Some fixed point theorems in 2-Banach spaces and 2-normed tensor product spaces

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Abstract: In this paper, we derive some fixed point theorems in 2-Banach spaces. Let $X$ be a 2-Banach space and $T$ be a self-mapping on $X$. Let $\psi : [0, \infty) \to [0, \infty)$, $\beta, \phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ and $\gamma : [0, \infty) \times [0, \infty) \times [0, \infty) \to [0, \infty)$ be continuous mappings having some specific characteristics. Using these mappings, we define some conditions for $T$ under which $T$ has a unique fixed point in $X$. The conditions for two self-mappings $T_1$ and $T_2$ on $X$ for having the common unique fixed point are also derived here with proper examples. Moreover, defining a 2-norm in the projective tensor product space, we derive a fixed point theorem here with a suitable example.

Keywords: 2-Banach space, fixed points, projective tensor product.

1 Introduction

In this paper, we derive some fixed point theorems for mappings on 2-Banach spaces satisfying some specific characteristics. The notion of 2-normed linear spaces and their topological structures was initiated by Gähler [10] in his paper “Linear 2-normed spaces”. He studied the special class of 2-metric spaces which is linear and defined a 2-norm on those spaces. Motivated by this work, several authors namely Iseki [11], Rhoads [27], White [29], etc., studied various aspects of the fixed point theory and proved some fixed point theorems in 2-metric and 2-Banach spaces. Cho et al. [3] investigated about common fixed points of weakly compatible mappings in 2-metric spaces. In 1993, Khan and Khan [12] derived some results on fixed points of involution maps in 2-Banach spaces. In 2013 [28], Saha et al. discussed some fixed point theorems for a class of for weakly C-contractive mappings in a setting of 2-Banach Space.

2 Preliminaries

Definition 1. Let $X$ be a real linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real valued function on $X \times X$ satisfying the following conditions:

(i) $\|x,y\| = 0$ if and only if $x$ and $y$ are linearly dependent,
(ii) $\|x,y\| = \|y,x\|$ for all $x,y \in X$,
(iii) $\|\alpha x,y\| = |\alpha| \|y,x\|$, $\alpha$ being real, $x,y \in X$,
(iv) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$, for all, $x,y,z \in X$

Then $\|\cdot,\cdot\|$ is called a 2-norm on $X$ and $(X,\|\cdot,\cdot\|)$ is called a linear 2-normed space.

Definition 2. A sequence $\{x_n\}$ in a 2-normed space $(X,\|\cdot,\cdot\|)$ is said to be a Cauchy sequence if $\lim_{n,m \to \infty} \|x_n - x_m,\alpha\| = 0$ for all $\alpha$ in $X$.

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Definition 3. A sequence \( \{x_n\} \) in a 2-normed space \( X \) is called a convergent sequence if there is an \( x \in X \) such that \( \lim_{n \to \infty} \|x_n - x, a\| = 0 \) for all \( a \) in \( X \).

Definition 4. A 2-normed space in which every Cauchy sequence is convergent is called a 2-Banach space.

Definition 5. Let \( X \) and \( Y \) be two linear 2-normed spaces. An operator \( T : X \to Y \) is said to be continuous at \( x \in X \) if for every sequence \( \{x_n\} \) in \( X \), \( \{x_n\} \to x \) as \( n \to \infty \) implies \( \{T(x_n)\} \to T(x) \) in \( Y \) as \( n \to \infty \).

Definition 6. Let \( f \) and \( g \) be two self-maps defined on a set \( X \). Then \( f \) and \( g \) are said to be weakly compatible if they commute at coincidence points, i.e., if \( f(u) = g(u) \) for some \( u \in X \), then \( f(g(u)) = g(f(u)) \).

3 Fixed point in 2-Banach spaces

Theorem 1. Let \( X \) be a 2-Banach space and \( T \) be a self map on \( X \). Let \( \psi : [0, \infty) \to [0, \infty) \) and \( \beta : [0, \infty) \times [0, \infty) \to [0, \infty) \) are continuous mappings satisfying the conditions: \( \psi(0) = 0 \), \( \psi \) is monotonically increasing;

\[
b\psi(s) \leq \beta(r, s) \Rightarrow bs \leq r, b \in \{1, 2\}; \beta(s, t) = 0 \Leftrightarrow s = t = 0.
\]

Let

\[
\psi(||Tx - Ty, a||) \leq \beta(||x - Tx, a||, ||y - Ty, a||) - \max[\psi(||x - Tx, a||), \psi(||y - Ty, a||)]
\]

where \( x, y, a \in X \). Then \( T \) has a unique fixed point on \( X \).

