Some properties of orders generated by uninorm and 2-uninorm

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Received: 4 February 2017, Accepted: 27 February 2017
Published online: 2 April 2017.

Abstract: In this paper, the order definition obtained from uninorm has been reorganized and some features have been examined in this way. Order-weakest uninorm and order-strongest uninorm was determined. Using the notions of order-weakest uninorm and order-strongest uninorm, order-weakest 2-uninorm and order-strongest 2-uninorm was also determined. A way to obtain partially ordered relation via orders obtained from uninorms on subinterval of bounded lattice is given. The relation between the order obtained 2-uninorm and this new construction method is investigated.

Keywords: Uninorm, 2-uninorm, bounded lattice, partial order.

1 Introduction

Uninorms can be seen as a more general class of t-norms and t-conorms. Since t-norms and t-conorms have been studied extensively, uninorms have been also studied extensively since they defined by Yager and Rybalov [17]. In addition to this, it can be said that they have extra interest because they have a lot of application areas [9, 18]. Although they were first described on unit real interval, they were also defined and studied by researchers on bounded lattice [4, 6, 12, 16]. How important is it that uninorms are a generalization of t-norms and t-conorms, 2- uninorms are also important for researchers to define and study on them [1, 2, 5].

Partially ordered relation obtained from logical operators has been investigated by researcher deeply [13, 14, 15]. In [13], a partial order defined by means of t-norms bounded lattices has been introduced. This partial order $\preceq_T$ is called a T-partial order on $L$. In addition, there have been some initiatives to define the order obtained by uninorms [11]. But, it was first defined in [7]. Again in [7], the order obtained by 2-uninorms is introduced on chain but without proof. Finally, the order obtained from 2-uninorms on bounded lattice is given with proof, the some properties of the order are examined [8].

In this study, the order definitions $\preceq_U$ and $\preceq_{U^2}$ have been reorganized. By this way, order-weakest uninorms and order-strongest uninorms were determined. In addition, it is showed that order-weakest uninorms and order-strongest uninorms may not be the only one. This new form of $\preceq_{U^2}$ also made it possible to obtain a new order definition from two orders obtained from two uninorms defined on subintervals of bounded lattice. The paper is organized as follows: I shortly recall some basic notions and results in Section 2. In Section 3, firstly, the order notion of $\preceq_U$ was reconsidered. In this way, order-weakest and order-strongest uninorm were studied. The example was given to show they dont need to be one. In same section, it was studied on $\preceq_{U^2}$ similarly. Using this new definition, a method was given to obtain partially ordered relation from two partially ordered relations on subintervals of bounded lattice.
2 Notations, definitions and a review of previous results

**Definition 1.** [12] Let \((L, \leq, 0, 1)\) be a bounded lattice. An operation \(U : L^2 \rightarrow L\) is called a uninorm on \(L\), if it is commutative, associative, increasing with respect to the both variables and has a neutral element \(e \in L\).

In this study, the notation \(\mathscr{U}(e)\) will be used for the set of all uninorms on \(L\) with neutral element \(e \in L\).

**Definition 2.** [13] An operation \(T (S)\) on a bounded lattice \(L\) is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element \(1 (0)\).

**Example 1.** Let \((L, \leq, 0, 1)\) be a bounded lattice. Smallest t-norm \(T_w\) and greatest t-norm \(T_\wedge\) on bounded lattice \(L\) are given respectively as follows.

\[
T_w (x, y) = \begin{cases} 
  y, & \text{if } x = 1 \\
  x, & \text{if } y = 1 \\
  0, & \text{otherwise}
\end{cases}
\]

\[
T_\wedge (x, y) = x \wedge y.
\]

Smallest t-conorm \(S_\lor\) and greatest t-norm \(S_w\) on bounded lattice \(L\) are given respectively as follows.

\[
S_\lor (x, y) = x \lor y
\]

\[
S_w (x, y) = \begin{cases} 
  y, & \text{if } x = 0 \\
  x, & \text{if } y = 0 \\
  1, & \text{otherwise.}
\end{cases}
\]

**Definition 3.** [13,14] A t-norm \(T\) (or a t-conorm \(S\)) on a bounded lattice \(L\) is divisible if the following condition holds. For all \(x, y \in L\) with \(x \leq y\) there is \(z \in L\) such that \(x = T(y, z)\) (or \(y = S(x, z)\)).

