A study on convergence of non-convolution type double singular integral operators

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Abstract: The aim of this paper is to investigate the pointwise convergence and the rate of convergence of the operators in the following form:

\[ L_\lambda(f;x,y) = \int_{\Omega} f(t,s)K_\lambda(t,s;x,y)\,ds\,dt, \quad (x,y) \in \Omega, \]

where \(\Omega = (A,B) \times (C,D)\) is an arbitrary closed, semi-closed or open region in \(\mathbb{R}^2\), at a \(\mu\)-generalized Lebesgue point of \(f \in L_p(\Omega)\) as \((x,y,\lambda) \to (x_0,y_0,\lambda_0)\).

Keywords: \(\mu\)-generalized Lebesgue points, pointwise convergence, rate of convergence.

1 Introduction

In [11], some pointwise approximation results for integral operators of the form:

\[ U_\lambda(f;x) = \int_{-\pi}^{\pi} f(t)K_\lambda(t-x)\,dt, \quad x \in (-\pi,\pi), \]

have been studied at a Lebesgue point of integrable functions. Then these type operators, convolution type singular integral operators depending on two parameters, were developed by Gadjiev [3], Rydzewska [7]. In [12], Taberski gave some theorems concerning the Riemann-Stieltjes, Lebesgue and Titchmarsh integrals on a given rectangle. Especially, he formulated a theorem of Faddeev’s type ([12], page 246) concerning the convergence of singular integrals of the form:

\[ U(x,y;\lambda,f) = \iint_{Q} f(t,s)\psi(x,y;\lambda)\,ds\,dt, \quad (x,y) \in Q, \]

for integrable functions \(f\), here \(Q\) denotes a given rectangle and \(\psi(x,y;\lambda)\) is the kernel satisfying suitable assumptions. In the same paper, he explored the some approximation theorems by using the following operators:

\[ V(x,y;\lambda,f) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t,s)K(t-x,s-y;\lambda)\,ds\,dt \]

In the summability theory of double Fourier series the integral operators of type (3) play an important role, where kernel \(K(t,s;\lambda)\) are bounded, measurable, even, \(2\pi\) periodic in each variable \(x, y\) separately. After the pioneering work of...
Taberski [12], in [8], Rydzewska estimated the order of convergence of double singular integrals of the form

$$V(x,y;\sigma,f) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(t,s)K(t-x,s-y;\sigma)dsdt$$

(4)

under various assumptions on \( f(s,t) \) and \( K(t,s;\sigma) \). Rydzewska considered the convergence of integral operators to a real integrable function \( f(t,s) \) at a generalized Lebesgue point. In particular, for further studies on the convergence of double singular integrals at the Lebesgue points, we address the reader to [9], [10], [14], [15] and [16]. For further reading we suggest the following papers: [1], [5], [17], [18], [19], and [21]-[24]. We present the pointwise convergence of non-convolution singular integral operators of the form

$$L_\lambda(f;x,y) = \iint f(t,s)K_\lambda(t,s;x,y)dsdt, \quad (x,y) \in \Omega,$$

(5)

where \( \Omega := (A,B) \times (C,D) \) is an arbitrary closed, semi-closed or open region in \( \mathbb{R}^2 \), at a \( \mu \)-generalized Lebesgue point of \( f \in L_p(\Omega) \) as \( (x,y,\lambda) \rightarrow (x_0,y_0,\lambda_0) \). Here \( L_p(\Omega) \) is the collection of all measurable functions \( f \) for which \( |f|^p \) is integrable on \( \Omega \).

The rest of the paper is organized as follows. Section 2 introduces the terminology used throughout this paper. Section 3 shows the existence of the operators of type (5). In Section 4, we give two theorems concerning the pointwise convergence of \( L_\lambda(f;x,y) \) on different regions. In Section 5 we estimate the rate of pointwise convergence of the operators of type (5) and give some examples.

2 Preliminaries

**Definition 1.** Let \( \varphi(x,y) \) be a function defined in the rectangle \( D \), let

$$P = \{ a = x_1 < x_2 < \ldots < x_i < \ldots < x_m < x_{m+1} = b \}$$

be a partition of \( D \) and

$$\triangle \varphi(x_i,y_j) = \varphi(x_i,y_j) - \varphi(x_{i+1},y_j) - \varphi(x_i,y_{j+1}) + \varphi(x_{i+1},y_{j+1}).$$

If \( \triangle \varphi(x_i,y_j) \geq 0 \) for any partition of \( P \), then it is said that \( \varphi(x,y) \) satisfies the condition \( \Omega \) in \( D \) [12].