Proof. For any fixed \( x_0 \in X \), we construct a sequence \( \{x_n\} \) by \( x_{n+1} = Tx_n, n = 0, 1, 2, \ldots \)

\[
\psi(||x_n - x_{n+1}, a||) \leq \beta(||x_n - x_{n+1}, a||) - \max[\psi(||x_n - x_{n+1}, a||), \psi(||x_{n+1} - x_{n+1}, a||)]
\]

Therefore we write,

\[
||x_n - x_{n+2}, a|| \leq ||x_n - x_{n+1}, a||.
\]

So, \( \{x_n - x_{n+1}, a\} \) is a monotonic decreasing sequence of real numbers and hence it converges to some \( r \), say, i.e., \( x_n - x_{n+1}, a \to r \) as \( n \to \infty \).

Now, \( \|x_n - x_{n+1}, a\| = \|Tx_{n-1} - Tx_n, a\|. \) So,

\[
\psi(r) = \psi(\lim_{n \to \infty} ||x_n - x_{n+1}, a||) = \lim_{n \to \infty} \psi(||Tx_{n-1} - Tx_n, a||)
\]

\[
\leq \lim_{n \to \infty} \beta(||x_{n-1} - Tx_{n-1}, a||, ||x_n - Tx_n, a||) - \max(\psi(||x_{n-1} - Tx_{n-1}, a||), \psi(||x_n - Tx_n, a||))
\]

\[
= \lim_{n \to \infty} \beta(||x_{n-1} - x_{n-1}, a||, ||x_n - x_{n+1}, a||) - \max(\psi(||x_{n-1} - x_{n-1}, a||), \psi(||x_n - x_{n+1}, a||))
\]

\[
= \beta(r, r) - \max(\psi(r), \psi(r))
\]

Thus, \( 2\psi(r) \leq \beta(r, r) \Rightarrow 2r \leq r, \) possible for \( r = 0 \). Hence, \( \|x_n - x_{n+1}, a\| \to 0 \) as \( n \to \infty \).

Next, we show that \( \{x_n\} \) is a Cauchy sequence in \( X \). If possible, let \( \{x_n\} \) be not a Cauchy sequence, and so, there exists
= 0 \implies |a| \leq 0 (\forall \alpha \in X)
\Rightarrow \|z - Tz, a\| = 0

Since $a$ is arbitrary, taking $a = 0$, we get, $z = Tz$.

To show the uniqueness: Let $Tz_1 = z_1$ and $Tz_2 = z_2$. Then

$$
\psi(\|Tz_1 - Tz_2, a\|) \leq \beta(\|z_1 - Tz_1, a\|, \|z_2 - Tz_2, a\|) - \max[\psi(\|z_1 - Tz_1, a\|, \psi(\|z_2 - Tz_2, a\|)]
\Rightarrow 2\psi(\|z - Tz, a\|) \leq \beta(0, \|z - Tz, a\|); [\text{taking } n \to \infty]
\Rightarrow 2\|z - Tz, a\| \leq 0, (\forall \alpha \in X)
\Rightarrow \|z - Tz, a\| = 0
$$

Therefore we write,

$$
\|Tz_1 - Tz_2, a\| = 0 \forall \alpha \in X \Rightarrow Tz_1 = Tz_2 \Rightarrow z_1 = z_2.
$$

The proof is completed.

**Example 1.** Let $X = \mathbb{R}^3$ and we consider the following 2-norm on $X$ (refer to [1])

$$
\|x, y\| = \left| \det \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|
$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3$; Then $(X, \|\cdot\|)$ is a 2-Banach space.

We fix $(e, f, g) \in \mathbb{R}^3$ and let $T$ be a self mapping on $\mathbb{R}^3$ defined by $T(x, y, z) = (e, f, g) \forall (x, y, z) \in \mathbb{R}^3$.

Let $\psi(s) = 2s, \beta(r, s) = \frac{r}{2} + s$; where $(r, s) \in [0, \infty)$. Now, $Tx = (e, f, g) = Ty$ therefore $\|Tx - Ty, a\| = 0$.

Hence all the conditions of Theorem 1 are satisfied. So, $T$ has a unique fixed point $(e, f, g) \in \mathbb{R}^3$.

For common fixed point of two self maps $T_1$ and $T_2$ on $X$, we prove.
Theorem 2. Let $X$ be a 2-Banach space and $T_1$ and $T_2$ be two self maps on $X$. Let $\psi$ and $\beta$ be as defined in Theorem 1 with $\beta(r,s) = \beta(s,r)$. Then $T_1$ and $T_2$ have common unique fixed point, if for $x, y, a \in X$
\[
\psi(\|T_1x - T_2y, a\|) \leq \beta(\|x - T_1x, a\|, \|y - T_2y, a\|) - \max(\psi(\|x - T_1x, a\|), \psi(\|y - T_2y, a\|))
\]

Proof. For a fixed point $x_0 \in X$, we construct a sequence $\{x_n\}$ by
\[
x_{2n+1} = T_1(x_{2n}) \text{ and } x_{2n+2} = T_2(x_{2n+1}), \quad n = 0, 1, 2, \ldots
\]
Now, it can be shown that $\{x_n\}$ is a Cauchy sequence in $X$, converging to some $z$ in $X$, which is the common fixed point for $T_1$ and $T_2$.