**Definition 4.** [13] Let \(L\) be a bounded lattice, \(T\) a t-norm on \(L\). The order defined by

\[
x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L
\]

is called a \(T\)− partial order (triangular order) for t-norm \(T\).

Similarly, the notion \(S\)− partial order can be defined as follows.

**Definition 5.** Let \(L\) be a bounded lattice, \(S\) be a t-conorm on \(L\). The order defined by is called a \(S\)− partial order for t-conorm \(S\).

\[
x \preceq_S y :\Leftrightarrow S(\ell, x) = y \text{ for some } \ell \in L
\]

is called a \(S\)− partial order for t-conorm \(S\).

Note that many properties satisfied for \(T\)− partial order are also satisfied for \(S\)− partial order.

**Definition 6.** [7] Let \((L, \leq, 0, 1)\) be a bounded lattice and \(U \in \mathscr{U}(e)\). Define the following relation, for \(x, y \in L\), as

\[
x \preceq_U y :\Leftrightarrow \begin{cases} 
  \text{if } x, y \in [0, e] & \text{and there exist } k \in [0, e] \text{ such that } U(k, y) = x \text{ or,} \\
  \text{if } x, y \in [e, 1] & \text{and there exist } \ell \in [e, 1] \text{ such that } U(x, \ell) = y \text{ or,} \\
  \text{if } (x, y) \in L^* & \text{and } x \leq y,
\end{cases}
\]

where \(L_e = \{ x \in L \mid x \| e \}\) and \(L^* = [0, e] \times [e, 1] \cup [0, e] \times I_e \cup [e, 1] \times [0, e] \cup I_e \times [0, e] \cup I_e \times [e, 1] \cup I_e \times L_e\).

Here, note that the notation \(x \| y\) denotes that \(x\) and \(y\) are incomparable.
Proposition 1. [7] The relation $\preceq_U$ defined in (1) is a partial order on $L$.

Definition 7. [5] Let $(L, \leq, 0, 1)$ be a bounded lattice. An operator $F : L^2 \rightarrow L$ is called 2-uninorm if it is commutative, associative, increasing with respect to both variables and fulfilling

$$\forall x \leq k F(e, x) = x \text{ and } \forall x \geq k F(f, x) = x,$$

where $e, k, f \in L$ with $0 \leq e \leq k \leq f \leq 1$.

By $U_{k(e,f)}$ we denote the class of all 2-uninorms on bounded lattice $L$.

Definition 8. [8] Let $U^2 \in U_{k(e,f)}$. Define the following relation. For every $x,y \in L$,

$$x \preceq_{U^2} y \iff \begin{cases} \exists \ell \leq e \text{ such that } U^2(\ell, y) = x, & \text{when } x, y \in [0, e] \text{ or}, \\ \exists m \in [e, k] \text{ such that } U^2(x, m) = y, & \text{when } x, y \in [e, k] \text{ or}, \\ \exists n \in [k, f] \text{ such that } U^2(y, n) = x, & \text{when } x, y \in [k, f] \text{ or}, \\ \exists p \in [f, 1] \text{ such that } U^2(x, p) = y, & \text{when } x, y \in [f, 1] \text{ or}, \\ x \leq y, & \text{otherwise}. \end{cases}$$

Proposition 2. [8] The relation $\preceq_{U^2}$ defined in (2) is a partial order on bounded lattice $L$.

3 Order-weakest and order strongest uninorms and 2-uninorms

In this section, the partially ordered relations obtained from uninorm and 2-uninorm on bounded lattice $L$ has been reorganized. By this way, order-weakest and order-strongest uninorms are determined. In addition, considering the relation between uninorms and 2-uninorms, order-weakest 2-uninorm and order-strongest 2-uninorms are also determined. Also, it is showed that order weakest uninorms or 2-uninorms and order-strongest uninorms or 2-uninorms dont need to be one. Further, the partially ordered relation obtained two partially ordered relations obtained from two uninorms on subintervals of bounded lattice is given.

Proposition 3. [12] Let $(L, \leq, 0, 1)$ be a bounded lattice, and $U$ a uninorm with a neutral element $e \in L$. Then

(i) $T^* = U \downarrow [0, e]^2 : [0, e] \rightarrow [0, e]$ is a t-norm on $[0, e]$.

