Warped product pseudo-slant submanifolds of 
\((LCS)_n\)-manifolds

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Abstract: The object of the present paper is to study warped product pseudo-slant submanifolds of \((LCS)_n\)-manifolds. We study the existence or non-existence of such submanifolds. The existence is also ensured by an example.

Keywords: Warped product, pseudo-slant submanifold, \((LCS)_n\)-manifold.

1 Introduction

The notion of warped product manifolds were introduced by Bishop and O’Neill [4] and later it was studied by many mathematicians and physicists. These manifolds are generalization of Riemannian product manifolds. The existence or non-existence of warped product manifolds plays some important role in differential geometry as well as physics.

The notion of slant submanifolds in a complex manifold was introduced and studied by B.-Y. Chen [8], which is a natural generalization of both invariant and anti-invariant submanifolds. B.-Y. Chen [8] also found examples of slant submanifolds of complex Euclidean space \(C^2\) and \(C^4\). Then Lotta [14] has defined and studied slant immersions of a Riemannian manifold into an almost contact metric manifold and proved some properties of such immersions. Thereafter, many authors studied slant submanifolds of almost contact metric manifolds.

In [18], N. Papaghiuc introduced the notion of semi-slant submanifolds of almost Hermitian manifolds. Then Cabrerizo et. al [5] defined and investigated semi-slant submanifolds of Sasakian manifolds. In this connection, it may be mentioned that Sahin [19] studied warped product semi-slant submanifolds of Kaehler manifolds. Also in [1], Atceken studied warped product semi-slant submanifolds in locally Riemannian product manifolds. Again Atceken [2] studied warped product semi-slant submanifolds in Kenmotsu manifolds. Beside these, Uddin and his co-authors studied warped product submanifolds in different context such as ([13], [28]) etc. Recently, Hui and Atceken [10] studied warped product semi-slant submanifolds of \((LCS)_n\)-manifolds.

Next, A. Carriazo [7] defined and studied bi-slant submanifolds in almost Hermitian manifolds and simultaneously gave the notion of pseudo-slant submanifolds in almost Hermitian manifolds. The contact version of pseudo-slant submanifolds has been defined and studied by Khan and Khan in [12]. In this connection it may be mentioned that Atceken and Hui [3] studied slant and pseudo-slant submanifolds of \((LCS)_n\)-manifolds. Recently, Khan and Chahal [11] have been studied warped product pseudo-slant submanifold of trans-Sasakian manifolds.
In 2003, Shaikh [20] introduced the notion of Lorentzian concircular structure manifolds (briefly, \((LCS)_n\)-manifolds), with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [15] and also by Mihai and Rosca [16]. Then Shaikh and Baishya ([22], [23]) investigated the applications of \((LCS)_n\)-manifolds to the general theory of relativity and cosmology. The \((LCS)_n\)-manifolds is also studied by Hui [9], Hui and Atceken ([3], [10]), Shaikh and his co-authors ([21], [24], [25], [26], [27]) and many others.

Motivated by the studies the object of the present paper is to study warped product pseudo-slant submanifolds of \((LCS)_n\)-manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries. In section 3, we study warped product pseudo-slant submanifolds of \((LCS)_n\)-manifolds. It is shown that there do not exist warped product pseudo-slant submanifolds of an \((LCS)_n\)-manifold \(M\) of the type \(M = N_\perp \times f N_\theta\) such that \(N_\perp\) and \(N_\theta\) are anti-invariant and proper slant submanifolds of \(M\), respectively such that \(\xi\) is tangent to \(N_\theta\), whereas the warped products of the form \(M = N_\perp \times f N_\theta\) exist, whenever \(\xi\) is tangent to \(N_\theta\). Finally, the existence of such submanifolds is ensured by an interesting example.

### 2 Preliminaries

An \(n\)-dimensional Lorentzian manifold \(\overline{M}\) is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric \(g\), that is, \(\overline{M}\) admits a smooth symmetric tensor field \(g\) of type \((0,2)\) such that for each point \(p \in \overline{M}\), the tensor \(g_p : T_p\overline{M} \times T_p\overline{M} \to \mathbb{R}\) is a non-degenerate inner product of signature \((-,+,\cdots,+\)\), where \(T_p\overline{M}\) denotes the tangent vector space of \(\overline{M}\) at \(p\) and \(\mathbb{R}\) is the real number space. A non-zero vector \(v \in T_p\overline{M}\) is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies \(g_p(v,v) < 0\) (resp, \(\leq 0, = 0, > 0\) [17].