Corollary 1. Let $X$ be a 2-Banach space and $T$ be a self mapping on $X$. Let $\psi, \beta$ be as defined in Theorem 3.1 satisfying
\[
\psi(\|Tx - Ty, a\|) \leq \frac{1}{c} \beta(\|x - Ty, a\|, \|y - Ty, a\|) - \max(\psi(\|x - Ty, a\|), \psi(\|y - Ty, a\|))
\]
where $c > 2$ and $x, y, a \in X$. Then $T$ has unique fixed point on $X$.

Corollary 2. If $\psi$ satisfies then also similar result holds for
\[
\psi(\|Tx - Ty, a\|) \leq \beta(\|x - y, a\|, \|y - Ty, a\|) - \max(\psi(\|x - y, a\|), \psi(\|y - Ty, a\|)), \forall x, y, a \in X.
\]

We now establish another fixed point theorem for $T$ using two other mappings $\gamma$ and $\phi$.

Theorem 3. Let $X$ be a 2-Banach space and $T$ be a self mapping on $X$. Let $\gamma : [0, \infty) \times [0, \infty) \times [0, \infty) \to [0, \infty)$ and $\phi : [0, \infty) \to [0, \infty)$ be continuous mapping satisfying $\gamma(r, 0, r + t) \leq kr$ and $\phi(r, t) \geq k'r$, where $k, k' \in [0, \infty)$ such that $k - k' < 1$. Let
\[
\|Tx - Ty, a\| \leq \gamma(\|x - y, a\|, \|y - Ty, a\|, \|x - Ty, a\|) - \phi(\|x - Ty, a\|, \|y - Ty, a\|), \forall x, y, a \in X
\]
Then $T$ has a fixed point.

Proof. For any fixed $x_0 \in X$, we construct a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. We get,
\[
\|x_n - x_{n+1}, a\| \leq \gamma(\|x_{n-1} - x_n, a\|, \|x_n - x_{n+1}, a\|, \|x_n - x_{n+1}, a\|) - \phi(\|x_{n-1} - x_n, a\|, \|x_n - x_{n+1}, a\|)\]
\[
\leq \gamma(\|x_{n-1} - x_n, a\|, 0, \|x_n - x_{n+1}, a\|) + \|x_n - x_{n+1}, a\| - \phi(\|x_{n-1} - x_n, a\|, \|x_n - x_{n+1}, a\|)\]
\[
\leq (k - k') \|x_{n-1} - x_n, a\|\]
\[
\leq (k - k')^2 \|x_{n-2} - x_{n-1}, a\| \leq \ldots \leq (k - k')^n \|x_0 - x_1, a\| \to 0 \text{ as } n \to \infty
\]
Hence $\{x_n\}$ is a Cauchy sequence in $X$ and so, it converges to some $z$ (say) in $X$.
\[
\|z - Tz, a\| \leq \|z - x_{n+1}, a\| + \|x_{n+1} - Tz, a\|
\leq \|z - x_{n+1}, a\| + \gamma(\|x_{n-1} - z, a\|, \|z - Tz, a\|) + \|x_n - Tz, a\| - \phi(\|x_n - Tz, a\|, \|z - Tz, a\|)\]
\[
\leq 0 + \gamma(0, 0, \|z - Tz, a\|) - \phi(\|z - Tz, a\|) \text{ [taking } n \to \infty]\]
Therefore we write $\|z - Tz, a\| = 0$, for all $a \in X$. Since $a$ is arbitrary, taking $a = 0$, we get, $z = Tz$.

Example 2. Let $\gamma(r, s, t) = k_1(r + s + t)$ and $\phi(r, s) = k_2(r + s)$, where $k_1$ and $k_2$ are two constants ($> 0$). Now we can find out $k, k' \in [0, \infty)$ with $k - k' < 1$ such that $\gamma(r, 0, r + t) = k_1(2r + t) \leq kr$ and $\phi(r, t) = k_2(r + t) \geq k'r$. Let $T$ and $X$ be as
defined in Example 1. Now, for \(x, y, a \in X\)
\[
\|Tx - Ty, a\| \leq \gamma ||x - y, a||, ||y - Tx, a||, ||x - Ty, a|| - \phi[||x - Tx, a||, ||y - Ty, a||]
\]
Hence by Theorem 3, \(T\) has a fixed point on \(X\).