(ii) $S^* = U \downarrow [e, 1]^2 : [e, 1] \rightarrow [e, 1]$ is a t-conorm on $[e, 1]$.

Considering Proposition 3, the definiton of $\preceq_U$ can be reorganized as follow.

Definition 9. Let $(L, \leq, 0, 1)$ be a bounded lattice and $U \in \mathcal{U}(e)$ such that $U \downarrow [0, e]^2 = T^*$ and $U \downarrow [e, 1]^2 = S^*$. (1) can be represented as following for $x,y \in L_*$ as

$$x \preceq_U y \iff \begin{cases} \text{if } x, y \in [0, e] \text{ and } x \preceq_T y \text{ or}, \\ \text{if } x, y \in [e, 1] \text{ and } x \preceq_S y \text{ or}, \\ \text{if } (x, y) \in L_* \text{ and } x \leq y, \end{cases}$$

where $L^* = [0, e] \times [0, e] \cup L \times [0, e] \times [0, e] \cup L \times L \times [0, e] \cup L \times L \times L \times L$.

Remark. [3] Let $T$ be a t-norm ans $S$ be a t-conorm on bounded lattice $L$ and consider the t-norms $T_W$ and $T_S$ and t-conorms $S_W$ and $S_S$.

$T_W$ is the order-weakest and $T_S$ is order-strongest t-norm on $L$, i.e.,

$$\preceq_{T_W} \subseteq \preceq_T \subseteq \preceq_{T_S}. $$

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Similarly, $S_v$ is the order-weakest and $S_W$ is order-strongest t-conorm on $L$, i.e.,

$$\preceq S_W \subseteq \preceq S \subseteq \preceq S_v.$$ 

**Proposition 4.** Let $(L, \leq, 0, 1)$ be a bounded lattice and $U_W \in \mathcal{W}(e)$ such that $U_W \downarrow [0, e]^2 = T_W$ and $U_W \downarrow [e, 1]^2 = S_W$. Then $\succeq U_W \subseteq \preceq U$ for all $U \in \mathcal{W}(e)$.

**Proof.** Let $U \in \mathcal{W}(e)$ be an arbitrary uninorm such that $U \downarrow [0, e]^2 = T^*$ and $U \downarrow [e, 1]^2 = S^*$. Let $(x, y) \in \preceq U_W$.

(i) $x, y \in [0, e]$. Then, it is obtained that $(x, y) \in \preceq T_W$. Since $\preceq T_W \subseteq \preceq T$ for any t-norm on $[0, e]$, $\preceq T_W \subseteq \preceq T^*$. Therefore, $(x, y) \in \preceq U$.

(ii) $x, y \in [e, 1]$. Then, it is obtained that $(x, y) \in \preceq S_W$. Since $\preceq S_W \subseteq \preceq S$ for any t-conorm on $[e, 1]$, $\preceq S_W \subseteq \preceq S^*$. Therefore, $(x, y) \in \preceq U$.

(iii) For other cases, $(x, y) \in \preceq U_W$ implies that $(x, y) \in \leq$. Therefore, $(x, y) \in \preceq U$.

Thus, it is obtained that $\preceq U_W \subseteq \preceq U$.

**Proposition 5.** Let $(L, \leq, 0, 1)$ be a bounded lattice and $U_{\wedge \vee} \in \mathcal{W}(e)$ such that $U_{\wedge \vee} \downarrow [0, e]^2 = T_{\wedge}$ and $U_{\wedge \vee} \downarrow [e, 1]^2 = S_{\vee}$. Then $\preceq U \subseteq \preceq U_{\wedge \vee}$ for all $U \in \mathcal{W}(e)$.

**Proof.** The proof can be done similar proof of Proposition 4.

**Corollary 1.** Let $(L, \leq, 0, 1)$ be a bounded lattice and $U \in \mathcal{W}(e)$ be an arbitrary uninorm on $L$. Then, $U_W$ is the order-weakest and $U_{\wedge \vee}$ is order-strongest uninorms on $L$, i.e.,

$$\preceq U_W \subseteq \preceq U \subseteq \preceq U_{\wedge \vee}.$$ 

**Remark.** $U_W$ is the order-weakest and $U_{\wedge \vee}$ is order-strongest uninorm on $L$ mentioned in Corollary 1 are not the necessarily the ones. Let show that following example:

**Example 2.** Consider the lattice $(L = \{0, a, b, c, d, e, 1\}, \leq, 0, 1)$ whose lattice diagram is displayed in Figure 1.