**Definition 1.** In a Lorentzian manifold \((\overline{M},g)\) a vector field \(P\) defined by

\[
g(X,P) = A(X),
\]

for any \(X \in \Gamma(T\overline{M})\), is said to be a concircular vector field [30] if

\[
(\nabla_X A)(Y) = \alpha \{g(X,Y) + \omega(X)A(Y)\}
\]

where \(\alpha\) is a non-zero scalar and \(\omega\) is a closed 1-form and \(\nabla\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\).

Let \(\overline{M}\) be an \(n\)-dimensional Lorentzian manifold admitting a unit timelike concircular vector field \(\xi\), called the characteristic vector field of the manifold. Then we have

\[
g(\xi,\xi) = -1.
\]

Since \(\xi\) is a unit concircular vector field, it follows that there exists a non-zero 1-form \(\eta\) such that for

\[
g(X,\xi) = \eta(X),
\]

the equation of the following form holds

\[
(\nabla_X \eta)(Y) = \alpha \{g(X,Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0)
\]

\[
\nabla_X \xi = \alpha \{X + \eta(X)\xi\}, \quad \alpha \neq 0
\]
for all vector fields $X$, $Y$, where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$ and $\alpha$ is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X \alpha) = d\alpha(X) = \rho \eta(X),$$

(5)

$\rho$ being a certain scalar function given by $\rho = -(\xi \alpha)$. If we put

$$\phi X = \frac{1}{\alpha} \nabla_X \xi,$$

(6)

then from (4) and (6) we have

$$\phi X = X + \eta(X)\xi,$$

(7)

g($\phi X, Y$) = $g(X, \phi Y)$

(8)

from which it follows that $\phi$ is a symmetric $(1,1)$ tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold $M$ together with the unit timelike concircular vector field $\xi$, its associated 1-form $\eta$ and an $(1,1)$ tensor field $\phi$ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold), [20]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [15]. In a $(LCS)_n$-manifold ($n > 2$), the following relations hold

$$\eta(\xi) = -1, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

(9)

$$\phi^2 X = X + \eta(X)\xi,$$

(10)

$$S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X),$$

(11)

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y],$$

(12)

$$R(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y],$$

(13)

$$\langle \nabla_X \phi \rangle Y = \alpha \{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\},$$

(14)

$$(X \rho) = d\rho(X) = \beta \eta(X),$$

(15)

$$R(X, Y)Z = \phi R(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi,$$

(16)

for all $X, Y, Z \in \Gamma(TM)$ and $\beta = -(\xi \rho)$ is a scalar function, where $R$ is the curvature tensor and $S$ is the Ricci tensor of the manifold.

Let $M$ be a submanifold of a $(LCS)_n$-manifold $\overline{M}$ with induced metric $g$. Also let $\nabla$ and $\nabla^\perp$ are the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$ respectively. Then the Gauss and Weingarten formulae are given by

$$\nabla_X Y = \nabla_X Y + h(X, Y)$$

(17)
and
\[ \nabla_X V = -A_V X + \nabla^\perp_X V \] (18)
for all \( X, Y \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \), where \( h \) and \( A_V \) are second fundamental form and the shape operator (corresponding to the normal vector field \( V \)) respectively for the immersion of \( M \) into \( M \). The second fundamental form \( h \) and the shape operator \( A_V \) are related by
\[ g(h(X,Y),V) = g(A_V X, Y), \] (19)
for any \( X, Y \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \).

For any \( X \in \Gamma(TM) \), we can write
\[ \phi X = EX + FX, \] (20)
where \( EX \) is the tangential component and \( FX \) is the normal component of \( \phi X \).

Also, for any \( V \in \Gamma(T^\perp M) \), we have
\[ \phi V = BV + CV, \] (21)
where \( BV \) and \( CV \) are also the tangential and normal components of \( \phi V \) respectively. From (20) and (21), we can derive the tensor fields \( E, F, B \) and \( C \) are also symmetric, because \( \phi \) is symmetric. Also from (8) and (20) we have
\[ g(EX,Y) = g(X,EY) \] (22)
for any \( X,Y \in \Gamma(TM) \).

Throughout the paper, we consider \( \xi \) to be tangent to \( M \). The submanifold \( M \) is said to be invariant if \( F \) is identically zero, i.e., \( \phi X \in \Gamma(TM) \) for any \( X \in \Gamma(TM) \). Also \( M \) is said to be anti-invariant if \( E \) is identically zero, that is \( \phi X \in \Gamma(T^\perp M) \) for any \( X \in \Gamma(TM) \).