Depending upon \(k_1\) and \(k_2\), the mapping \(T\) is of different types. From the given condition,
\[
\|Tx - Ty, a\| \leq \gamma ||x - y, a||, ||y - x, a|| + ||x - Tx, a||, ||y - y, a|| + \phi[||x - Tx, a||, ||y - Ty, a||]
\]
\[
\leq k_1[3||x - y, a|| + ||x - Tx, a|| + ||y - Ty, a||] - k_2[||x - Tx, a|| + ||y - Ty, a||].
\]
So, if \(k_1 = k_2\), then \(\|Tx - Ty, a\| \leq 3k_1||x - y, a||\) which is a contraction mapping for \(k_1 < \frac{1}{3}\) (and has a unique fixed point) and nonexpansive for \(k_1 = \frac{1}{3}\).

Next, we discuss common fixed point for four mappings in 2-Banach spaces.

4 2-Norm for projective tensor product

4.1 Algebraic tensor product

[2]. Let \(X, Y\) be normed spaces over \(F\) with dual spaces \(X^*\) and \(Y^*\) respectively. Given \(x \in X, y \in Y, \) let \(x \otimes y\) be the element of \(BL(X^*, Y^*; F)\) (which is the set of all bounded bilinear forms from \(X^* \times Y^*\) to \(F\)), defined by
\[
x \otimes y(f, g) = f(x)g(y), (f \in X^*, g \in Y^*)
\]
The algebraic tensor product of \(X, Y, X \otimes Y\) is defined to be the linear span of \(\{x \otimes y : x \in X, y \in Y\}\) in \(BL(X^*, Y^*; F)\).

4.2 Projective tensor product

[2]. Given normed spaces \(X, Y\), the projective tensor norm \(\gamma\) on \(X \otimes Y\) is defined by
\[
\|u\|_\gamma = \inf \left\{ \sum_i \|x_i\|_Y \mid u = \sum_i x_i \otimes y_i \right\}
\]
where the infimum is taken over all (finite) representations of \(u\). For the normed spaces \(X, Y\), in the projective tensor product \(X \otimes_\gamma Y\), we take
\[
\|u, v\| = \|u\|_\gamma \|v\|, u, v \in X \otimes_\gamma Y
\]
Following White [29], we can say that \(X \otimes_\gamma Y\) is a 2-Banach space up to linear dependence (i.e., \(X \otimes_\gamma Y\) satisfies all the conditions for being a 2-Banach space except \(u\) and \(v\) may be linearly dependent and yet \(\|u, v\| \neq 0\)).

Let \(D_X, D_Y\) and \(D_X \otimes_\gamma Y\) denote a closed and bounded subset of \(X, Y\) and \(X \otimes_\gamma Y\) respectively. Let \(T_1\) and \(T_2\) be two pairs of mappings where \(T_1 : D_X \otimes_\gamma Y \rightarrow D_X\) and \(T_2 : D_X \otimes_\gamma Y \rightarrow D_Y\) be such that for any \(u, v \in D_X \otimes_\gamma Y\) and \(a \otimes b \in D_X \otimes_\gamma Y\) with \(\|a\| \geq 1\) and \(\|b\| \geq 1\).

\((E)\) \(\|T_1(u) - T_1(v)\| \leq \frac{1}{KM_2}(k\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|))\)

\((F)\) \(\|T_2(u) - T_2(v)\| \leq \frac{1}{KM_1}(k'\|u - v, a \otimes b\| - \psi(k'\|u - v, a \otimes b\|))\)
where

(i) \( \psi: [0, \infty) \rightarrow [0, \infty) \) is continuous and non-decreasing, \( \psi(0) = 0 \)

(ii) \( \|T_1u\| \leq M_1 \) and \( \|T_2u\| \leq M_2 \), \( \forall u \in D_{X \otimes Y} \).

Here, \( D_{X \otimes Y} \) is bounded by \( K \) and \( k, k' \) are positive. From the mappings \( T_1 \) and \( T_2 \) we define a mapping \( T: D_{X \otimes Y} \rightarrow D_{X \otimes Y} \) such that \( Tu = T_1u \otimes T_2u \).

**Theorem 4.** The mapping \( T \) derived by the pair of mappings \( (T_1, T_2) \) satisfying (E) and (F) has a unique fixed point in \( D_{X \otimes Y} \) if \( k + k' \leq 1 \).

**Proof.** For \( u, v \in D_{X \otimes Y}, \ a \in X \) and \( b \in Y \) and \( a \otimes b \in D_{X \otimes Y} \) with \( \|a\| \geq 1 \) and \( \|b\| \geq 1 \)

\[
\|Tu - Tv, a \otimes b\| = \|T_1u \otimes T_2u - T_1v \otimes T_2v, a \otimes b\|
\leq \|T_1u - T_1v\|,\|T_2u \otimes a \otimes b\| + \|T_1v \otimes (T_2u - T_2v), a \otimes b\|
= \|T_1u - T_1v\|,\|T_2u\|,\|a \otimes b\| + \|T_1v\|,\|T_2u - T_2v\|,\|a \otimes b\|
\leq \frac{1}{K} [k,\|u, a \otimes b\| - \psi(\|u, a \otimes b\|)]M_2
+ \frac{1}{KM_1} [k',\|u, a \otimes b\| - \psi(\|k',\|u, a \otimes b\|)]MK_1
= (k + k')\|u, a \otimes b\| - \psi(\|k',\|u, a \otimes b\|) - \psi(\|k,\|u, a \otimes b\|)
\leq \|u, a \otimes b\| - \{\psi(\|u, a \otimes b\|) + \psi(\|k',\|u, a \otimes b\|)\}

