(x), \nabla w(x)) \, dx, \quad v, w \in H^1_0(\Omega).
\] (123)

Since \( u^\epsilon \in L^2(0,T;H^1_0(\Omega)), \) by (122) and (114), it follows that

\[
\int_{Q_{0,T}} \left[ -\epsilon u^\epsilon_r u^\epsilon_r + |u^\epsilon|^{r-2}u^\epsilon_r^2 + a(\nabla u^\epsilon, \nabla u^\epsilon_r) + G u^\epsilon u^\epsilon_r + \phi(E u^\epsilon)u^\epsilon_r \right] \, dx \, dt = \int_{Q_{0,T}} f u^\epsilon \, dx \, dt.
\] (124)

Integrating by parts (see (117)) and using (63), we obtain

\[
\int_{Q_{0,T}} -\epsilon u^\epsilon_r u^\epsilon_r \, dx \, dt = -\frac{\epsilon}{2} \int_{\Omega} |u^\epsilon_r|^{2} \, dx \bigg|_{t=0}^{T} = -\frac{\epsilon}{2} \int_{\Omega} |u^\epsilon_r(0)|^{2} \, dx \geq 0.
\]

From the Young inequality (note that \( \frac{1}{2} + \frac{1}{r} + \frac{r-2}{2r} = 1 \)), it follows that

\[
|fu^\epsilon| = |u^\epsilon|^q(\cdot) |f| |u^\epsilon|^{q-2} f \leq \epsilon_1 |u^\epsilon|^{q-2} |u^\epsilon_r|^2 + C_{63}(\epsilon_1) (|f|^r + |u^\epsilon|^{(1-\frac{2}{r})
\frac{2}{r-2}}^r)
\]

\[
= \epsilon_1 |u^\epsilon|^{q-2} |u^\epsilon_r|^2 + C_{63}(\epsilon_1) (|f|^r + |u^\epsilon|^r), \quad \epsilon_1 > 0,
\]
Assume that the constant \( \gamma \in (\frac{4}{5}, 2) \) is defined by (11). Then the Young inequality with \( \frac{2}{4-\gamma}, \frac{4}{4-\gamma} > 1 \) implies that

\[
|u^\epsilon|^\gamma = |u^\epsilon|^{\frac{2(2-\gamma)}{4-\gamma}} |u^\epsilon|^{\frac{4}{4-\gamma}} \leq \kappa_2 |u^\epsilon|^{2(2-\gamma)} |u^\epsilon|^2 + C_{64}(\kappa_2) |u^\epsilon|^2, \quad \kappa_2 > 0. \tag{125}
\]

Whence, since \( \phi \) is bounded, the Young inequality and (104) yield that

\[
\left| \int_{Q_{0,T}} \phi(Eu^\epsilon)u^\epsilon \, dx \, dt \right| \leq C_{65} \int_{Q_{0,T}} |u^\epsilon| \, dx \, dt \leq \int_{Q_{0,T}} |u^\epsilon|^\gamma \, dx \, dt + C_{66} \leq \kappa_2 \int_{Q_{0,T}} |u^\epsilon|^{2(2-\gamma)} |u^\epsilon|^2 \, dx \, dt + C_{67}(\kappa_2).
\]

Taking into account the transformations above, by (124), (118), and (119), we obtain

\[
\int_{Q_{0,T}} \left( (1 - \kappa_1 - \kappa_2) |u^\epsilon|^{2(2-\gamma)} |u^\epsilon|^2 + \frac{\alpha}{2} (|\nabla u^\epsilon|^2)_t + \frac{g(x,t)}{q(x)} (|u^\epsilon|^q)_t \right) \, dx \, dt \leq C_{68}(\kappa_1, \kappa_2). \tag{126}
\]