![Fig. 1: $\preceq L$](image)

Let define the following $U_1$ and $U_2$ uninorms with neutral element $c$ given in Table 1 and Table 2 respectively:
One can easily check that $U_1$ and $U_2$ satisfies the conditions of Proposition 4, thus $U_1$ and $U_2$ can be seen as $U_W$ but $U_1 \neq U_2$.

Let define the following $U_3$ and $U_4$ uninorms with neutral element $e$ given in Table 3 and Table 4 respectively.

<table>
<thead>
<tr>
<th>$U_3$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0</td>
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<td>c</td>
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<tr>
<td>d</td>
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</tbody>
</table>

**Table 3:** $U_3$ Uninorm.

<table>
<thead>
<tr>
<th>$U_4$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>1</th>
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</thead>
<tbody>
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<td>0</td>
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<tr>
<td>a</td>
<td>0</td>
<td>a</td>
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<td>a</td>
<td>0</td>
<td>a</td>
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<tr>
<td>b</td>
<td>0</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
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<td>0</td>
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<tr>
<td>c</td>
<td>0</td>
<td>c</td>
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<td>d</td>
<td>0</td>
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</tbody>
</table>

**Table 4:** $U_4$ Uninorm.

One can easily check that $U_3$ and $U_4$ satisfies the conditions of Proposition 5, thus $U_3$ and $U_4$ can be seen as $U_{\leq W}$ but $U_3 \neq U_4$.

Let $U^2 \in U_{k(e,f)}$. It is well known that $U^2 \downarrow [0,k]^2$ is an uninorm on $[0,k]$ with neutral element $e$ and $U^2 \downarrow [1,k]^2$ is an uninorm on $[1,k]$ with neutral element $f$. Let we call $U^2 \downarrow [0,k]^2$ as $U^2_1$ and $U^2 \downarrow [1,k]^2$ as $U^2_2$.

Similar to the reorganization of the order $\leq_U$, one can reorganize $\leq_{U^2}$ as follow.

**Definition 10.** Let $U^2 \in U_{k(e,f)}$ such that $U^2 \downarrow [0,k]^2 = U^2_1$ and $U^2 \downarrow [1,k]^2 = U^2_2$. Define the following relation: For every $x,y \in L$, $x \leq_{U^2} y :\iff \begin{cases} 
\text{if } x,y \in [0,k] \text{ and } x \leq_U y \text{ or, } \\
\text{if } x,y \in [1,k] \text{ and } x \leq_{U^2} y \text{ or, } \\
x \leq y, & \text{otherwise.}
\end{cases} \tag{4}
$

**Proposition 6.** Let $(L,\leq,0,1)$ be a bounded lattice and $U^{2W} \in U_{k(e,f)}$ be an $2$-uninorm on $L$ such that $U^{2W}_1 = U_W$ on $[0,k]^2$ and $U^{2W}_2 = U_W$ on $[k,1]^2$. Then, $\leq_{U^{2W}} \leq_{U^2}$ for all $F^2 \in U_{k(e,f)}$.

**Proof.** $F^2 \in U_{k(e,f)}$ arbitrary $2$-uninorm. Let $(x,y) \in \leq_{U^{2W}}$.

(i) $x,y \in [0,k]$. Then, $(x,y) \in \leq_{U^{2W}}$ implies that $(x,y) \in \leq_{U^2_1}$ and $(x,y) \in \leq_{U^2_2}$. Since $\leq_{U^2} \subseteq \leq_{U^2}$ for any uninorm $U$ on $[0,k]$, $\leq_{U^{2W}} \subseteq \leq_{U^2}$. Therefore, $(x,y) \in \leq_{U^2}$.

(ii) $x,y \in [1,k]$. Then, $(x,y) \in \leq_{U^{2W}}$ implies that $(x,y) \in \leq_{U^2_1}$ and $(x,y) \in \leq_{U^2_2}$. Since $\leq_{U^2} \subseteq \leq_{U^2}$ for any uninorm $U$ on $[k,1]$, $\leq_{U^2} \subseteq \leq_{U^2}$. Therefore, $(x,y) \in \leq_{U^2}$.