Let \( x_0 \in D_{X \otimes Y} \) be fixed. We take \( x_{n+1} = Tx_n \). Now,

\[
\|x_{n+1} - x_n, a \otimes b\| = \|Tx_n - Tx_{n-1}, a \otimes b\|
\leq \|x_n - x_{n-1}, a \otimes b\| - \psi(\|x_n - x_{n-1}, a \otimes b\|)
\leq \|x_n - x_{n-1}, a \otimes b\|
\]

Hence \( \{\|x_{n+1} - x_n, a \otimes b\|\} \) is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say \( r \). Taking \( n \to \infty \), we get

\[
r \leq r - \{\psi(\|k\|) + \psi(\|k'\|)\}\,.(by\,continuity\,of\,\psi)\,\,Then,\,\,\psi(\|k\|) + \psi(\|k'\|) \leq 0,
\]

this is possible only when \( r = 0 \). So,

\[
\lim_{n \to \infty} x_{n+1} - x_n, a \otimes b\| = 0
\]

\[
\Rightarrow \lim_{n \to \infty} \|x_{n+1} - x_n\|,\|a \otimes b\| = 0 \Rightarrow \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0
\]

\[
\Rightarrow \lim_{n \to \infty} \|x_{n+1} - x_n, a \otimes b\| = 0 \forall u \in D_{X \otimes Y}
\]

Hence, \( \{x_n\} \) is a Cauchy sequence in the 2-Banach space \( D_{X \otimes Y} \). Let it converge to some \( z \in D_{X \otimes Y} \). Now,

\[
\|z - Tz, u\| \leq \|z - x_{n+1}, u\| + \|x_{n+1} - Tz, u\| = \|z - x_{n+1}, u\| + \|Tx_n - Tz, u\|
\leq \|z - x_{n+1}, u\| + \|x_n - z, u\| - \psi(\|x_n - z, u\|) + \psi(\|x_n - z, u\|)
\rightarrow 0\, as\, n \to \infty
Hence, \( \|z - Tz, u\| = 0 \Rightarrow z = Tz \). To show the uniqueness. Let \( z_1 \) and \( z_2 \) be two distinct fixed points for \( T \) in \( D_{X \otimes Y} \). Now,

\[
\|z_1 - z_2, u\| = \|Tz_1 - Tz_2, u\| \leq \|z_1 - z_2, u\| + \|\psi(k\|z_1 - z_2, u\|) + \psi(k'\|z_1 - z_2, u\|)\| \\
\Rightarrow \psi(k\|z_1 - z_2, u\|) + \psi(k'\|z_1 - z_2, u\|) \leq 0
\]

which is contradiction. So, \( z_1 = z_2 \). Thus, \( T \) has a unique fixed point in the closed and bounded subset \( D_{X \otimes Y} \) of \( X \otimes Y \).

**Example 3.** Let \( D_{l^1 \otimes K} \) (with the same 2-norm as defined above in the tensor product space), \( D_{l^1} \) and \( D_K \) denote a closed and bounded subset of \( l^1 \otimes K \), \( l^1 \) and \( K \), bounded by \( K \), \( \sqrt{K} \) and \( \sqrt{K} \) respectively \( (K > 0) \).

We define \( T_1 : D_{l^1 \otimes K} \to D_{l^1} \) by

\[
T_1\left( \sum_i a_i \otimes x_i \right) = \frac{1}{2K} \sum_i \{a_i,x_i\}, \text{ where } a_i = \{a_{ik}\}_n
\]

and \( T_2 : D_{l^1 \otimes K} \to D_K \) by \( T_2(\sum_i a_i \otimes x_i) = \frac{1}{4} \sum_i \|a_i\|.x_i \). For arbitrary \( b_k = \{b_{kn}\} \in D_{l^1}, b \in D_K \) with \( \|b_k\| \geq 1 \) and \( |b| \geq 1 \),

\[
\|T_1(\sum_i a_i \otimes x_i)\| = \frac{1}{2K} \sum_i \|a_i,x_i\| \leq \frac{1}{2K} \sum_i \|a_i,x_i\| \|b_k\||b| \\
\leq \frac{1}{2K} \sum_i \|a_i \otimes x_i\| \|b_k \otimes b\| = \frac{K^2}{2} = (M_1),
\]

and

\[
\|T_2(\sum_i a_i \otimes x_i)\| \leq \frac{1}{4} \sum_i \|a_i \otimes x_i\| \|b_k \otimes b\| \leq \frac{K^2}{4} = (M_2)
\]