Choosing \( \kappa_1, \kappa_2 > 0 \) small enough and integrating by parts, we get

\[
\int_{\Omega_T} \left[ \frac{\alpha}{2} |\nabla u^\epsilon|^2 + \frac{g_0}{q_0^d} |u^\epsilon|^q(x) \right] \, dx + \frac{1}{2} \int_{Q_{0,T}} |u^\epsilon|^{2(2-\gamma)} |u^\epsilon|^2 \, dx \, dt \leq C_{69} \left( 1 + \int_{\Omega} \left[ |\nabla u_0|^2 + |u_0|^q(x) \right] \, dx + \int_{Q_{0,T}} \frac{|g_1(x,t)|}{q(x)} |u^\epsilon|^q(x) \, dx \, dt \right).
\]

Thus,

\[
\int_{Q_{0,T}} |u^\epsilon|^{2(2-\gamma)} |u^\epsilon|^2 \, dx \, dt \leq C_{70}, \tag{127}
\]

and so (113) yields that

\[
\int_{Q_{0,T}} \left| \left( |u^\epsilon|^{2-2} u^\epsilon \right)_t \right|^2 \, dx \, dt = \frac{|r|^2}{4} \int_{Q_{0,T}} |u^\epsilon|^{2(2-\gamma)} |u^\epsilon|^2 \, dx \, dt \leq C_{71}. \tag{128}
\]

Moreover, from (104) it follows that

\[
\int_{Q_{0,T}} \left| \left( |u^\epsilon|^{\frac{2}{2-1}} u^\epsilon \right)_t \right|^2 \, dx \, dt = \int_{Q_{0,T}} |u^\epsilon|^\gamma \, dx \, dt \leq C_{72}. \tag{129}
\]

In addition, (125), (127), and (104) imply that

\[
\int_{Q_{0,T}} |u^\epsilon|^\gamma \, dx \, dt \leq C_{73}. \tag{130}
\]

Here \( C_{63}, \ldots, C_{73} > 0 \) are independent of \( \epsilon \).

Estimates (104) and (127)-(130) yield that

\[
|u^\epsilon|^{2-1} u^\epsilon_j \rightharpoonup \chi_5 \text{ weakly in } H^1(0,T;L^2(\Omega)), \tag{131}
\]

\[
|u_j^\epsilon| \rightharpoonup u \text{ weakly in } W^{1,\gamma}(Q_{0,T}). \tag{132}
\]

Hence, using the Rellich-Kondrachov theorem (see Lemma 1.28 [27, p. 47]) and Lemma 1.18 [27, p. 39], we obtain

\[
u^\epsilon_j \rightharpoonup u \text{ strongly in } L^\gamma(Q_{0,T}) \text{ and almost everywhere in } Q_{0,T}. \tag{133}
\]
Therefore, (see Proposition 4, (109), (110), and (131)) \( \chi_1 = Gu, \chi_2 = R_u, \text{ and } \chi_5 = |u|^{r-1} u \). By (132) and the Aubin theorem (see Proposition 7), it follows that

\[ u^{(j)} \to u \quad \text{in } C([0,T];L^r(\Omega)). \quad (134) \]

It is easy to prove that the operator \( E \) (see (3)) is continuous from \( L^r(Q_0,T) \) into \( L^r(Q_0,T) \). Then, (\( \Phi \)) and (134) imply that \( \phi(Eu^{(j)} \to \phi(Eu) \quad \text{strongly in } L^r(Q_0,T) \). Thus, \( \chi_3 = \phi(Eu) \).

**Step 3.** We substitute \( \varepsilon = e_j \) in (66) and tend \( j \to \infty \) if \( v \in H^1_0(Q_0,T) \). Taking into account (108)-(112), we obtain (15). By (132) and condition (63), it follows that the function \( u \) satisfies (2). Thus, \( u \) is a weak solution to problem (1)-(2). By (131) and (132), we get \( |u|^{r-1} u \in H^1(0,T;L^2(\Omega)) \) and \( u \in W^{1,\gamma}(Q_0,T) \). This completes the proof of Theorem 1. \( \Box \)

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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