(iii) For other cases, $(x,y) \in \leq_{U^{2W}}$ implies that $(x,y) \in \leq_{U^2}$. Therefore, $(x,y) \in \leq_{U^2}$.
Thus, it is obtained that $\preceq_{U^2W} \subseteq \preceq_{U^2}$. 

**Proposition 7.** Let $(L, \preceq_{0, 1})$ be a bounded lattice and $U^{2S} \in U^{2}_{k(e,f)}$ be an 2-uninorm on $L$ such that $U^{2S}_1 = U_{\vee \wedge}$ on $[0, k]^2$ and $U^{2S}_2 = U_{\wedge \vee}$ on $[k, 1]^2$. Then, $\preceq_{U^2} \subseteq \preceq_{U^{2S}}$ for all $G^2 \in U^{2}_{k(e,f)}$. 

**Proof.** The proof can be done similar proof of Proposition 6.

**Corollary 2.** Let $(L, \preceq_{0, 1})$ be a bounded lattice and $U^2 \in U^{2}_{k(e,f)}$ be an arbitrary 2-uninorm on $L$. Then, $U^{2W}$ is the order-weakest and $U^{2S}$ is order-strongest 2-uninorms on $L$ i.e.,

$$\preceq_{U^{2W}} \subseteq \preceq_{U^2} \subseteq \preceq_{U^{2S}}.$$ 

**Remark.** The order-weakest 2-uninorm $U^{2W}$ and order-strongest 2-uninorm $U^{2S}$ on $L$ mentioned in Corollary 2 are not the necessarily the ones. This argues clearly obtained that Remark 3, Proposition 6 and Proposition 7.

**Remark.** The relation (4) can be seen as a way to obtain order from two uninorms defined on subintervals $[0, k]$ and $[k, 1]$ of $L$. Check following proposition.

**Proposition 8.** Let $(L, \preceq_{0, 1})$ be a bounded lattice, $U_1$ uninorm on subinterval $[0, k]$ of $L$ with neutral element $e$ and $U_2$ uninorm on subinterval $[k, 1]$ of $L$ with neutral element $f$. Following relation is partially ordered relation on $L$.

For every $x, y \in L$,

$$x \preceq_{U_1 U_2} y \iff \begin{cases} \text{if } x, y \in [0, k] & \text{and } x \preceq_{U_1} y \text{ or,} \\ \text{if } x, y \in [k, 1] & \text{and } x \preceq_{U_2} y \text{ or,} \\ x \preceq y, & \text{otherwise.} \end{cases}$$

(5)

**Corollary 3.** (5) In Proposition 8 coincides with $\preceq_{U^2}$ if 2-uninorm $U^2 \in U^{2}_{k(e,f)}$ provide that $U^{2S}_1 = U_1$ and $U^{2S}_2 = U_2$, i.e., $\preceq_{U_1 U_2} = \preceq_{U^2}$.

**Proposition 9.** Let $L$ be a lattice and $U \in U(e)$ such that $k \in L \setminus \{0, 1\}$ is comparable with all elements of $L$. Then, $([0, k], \preceq_{U_1})$ and $([k, 1], \preceq_{U_2})$ are lattices if and only if $(L, \preceq_{U_1 U_2})$ is a lattice.

**Proof.** Suppose that $([0, k], \preceq_{U_1})$ and $([k, 1], \preceq_{U_2})$ are lattices.

(i) Let $x, y \in [0, k]$ be arbitrary. Since $([0, k], \preceq_{U_1})$ is a lattice, $x \vee_{U_1} y$ and $x \wedge_{U_1} y$ exist. Let call $x \vee_{U_1} y = a \in [0, k]$ and $x \wedge_{U_1} y = b \in [0, k]$. Since $x \vee_{U_1} y = a$, $x \preceq_{U_1} a$ and $y \preceq_{U_1} a$. Thus, it is obtained that $x \preceq_{U_1 U_2} a$ and $y \preceq_{U_1 U_2} a$, that is, $a \in \{x, y\}_{U_1 U_2}$.