For \( u = \sum_i a_i \otimes x_i \) and \( v = \sum_i d_i \otimes y_i \) in \( D_{l^1 \otimes K} \), we have,

\[
\|T_1u - T_1v\| = \frac{1}{2K} \sum_i \|a_i,x_i\| - \frac{1}{2K} \sum_i \|d_i,y_i\| \\
= \frac{1}{4} \sum_i \|a_i \otimes x_i - d_i \otimes y_i\| \leq \frac{1}{4} \|u - v\| \|b_k \otimes b\| \\
\leq 2 \left[ \frac{1}{2} \|u - v, b_k \otimes b\| - \frac{1}{2} \|u - v, b_k \otimes b\| \right] \\
= \frac{1}{KM_2} \left[ \frac{1}{2} \|u - v, b_k \otimes b\| - \psi \left( \frac{1}{2} \|u - v, b_k \otimes b\| \right) \right]; \text{ where } \psi(t) = \frac{t}{2} \cdot k = \frac{1}{2},
\]

and

\[
\|T_2u - T_2v\| = \frac{1}{4} \sum_i \|a_i\|.x_i - \frac{1}{4} \sum_i \|d_i\|.y_i\| \leq \frac{1}{4} \sum_i \|a_i\|.x_i - \sum_i \|d_i\|.y_i\| \|b_k\| \|b\|.
\]
Taking the projective tensor norm,
\[
\|T_2u - T_2v\| \leq \frac{1}{4} \|u - v\| \|b_k \otimes b\| + \frac{1}{2} \|u - v\| \|b_k \otimes b\| = \frac{1}{4} \|u - v\| \|b_k \otimes b\|
\]
\[
\leq \frac{1}{4} \|u - v\| \|b_k \otimes b\| - \frac{1}{2} \left( \frac{1}{2} \|u - v\| \|b_k \otimes b\| \right)
\]
\[
= \frac{1}{2K} \left( \frac{1}{2} \|u - v\| \|b_k \otimes b\| \right)
\]
\[
\leq \frac{1}{2K} \|u - v\| \|b_k \otimes b\| - \|D\| \left( \frac{1}{2} \|u - v\| \|b_k \otimes b\| \right)
\]
\[
= \frac{1}{2K} \|u - v\| \|b_k \otimes b\| - \|D\| \left( \frac{1}{2} \|u - v\| \|b_k \otimes b\| \right)
\]
\[
= \frac{1}{2K} \|u - v\| \|b_k \otimes b\| - \|D\| \left( \frac{1}{2} \|u - v\| \|b_k \otimes b\| \right)
\]
\[
= \frac{1}{2K} \|u - v\| \|b_k \otimes b\| - \|D\| \left( \frac{1}{2} \|u - v\| \|b_k \otimes b\| \right)
\]

Therefore, \((T_1, T_2)\) satisfies the conditions (a) and (b). Also, \(k + k' = \frac{1}{2} + \frac{1}{2} = 1\). So, the mapping \(T : D_{X_1 \otimes Y} \to D_{X_2 \otimes Y}\)
has a unique fixed point in \(D_{X_1 \otimes Y}\).

Let \(T_1, S_1, P_1, T_2, S_2, P_2\) be some mappings where \(T_1, S_1, P_1 : D_X \otimes Y \to D_X\) and \(T_2, S_2, P_2 : D_X \otimes Y \to D_Y\) be two mappings such that for any \(u, v \in D_X \otimes Y\) and \(a \otimes b \in X \otimes Y\),
\[
(G) \quad \|T_1(u) - S_1(v)\| \leq \frac{1}{MM_2} \|k\| \|Pu - Tv, a \otimes b\| + \|Pu - Sv, a \otimes b\| - \psi(k\|Pu - Tv, a \otimes b\|, k\|Pu - Sv, a \otimes b\|)
\]
\[
(H) \quad \|T_2(u) - S_2(v)\| \leq \frac{1}{MM_1} \|k\| \|Pu - Tv, a \otimes b\| + \|Pu - Sv, a \otimes b\| - \psi(k\|Pu - Tv, a \otimes b\|, k\|Pu - Sv, a \otimes b\|)
\]

where

(i) \(\psi : [0, \infty) \to [0, \infty)\) is continuous and non-decreasing, \(\psi(0) = 0\)
(ii) \(\max \|T_1u\|, \|S_1v\| \leq M_1\) and \(\max \|T_2u\|, \|S_2v\| \leq M_2\), \(\forall u, v \in D_X \otimes Y\), \(a \in X\) and \(b \in Y\). Here, \(D_X \otimes Y\) is bounded by \(M\) and \(k, k'\) are positive.