For any \( X, Y \in \Gamma(TM) \), we have from (7), (17) and (20) that
\[ \nabla_X \xi = \alpha EX, \] (23)
\[ h(X, \xi) = \alpha FX. \] (24)

**Definition 2.** Let \( M \) be a submanifold of \((LCS)_n\)-manifold \( \overline{M} \). For each non-zero vector \( X \) tangent to \( M \) at \( x \), the angle \( \theta(x) \), \( 0 \leq \theta(x) \leq \frac{\pi}{2} \) between \( \phi X \) and \( EX \) is called the slant angle or the Wirtinger angle. If the slant angle is constant then the submanifold is called the slant submanifold. Invariant and anti-invariant submanifolds are particular slant submanifolds with slant angle \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \) respectively. A slant submanifold is said to be proper slant if the slant angle \( \theta \) lies strictly between 0 and \( \frac{\pi}{2} \), i.e., \( 0 < \theta < \frac{\pi}{2} \) [6].

**Theorem 1.[2]** Let \( M \) be a submanifold of a \((LCS)_n\)-manifold \( \overline{M} \) such that \( \xi \) is tangent to \( M \). Then \( M \) is slant submanifold if and only if there exists a constant \( \lambda \in [0,1] \) such that
\[ E^2 = \lambda (I + \eta \otimes \xi). \] (25)
Moreover, if $\theta$ is the slant angle of $M$, then $\lambda = \cos^2 \theta$.

Also from (25) we have

\[ g(EX, EY) = \cos^2 \theta [g(X, Y) + \eta(X)\eta(Y)], \]  

(26)

\[ g(FX, FY) = \sin^2 \theta [g(X, Y) + \eta(X)\eta(Y)] \]  

(27)

for any $X, Y$ tangent to $M$.

**Definition 3.** Let $M$ be a $(LCS)_n$-manifold and $M$ be an immersed submanifold in $\overline{M}$. Then $M$ is said to be pseudo-slant submanifold of $\overline{M}$ if there exist two orthogonal complementary distributions $D_\theta$ and $D^\bot$ such that

(i) $TM = D^\bot \oplus D_\theta < \xi >$,  

(ii) the distribution $D^\bot$ is anti-invariant, that is, $\phi(D^\bot) \subseteq (T^\bot M)$,  

(iii) the distribution $D_\theta$ is slant with slant angle $\theta \neq \frac{\pi}{2}$.

From the above definition, it is obvious that if $\theta = 0$ and $\theta = \frac{\pi}{2}$, then the pseudo slant submanifold becomes semi-invariant submanifold and anti-invariant submanifold, respectively. On the other hand, if we denote the dimensions of $D_\theta$ and $D^\bot$ by $d_1$ and $d_2$, respectively, then we have the following cases.

(i) if $d_1 = 0$, then $M$ is an anti-invariant submanifold,  

(ii) if $d_2$ and $\theta = 0$, then $M$ is an invariant submanifold,  

(iii) if $d_2 = 0$ and $\theta \neq 0$, then $M$ is a proper slant submanifold.

A pseudo submanifold is called proper if $d_1, d_2 \neq 0$, $\theta \neq 0$ and $\theta \neq \frac{\pi}{2}$.

In this connection it may be mentioned that Atceken and Hui [3] studied pseudo-slant submanifolds of $(LCS)_n$-manifolds.

The notion of warped product manifolds were introduced by Bishop and O’Neill [4].

**Definition 4.** Let $(N_1, g_1)$ and $(N_2, g_2)$ be two Riemannian manifolds with Riemannian metric $g_1$ and $g_2$ respectively and $f$ be a positive definite smooth function on $N_1$. The warped product of $N_1$ and $N_2$ is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

\[ g = g_1 + f^2 g_2. \]  

(28)

A warped product manifold $N_1 \times_f N_2$ is said to be trivial if the warping function $f$ is constant.

More explicitly, if the vector fields $X$ and $Y$ are tangent to $N_1 \times_f N_2$ at $(p, q)$ then

\[ g(X, Y) = g_1(\pi_1 * X, \pi_1 * Y) + f^2(p)g_2(\pi_2 * X, \pi_2 * Y), \]

where $\pi_i$ ($i = 1, 2$) are the canonical projections of $N_1 \times N_2$ onto $N_1$ and $N_2$ respectively and $*$ stands for the derivative map.

Let $M = N_1 \times_f N_2$ be warped product manifold, which means that $N_1$ and $N_2$ are totally geodesic and totally umbilical submanifolds of $M$ respectively.

For warped product manifolds, we have [17].