In other words, if \( \triangle \varphi(x_i,y_j) \geq 0 \) for all partitions of \( D \) then it is said that \( \varphi(x,y) \) is bimonotonically increasing and if \( \triangle \varphi(x_i,y_j) \leq 0 \) for all partitions of \( D \), then it is said that \( \varphi(x,y) \) is bimonotonically decreasing [4].

The definition of the \( \mu \)-generalized Lebesgue point is the special form of the definition of the Lebesgue point in [8].

**Definition 2.** A point \( (x_0,y_0) \in D \) is called a \( \mu \)-generalized Lebesgue point of function \( f \in L_p(D) \) if

$$\lim_{(h,k) \rightarrow (0,0)} \frac{1}{\mu_1(h)\mu_2(k)} \int_0^h \int_0^k |f(t+x_0,s+y_0) - f(x_0,y_0)|^p dsdt = 0$$

where \( \mu_1(t):\mathbb{R} \rightarrow \mathbb{R} \), absolutely continuous on \( [-\delta_0,\delta_0] \), increasing on \([0,\delta_0] \) and \( \mu_1(0) = 0 \) and also \( \mu_2(s):\mathbb{R} \rightarrow \mathbb{R} \), absolutely continuous on \([-\delta_0,\delta_0] \), increasing on \([0,\delta_0] \) and \( \mu_2(0) = 0 \). Here, \( 0 < h,k < \delta_0 \).
The following definition was inspired by the definition of class A in [13].

**Definition 3.** (Class A) Let \( \Omega = \mathbb{R}^2 \times \mathbb{R}^2 \), \( \Psi = \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) be an index set and \( k_0 \in \Lambda \) be an accumulation point of \( k \). If the following conditions are satisfied, then \( K_k : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \) belongs to class A; i.e.,

(a) For fixed \( (x_0, y_0) \in \Omega \), \( K_k (x_0, y_0) \) tends to infinity as \( k \) tends to \( k_0 \) for any fixed \( (x, y) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \).

(b) \( \lim_{(x, y, \lambda) \to (x_0, y_0, k_0)} \int\int K_k (t, s; x, y) \, dsdt = 1 \).

(c) For any fixed \( (x, y) \in \Psi \) there exists a point \( (x_0, y_0) \in \Omega \) such that
\[
\lim_{\lambda \to k_0} \int\int_{\mathbb{R}^2 \times N} K_k (t, s; x, y) \, dsdt = 0, \quad \forall N \in \mathcal{N}(x_0, y_0),
\]
where \( \mathcal{N}(x_0, y_0) \) stands for the family of all neighborhoods of \( (x_0, y_0) \) in \( \mathbb{R}^2 \).

(d) For any fixed \( (x, y) \in \Psi \) there exists a point \( (x_0, y_0) \in \Omega \) such that
\[
\lim_{\lambda \to k_0} \sup_{(x, y, \lambda) \in \Omega \setminus \mathcal{N}(x_0, y_0)} K_k (t, s; x, y) = 0, \quad \forall \delta > 0
\]
where \( \mathcal{N}(x_0, y_0) = \{(x_0 - \delta, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta)\} \).

(e) For any fixed \( x \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) there exists a point \( x_0 \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) such that \( K_k (t, s; x, y) \) is monotonically increasing with respect to \( t \) on \( (x_0, x_0 - \delta) > 0 \) and monotonically decreasing on \( (x_0, x_0 + \delta) > 0 \) and similarly, for any fixed \( y_0 \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) there exists a point \( y_0 \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) such that \( K_k (t, s; x, y) \) is monotonically increasing with respect to \( s \) on \( (y_0, y_0 - \delta) > 0 \) and monotonically decreasing on \( (y_0, y_0 + \delta) > 0 \) for any \( \lambda \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \).

Throughout this paper, we suppose that the kernel \( K_k (t, s; x, y) \) belongs to class A.

### 3 Existence of operator

**Lemma 1.** Let \( \|K_k (\cdot; x, y)\|_{L_1(\mathbb{R}^2)} \leq M, \forall k \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) and \( \forall (x, y) \in \Psi \). If \( f \in L_1(\Omega) \) then \( L_k (f; x, y) \) defines a continuous transformation acting on \( L_1(\Omega) \).