From the mappings \(T_1, S_1, P_1, T_2, S_2, P_2\) we define some mappings \(T : D_X \otimes Y \to D_X \otimes Y\) such that \(Tu = T_1u \otimes T_2u\);
\(S : D_X \otimes Y \to D_X \otimes Y\) such that \(Su = S_1u \otimes S_2u\) and \(P : D_X \otimes Y \to D_X \otimes Y\) such that \(Pu = P_1u \otimes P_2u\).

**Theorem 5.** Let \(T, S\) and \(P\) be self mappings as defined above such that

(i) \(\{T, P\}\) and \(\{S, P\}\) are weakly compatible
(ii) \(T(X \otimes Y) \subseteq P(X \otimes Y)\) and \(S(X \otimes Y) \subseteq P(X \otimes Y)\)
(iii) satisfy (G) and (H), then \(T, S\) and \(P\) have a common unique fixed point on \(D_X \otimes Y\) if \(k + k' \leq \frac{1}{4}\).

**Proof.** Let \(x_0 \in D_X \otimes Y\) be fixed. We define
\[
y_n = Tx_n = PX_{n+1}, y_{n+1} = Sx_{n+1} = PX_{n+2}
\]
Now, for any \(a \otimes b \in D_X \otimes Y\),
\[
\|Tu - Sv, a \otimes b\| \leq \|T_1u - S_1v\| \|T_2u\| \|a \otimes b\| + \|S_1v\| \|T_2u - S_2v\| \|a \otimes b\|
\]
\[
\leq \frac{1}{4} \|Pu - Tv, a \otimes b\| + \|Pu - Sv, a \otimes b\| - \psi(k\|Pu - Tv, a \otimes b\|, k'\|Pu - Sv, a \otimes b\|).
Now, proceeding as in Theorem 4.3 we have \( \lim_{n \to \infty} x_n = x \)
\[ \|y_n - y_{n+1} + \alpha \otimes b\| = \|T x_n - S x_{n+1} + \alpha \otimes b\| \]
\[ \leq \frac{1}{4} \left( \|P x_n - T x_{n+1} + \alpha \otimes b\| + \|P x_n - S x_{n+1} + \alpha \otimes b\| \right) \]
\[ - \psi(k \|P x_n - T x_{n+1} + \alpha \otimes b\|, k \|P x_n - S x_{n+1} + \alpha \otimes b\|) \]
\[ - \psi(k' \|P x_n - T x_{n+1} + \alpha \otimes b\|, k' \|P x_n - S x_{n+1} + \alpha \otimes b\|) \]
\[ = \frac{1}{4} \left( \|y_n - y_{n+1} + \alpha \otimes b\| + \|y_n - y_{n+1} + \alpha \otimes b\| \right) \]
\[ - \psi(k \|y_n - y_{n+1} + \alpha \otimes b\|, k \|y_n - y_{n+1} + \alpha \otimes b\|) \]
\[ - \psi(k' \|y_n - y_{n+1} + \alpha \otimes b\|, k' \|y_n - y_{n+1} + \alpha \otimes b\|) \]
\[ = \frac{1}{2} \left( \|y_n - y_{n+1} + \alpha \otimes b\| + \|y_n - y_{n+1} + \alpha \otimes b\| \right) \]
\[ \leq \frac{1}{2} \left( \|y_n - y_{n+1} + \alpha \otimes b\| + \|y_n - y_{n+1} + \alpha \otimes b\| \right) \]

Hence \( \{\|y_n - y_{n+1} + \alpha \otimes b\|\} \) is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say \( r \). If \( r \neq 0 \), then
\[ \|y_n - y_{n+1} + \alpha \otimes b\| \leq \frac{1}{2} \left( \|y_n - y_{n+1} + \alpha \otimes b\| + \|y_n - y_{n+1} + \alpha \otimes b\| \right) \]

Taking \( n \to \infty \), we get \( \|y_n - y_{n+1} + \alpha \otimes b\| \to 2r \) and
\[ r \leq 2 \left( \psi(2kr, 2kr) + \psi(2k' r, 2k' r) \right) \] (by continuity of \( \psi \)), therefore
\[ 2 \left( \psi(2kr, 2kr) + \psi(2k' r, 2k' r) \right) \leq 0, \]
this is possible only when \( r = 0 \). So,
\[ \lim_{n \to \infty} \|y_n - y_{n+1} + \alpha \otimes b\| = 0. \] (1)