**Proposition 1.** Let $M = N_1 \times_f N_2$ be a warped product manifold. Then
(i) $\nabla_X Y \in TN_1$ is the lift of $\nabla_X Y$ on $N_1$,
(ii) $\nabla_U X = \nabla_X U = (X \ln f) U$,
(iii) $\nabla_U V = \nabla'_U V - g(U, V) \nabla \ln f$, for any $X, Y \in \Gamma(TN_1)$ and $U, V \in \Gamma(TN_2)$, where $\nabla$ and $\nabla'$ denote the Levi-Civita connections on $N_1$ and $N_2$, respectively.

3 Warped product pseudo-slant submanifolds of $(LCS)_n$-manifolds

Let us suppose that $M = N_1 \times_f N_2$ be a warped product pseudo-slant submanifold of a $(LCS)_n$-manifold $\overline{M}$. Such submanifolds are always tangent to the structure vector field $\xi$. If $N_\theta$ and $N_\perp$ are proper slant submanifolds and anti-invariant submanifolds of a $(LCS)_n$-manifold $\overline{M}$ then their warped product pseudo-slant submanifolds may be given by one of the following:
(i) $N_\perp \times_f N_\theta$, (ii) $N_\theta \times_f N_\perp$

We now prove the following.

**Theorem 2.** Let $\overline{M}$ be a $(LCS)_n$-manifold. Then there does not exist warped product pseudo-slant submanifold of $\overline{M}$ of the type $M = N_\perp \times_f N_\theta$ in $\overline{M}$ where $N_\perp$ is an anti-invariant submanifold and $N_\theta$ is a proper slant submanifold of $\overline{M}$ such that $\xi$ is tangent to $N_\theta$.

**Proof.** From Proposition 1, we have
$$\nabla_X Z = \nabla_Z X = (Z \ln f) X$$
for any vector fields $X \in \Gamma(TN_\theta)$ and $Z \in \Gamma(TN_\perp)$. If $\xi \in \Gamma(TN_\theta)$, then we have
$$\nabla_Z \xi = (Z \ln f) \xi.$$  
(30)

On the other hand, from (4), (17) and Proposition 1, we have
$$Z(\ln f) \xi = \alpha Z$$
(31)
Taking the inner product with $\xi$ in (31), we get $Z(\ln f) = 0$, which means that $f$ is constant on $M$ and hence the proof is complete.

**Theorem 3.** Let $\overline{M}$ be a $(LCS)_n$-manifold. Then there exist warped product pseudo-slant submanifolds of $\overline{M}$ of the type $M = N_\theta \times_f N_\perp$ in $\overline{M}$ such that $N_\theta$ is a proper slant submanifold tangent to $\xi$ and $N_\perp$ is an anti-invariant submanifold of $\overline{M}$.

**Proof.** For any vector fields $X \in \Gamma(TN_\theta)$ and $Z \in \Gamma(TN_\perp)$, from Proposition 1, we get the relation (29). Then for $\xi \in \Gamma(TN_\theta)$ we have from (29) that
$$\nabla_Z \xi = (\xi \ln f) Z.$$  
(32)
Again, from (4) and (17), we get
$$\nabla_Z \xi = \alpha Z,$$
(33)
$$h(Z, \xi) = 0.$$  
(34)
From (32) and (33), we get $\xi \ln f = \alpha (\neq 0)$ for all $Z \in \Gamma(TN_\perp)$. That means we get a non-zero and non-constant warping function $f$. Hence such a structure exist and consequently the theorem is proved.
Example 1. Consider the semi-Euclidean space $\mathbb{R}^{11}$ with the cartesian coordinates $(x_1, y_1, \ldots, x_5, y_5, t)$ and paracontact structure

$$\phi \left( \frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial y_i}, \quad \phi \left( \frac{\partial}{\partial y_j} \right) = \frac{\partial}{\partial x_j}, \quad \phi \left( \frac{\partial}{\partial t} \right) = 0, \quad 1 \leq i, j \leq 5.$$ 