**Proof.** By the linearity of the operator \( L_k (f; x, y) \), it is sufficient to show that
\[
\|L_k\|_1 = \sup_{f \neq 0} \frac{\|L_k (f; x, y)\|_{L_1(\Omega)}}{\|f\|_{L_1(\Omega)}} < \infty
\]
remains bounded. Now, using Fubini Theorem [2] we can write
\[
\|L_k (f; x, y)\|_{L_1(\Omega)} = \iint \left( \int \int \Omega f (t, s) K_k (t, s; x, y) \, dsdt \right) \, dydx
\]
\[
\leq \iint \Omega f (t, s) \left( \int \int \mathbb{R}^2 K_k (t, s; x, y) \, dydx \right) \, dsdt
\]
\[
\leq M \|f\|_{L_1(\Omega)}.
\]
Thus the proof is completed.
Lemma 2. Let $1 < p < \infty$ and $\|K_\lambda\|_{L_p(R^2 \times R^2)} \leq M$, $\forall \lambda \in \Lambda$ whenever $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p(\Omega)$ then $L_\lambda (f; x, y)$ defines a continuous transformation from $L_p(\Omega)$ to $L_q(\Omega)$.

Proof. We assume that $1 < p < \infty$. By the linearity of the operator $L_\lambda (f; x, y)$, it is sufficient to show that

$$\|L_\lambda\|_q = \sup_{f \neq 0} \frac{\|L_\lambda (f; x, y)\|_{L_p(\Omega)}}{\|f\|_{L_p(\Omega)}}$$

is bounded. Let us define a new function by

$$g(t, s) = \begin{cases} f(t, s), (t, s) \in \Omega, \\ 0, (t, s) \in R^2 \setminus \Omega. \end{cases}$$

Rearranging and rewriting the norm as follows

$$\|L_\lambda (f; x, y)\|_{L_q(\Omega)} = \|L_\lambda (g; x, y)\|_{L_q(\Omega)} = \left( \int_{R^2} \int_{R^2} g(t, s) K_\lambda (t, s; x, y) ds dt \right)^q dy dx.$$

Applying Hölder’s inequality [2] to the last equality we obtain

$$\|L_\lambda (f; x, y)\|_{L_q(\Omega)} \leq \left( \int_\Omega \left( \int_{R^2} \left( \int_{R^2} |g(t, s)|^p ds \right)^q ds \right)^{\frac{1}{q}} \left( \int_{R^2} \left( \int_{R^2} |K_\lambda (t, s; x, y)|^q ds \right)^{\frac{1}{q}} \right)^q dy dx \right)^{\frac{1}{p}}$$

$$\leq \left( \int_\Omega \left( \|f\|_{L_p(\Omega)} \int_{R^2} \left( \|K_\lambda (t, s; x, y)\|^{\frac{q}{p}} ds \right)^q \right)^{\frac{1}{q}} ds \right)^{\frac{1}{q}} dy dx \leq \|f\|_{L_p(\Omega)} \|K_\lambda\|_{L_q(R^2 \times R^2)} \leq M \|f\|_{L_p(\Omega)}.$$

Hence the proof is completed.

4 Convergence at characteristic points

We are now ready to prove our first main result.

Theorem 1. If $(x_0, y_0)$ be a generalized Lebesgue point of function $f \in L_p(\Omega)$ then

$$\lim_{(x, y, \lambda) \to (x_0, y_0, \lambda_0)} L_\lambda (f; x, y) = f(x_0, y_0)$$

on any set $Z$ on which the function

$$\int_{x_0 - \delta y_0 - \delta}^{x_0 + \delta y_0 + \delta} \int_{x_0 - \delta y_0 - \delta}^{x_0 + \delta y_0 + \delta} K_\lambda (t, s; x, y) \left( \left\{ \mu_1 (|t - x_0|) \right\}_s \right) \left( \left\{ \mu_2 (|s - y_0|) \right\}_s \right) ds dt$$

is bounded as $(x, y, \lambda) \to (x_0, y_0, \lambda_0)$.