Let $t \in \{x, y\}_{U_1 U_2}$ be arbitrary. Then, $x \preceq_{U_1 U_2} t$ and $y \preceq_{U_1 U_2} t$.

Since $k$ is comparable with the elements of $L$, either $t \leq k$ or $k \leq t$.

Suppose that $t \leq k$. Then $x \preceq_{U_1} t$ and $y \preceq_{U_1} t$, that is, we have that $t \in \{x, y\}_{U_1}$. Since $x \vee_{U_1} y = a, a \preceq_{U_1} t$. Then, it is obtained that $a \preceq_{U_1 U_2} t$ since $a, t \in [0, k]$. So, $x \vee_{U_1 U_2} y = a$. Similarly, it can be shown that $x \wedge_{U_1 U_2} y = b$.

(ii) Let $x, y \in [k, 1]$ be arbitrary. Since $([k, 1], \preceq_{U_2})$ is a lattice, $x \vee_{U_2} y$ and $x \wedge_{U_2} y$ exist. Let call $x \vee_{U_2} y = a^* \in [k, 1]$ and $x \wedge_{U_2} y = b^* \in [k, 1]$. Similarly, it is obtained that $x \vee_{U_1 U_2} y = a^*$ and $x \wedge_{U_1 U_2} y = b^*$.

(iii) Let $x \leq k$ and $k \leq y$. Then it is clear that $x \vee_{U_1 U_2} y = y$ and $x \wedge_{U_1 U_2} y = x$.

(iv) Let $k \leq x$ and $y \leq k$. Then it is clear that $x \vee_{U_1 U_2} y = x$ and $x \wedge_{U_1 U_2} y = y$.

Therefore, $(L, \preceq_{U_1 U_2})$ is a lattice if $([0, k], \preceq_{U_1})$ and $([k, 1], \preceq_{U_2})$ are lattices.

Conversely, if $(L, \preceq_{U_1 U_2})$ is a lattice, it is clear that $([0, k], \preceq_{U_1})$ and $([k, 1], \preceq_{U_2})$ are lattices.
**Remark.** If we drop the condition given in Proposition 9, that is, if $k$ is not comparable with all elements of $L$, then the claim need not be satisfied. Check the following example.

**Example 3.** Consider the lattice $(L, \leq, 0, 1)$ given its lattices diagram as follows.

![Diagram](image1)

**Fig. 2:** $(L, \leq)$

Take the following uninorm $U_1$ on $[0, d]$ with neutral element $a$ and its lattice diagram are as follows.

![Table](image2)

**Table 5:** The uninorm $U_1$.

Also, uninorm $U_2$ on $[d, 1]$ with neutral element $f$ and its lattice diagram are as follows.

![Table](image3)

**Table 6:** The uninorm $U_2$.

Finally, the order $\preceq_{U_1U_2}$ is depicted as follows.

![Diagram](image4)

**Fig. 4:** $([d, 1], \preceq_{U_1})$. 
As it is easily seen in the figures, although \(((0,d], \preceq_{U1})\) and \(((d,1], \preceq_{U2})\) are lattices, \((L, \preceq_{U1} U_2)\) is not.

**Corollary 4.** Let \((L, \leq, 0, 1)\) be a bounded lattice, \(U_1\) uninorm on subinterval \([0,k]\) of \(L\) with neutral element \(e\) such that \(U_1 \downarrow [0,e]^2\) divisible t-norm, \(U_1 \downarrow [e,k]^2\) divisible t-conorm and \(U_2\) uninorm on subinterval \([k,1]\) of \(L\) with neutral element \(f\) such that \(U_2 \downarrow [k,f]^2\) divisible t-norm, \(U_2 \downarrow [f,1]^2\) divisible t-conorm. Then, \(\preceq_{U1 U_2} = \leq\).

### 4 Conclusion

The order definition of \(\leq_{U}\) has been reorganized. By this way, order-weakest uninorms and order-strongest uninorms are determined. In addition, order-weakest uninorms and order-strongest uninorms may not be the only one. Similarly, the order definition of \(\leq_{U2}\) has been reorganized through the underlying uninorms. This new form also made it possible to obtain a new order definition from two orders obtained from two uninorms defined on subintervals of bounded lattice.

### Competing interests

The authors declare that they have no competing interests.

### Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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