Now, proceeding as in Theorem 4.3 we have \( \lim_{n \to \infty} \|y_{n+1} - y_n + q\| = 0 \ \forall q \in \mathcal{X} \otimes \mathcal{Y} \) and \( \{y_n\} \) is a Cauchy sequence in \( \mathcal{X} \otimes \mathcal{Y} \). Let it converge to some \( z \in \mathcal{X} \otimes \mathcal{Y} \) i.e.,
\[ \lim_{n \to \infty} y_n = z \Rightarrow \lim_{n \to \infty} x_n = \lim_{n \to \infty} P x_{n+1} = z \] and
\[ \lim_{n \to \infty} S x_{n+1} = \lim_{n \to \infty} P x_{n+2} = z. \]
Since $S(X) \subseteq P(X)$ and $T(X) \subseteq P(X)$, so there exists a point $u \in DX_{x,y}$ such that $z = Pu$. Now,

$$
||Tu - z, q|| \leq ||Tu - Sx_{n+1}, q|| + ||Sx_{n+1} - z, q||
$$

$$
\leq \frac{1}{4}(||Pu - T_{x_{n+1}}, q|| + ||Pu - Sx_{n+1}, q||)
$$

$$
- \psi(k||Pu - T_{x_{n+1}}, q||, k||Pu - Sx_{n+1}, q||)
$$

$$
- \psi(k'||Pu - T_{x_{n+1}}, q||, k'||Pu - Sx_{n+1}, q||) + ||Sx_{n+1} - z, q||.
$$

Taking $n \to \infty$,

$$
||Tu - z, q|| \leq 0 \Rightarrow ||Tu - z, q|| = 0
$$

Therefore $Tu = z$. So, $Pu = Tu = z$, i.e., $u$ is a coincidence point of $P$ and $T$. Since the pair of mappings are weakly compatible, so,

$$
PTu = TPu \Rightarrow Pz = Tz.
$$

Again for $z = Pu$ we have,

$$
||z - Su, q|| = ||Tu - Su, q||
$$

$$
\leq \frac{1}{4}(||Pu - Tu, q|| + ||Pu - Su, q||)
$$

$$
- \psi(k||Pu - Tu, q||, k||Pu - Su, q||)
$$

$$
- \psi(k'||Pu - Tu, q||, k'||Pu - Su, q||)
$$

$$
= ||z - Su, q||.
$$

Thus, $||z - Su, q|| = 0$. So, $Su = z$. Thus $Pu = Su = z$, i.e., $w$ is a coincidence point of $P$ and $S$. Since the pair of mappings are weakly compatible, so,

$$
PSu = SPu \Rightarrow Pz = Sz
$$

Now, we show that $z$ is a fixed point of $T$

$$
||Tz - z, q|| = ||Tz - Su, q||
$$

$$
\leq \frac{1}{4}(||Pz - Tu, q|| + ||Pz - Su, q||)
$$

$$
- \psi(k||Pz - Tu, q||, k||Pz - Su, q||)
$$

$$
- \psi(k'||Pz - Tu, q||, k'||Pz - Su, q||)
$$

$$
= \frac{1}{4}||Tz - z, q||
$$

possible only for $||Tz - z, q|| = 0 \Rightarrow Tz = z$ therefore $Tz = Pz = z$. Now, we show that $z$ is a fixed point of $S$

$$
||z - Sz, q|| = ||Tz - Sz, q||
$$

$$
\leq \frac{1}{4}(||Pz - Tz, q|| + ||Pz - Sz, q||)
$$

$$
- \psi(k||Pz - Tz, q||, k||Pz - Sz, q||)
$$

$$
- \psi(k'||Pz - Tz, q||, k'||Pz - Sz, q||)
$$

$$
\Rightarrow ||Sz - z, q|| = 0
$$

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possible only for \( Sz = z \) therefore \( Sz = Pz = z \). Hence, \( Tz = Pz = z = Sz \). Uniqueness can be shown in a similar manner. Thus \( z \) is a common unique fixed point for the mappings \( T, S \) and \( P \).

5 Conclusion

Thus, in this paper, we have derived different fixed point theorems in 2-Banach spaces and also in the tensor product of normed spaces as 2-Banach spaces.

In the paper of Misiak [17], in 1989, the idea of \( n \)-normed spaces can be found. Some recent results and related works in \( n \)-normed spaces can be found in [13], [16]. Let \( n \in \mathbb{N} \) and \( X \) be a real vector space of dimension \( d \geq n \). A real-valued function \( \| \cdot, \ldots, \cdot \| \) on \( X^n \) satisfying the following four properties,

1. \( \| x_1, \ldots, x_n \| = 0 \) if and only if \( x_1, \ldots, x_n \) are linearly dependent;
2. \( \| x_1, \ldots, x_n \| \) is invariant under permutation;
3. \( \| x_1, \ldots, x_{n-1}, \alpha x_n \| = \| \alpha \| x_1, \ldots, x_{n-1}, x_n \| \) for any \( \alpha \in \mathbb{R} \);
4. \( \| x_1, \ldots, x_{n-1}, y + z \| \leq \| x_1, \ldots, x_{n-1}, y \| + \| x_1, \ldots, x_{n-1}, z \| \),

is called an \( n \)-norm on \( X \) and the pair \((X, \| \cdot, \ldots, \cdot \|)\) is called an \( n \)-normed space. Considering the study of fixed points, the following problem can be raised.

Can we make analogous study concerning fixed points for a mapping \( T \) in the \( n \)-normed spaces and their tensor product?

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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