It is clear that $\mathbb{R}^{11}$ is a Lorentzian metric manifold with usual semi-Euclidean metric tensor. Let $M$ be a submanifold of $\mathbb{R}^{11}$ defined by

$$\chi(u, v, w, t) = (v\cos u, v\sin u, w\cos u, w\sin u, v + 2w, -2v + w, -w\cos u, w\sin u, -v\cos u, v\sin u, t)$$

with non-zero $u, v, w$ and $u \in \left(0, \frac{\pi}{2}\right)$. Then the tangent space of $M$ is spanned by the following vectors

$$Z_1 = -v\sin u \frac{\partial}{\partial x_1} + v\cos u \frac{\partial}{\partial y_1} - w\sin u \frac{\partial}{\partial x_2} + w\cos u \frac{\partial}{\partial y_2} + w\sin u \frac{\partial}{\partial x_4} + w\cos u \frac{\partial}{\partial y_4} + v\cos u \frac{\partial}{\partial y_5},$$

$$Z_2 = \cos u \frac{\partial}{\partial x_1} + \sin u \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_3} - 2\frac{\partial}{\partial x_3} - \cos u \frac{\partial}{\partial y_5} + \sin u \frac{\partial}{\partial x_5},$$

$$Z_3 = \cos u \frac{\partial}{\partial x_2} + \sin u \frac{\partial}{\partial y_2} + \frac{\partial}{\partial x_3} + \cos u \frac{\partial}{\partial y_3} - \sin u \frac{\partial}{\partial y_4},$$

$$Z_4 = \frac{\partial}{\partial t}.$$

Then with respect to paracontact structure on $\mathbb{R}^{11}$, we get

$$\phi Z_1 = -v\sin u \frac{\partial}{\partial y_1} + v\cos u \frac{\partial}{\partial x_1} - w\sin u \frac{\partial}{\partial y_2} + w\cos u \frac{\partial}{\partial x_2} + w\sin u \frac{\partial}{\partial y_4} + w\cos u \frac{\partial}{\partial x_4} + v\cos u \frac{\partial}{\partial x_5},$$

$$\phi Z_2 = \cos u \frac{\partial}{\partial y_1} + \sin u \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_3} - 2\frac{\partial}{\partial x_3} - \cos u \frac{\partial}{\partial x_5} + \sin u \frac{\partial}{\partial y_5},$$

$$\phi Z_3 = \cos u \frac{\partial}{\partial y_2} + \sin u \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_3} + \cos u \frac{\partial}{\partial x_3} - \sin u \frac{\partial}{\partial x_4},$$

$$\phi Z_4 = 0.$$

It is easy to see that $\mathcal{D}^\theta = \text{Span}\{Z_2, Z_3\}$ is a slant distribution with slant angle $\theta = \cos^{-1} \left( \frac{1}{2} \right)$ and $\mathcal{D}^\perp = \text{Span}\{Z_1\}$ is an anti-invariant distribution. Thus $M$ is a pseudo-slant submanifold of $\mathbb{R}^{11}$. It is easy to see that both the distributions are integrable. We denote the integral manifolds of $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ by $M_\theta$ and $M_\perp$, respectively. Then the product metric $g$ of $M$ is given by

$$g = -dt^2 + 7\left( dv^2 + dw^2 \right) + 2\left( v^2 + w^2 \right) du^2.$$

Hence $M$ is a warped product pseudo-slant submanifold of $\mathbb{R}^{11}$ of the type $M_\theta \times_f M_\perp$ with warping function $f = \sqrt{2(v^2 + w^2)}$.

4 Conclusion

Let $N_\theta$ and $N_\perp$ be proper slant and anti-invariant submanifolds of a $(LCS)_n$-manifold $\overline{M}$ then their warped product pseudo-slant submanifolds may be given by one of the following.

(i) $N_\perp \times_f N_\theta$.

(ii) $N_\theta \times_f N_\perp$. Here we prove two theorems. Theorem 2 states that there does not exist warped product pseudo-slant submanifolds of $(LCS)_n$-manifold $\overline{M}$ of the type $M = N_\perp \times_f N_\theta$ such that $N_\perp$ and $N_\theta$ are anti-invariant and proper slant submanifolds of $\overline{M}$ so that $\xi$ is tangent to $N_\theta$. And theorem 3 states that there exist warped product pseudo-slant submanifolds of a $(LCS)_n$-manifold $\overline{M}$ of the type $M = N_\theta \times_f N_\perp$ such that $N_\theta$ is a proper slant submanifold tangent to $\xi$ and $N_\perp$ is an anti-invariant submanifold of $\overline{M}$. The example 1 also support the Theorem 3. So there is a
natural question arises. Does there exist warped product pseudo-slant submanifolds of a \((LCS)_n\)-manifold \(\overline{M}\) of the type,

(iii) \(M = N_\perp \times_f N_\theta\) and (iv) \(M = N_\theta \times_f N_\perp\) such that \(N_\theta\) is a proper slant submanifold and \(N_\perp\) is an anti-invariant submanifold tangent to \(\xi\) of \(\overline{M}\)? These problems are still open.

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The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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