Proof. Suppose that $\hat{N}_\delta(x_0, y_0) \subset \Omega$ and $(x_0, y_0)$ be $\mu$-generalized Lebesgue point of $f \in L_p(\Omega)$. For the case $p = 1$, the proof is quite similar to for $p > 1$, therefore we will prove the theorem for the case $1 < p < \infty$. Since $(x_0, y_0) \in \Omega$
be $\mu$–generalized Lebesgue point of $f \in L_p(\Omega)$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $h$ and $k$ satisfying $0 < h, k \leq \delta$, the following inequalities

$$\int_{x_0-\delta y_0 - \delta}^{x_0+\delta y_0 + \delta} |f(t,s) - f(x_0, y_0)|^p \, ds \, dt < \varepsilon \mu_1(h) \mu_2(k),$$

for all $s,t \in \Omega$. This gives

$$\int_{x_0-\delta y_0 - \delta}^{x_0+\delta y_0 + \delta} |f(t,s) - f(x_0, y_0)|^p \, ds \, dt < \varepsilon \mu_1(h) \mu_2(k).$$

By condition (c) of class $A$, we shall write

$$|L_\lambda(f;x,y) - f(x_0,y_0)| = \int_{\Omega} f(t,s)K_\lambda(t,s;x,y) \, ds \, dt.$$

Using Hölder’s inequality we have the following

$$I + II \leq \left( \int_{\Omega} |f(t,s) - f(x_0, y_0)|^p |K_\lambda(t,s;x,y)| \, ds \, dt \right)^\frac{1}{p} \times \left( \int_{\Omega} |K_\lambda(t,s;x,y)|^q \, ds \, dt \right)^\frac{1}{q}$$

$$+ |f(x_0, y_0)| \left( \int_{\Omega} |K_\lambda(t,s;x,y)|^q \, ds \, dt \right)^\frac{1}{q}.$$

Since whenever $m,n$ positive numbers the inequality $(m+n)^p \leq 2^p (m^p + n^p)$ holds [6], by taking the $p-th$ power of both sides we have

$$(I + II)^p \leq 2^p \int_{\Omega} |f(t,s) - f(x_0, y_0)|^p |K_\lambda(t,s;x,y)| \, ds \, dt$$

$$\times \left( \int_{\Omega} |K_\lambda(t,s;x,y)|^q \, ds \, dt \right)^\frac{p}{q} + 2^p |f(x_0, y_0)|^p \left( \int_{\Omega} |K_\lambda(t,s;x,y)|^q \, ds \, dt \right)^\frac{p}{q}$$

$$= 2^p (I_1 \times I_2 + I_3).$$
Observe that by condition (c) of class A, the integral $I^*$ tends to 1 as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$ and the integral $I_2$ tends to zero. Now we investigate the integral $I_1$.

$$I_1 = \left\{ \left\{ \int_{\Omega \setminus S_\delta(x_0, y_0)} + \int_{S_\delta(x_0, y_0)} \right\} |f(t, s) - f(x_0, y_0)|^p K_\lambda(t, s; x, y) \, ds \right\} \, dt = I_{11} + I_{12}.$$

The following inequality holds for the integral $I_{11}$ i.e.:

$$I_{11} \leq \sup_{\Omega \setminus S_\delta(x_0, y_0)} K_\lambda(t, s; x, y) \left[ \|f\|_{L_1(\Omega)}^p + |f(x_0, y_0)|^p |B - A| |D - C| \right].$$

Hence by condition (d) of class A, $I_{11} \rightarrow 0$ as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$.

Next, we can show that $I_{12}$ tends to zero as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$ on $S_\delta(x_0, y_0)$.

$$I_{12} = \int_{\Omega \setminus S_\delta(x_0, y_0)} \int_{x_0 - \delta y_0 - \delta} |f(t, s) - f(x_0, y_0)|^p K_\lambda(t, s; x, y) \, ds \, dt$$

$$= \left\{ \left\{ \int_{x_0 - \delta y_0 - \delta} |f(t, s) - f(x_0, y_0)|^p K_\lambda(t, s; x, y) \, ds \right\} + \left\{ \int_{x_0 - \delta y_0 - \delta} |f(t, s) - f(x_0, y_0)|^p K_\lambda(t, s; x, y) \, ds \right\} \right\}$$

$$= I_{121} + I_{122} + I_{123} + I_{124}.$$

Hence we can evaluate the integral $I_{121}$. From [12] (see 2.5 p.101), we can write the following:

$$I_{121} = \int_{x_0 - \delta y_0 - \delta}^{x_0} \int_{y_0} |f(t, s) - f(x_0, y_0)|^p K_\lambda(t, s; x, y) \, ds \, dt.$$
The following theorem gives a pointwise approximation of the integral operators type (3) to the function

Thus, the proof is finished.

For the integrals

where

Lebesgue point of \( f \) from \( x_0 \).

Collecting the estimates

Therefore, if the points \((x, y, \lambda)\) are sufficiently near to \((x_0, y_0, \lambda_0)\), we have

\[
I_{12} \leq \varepsilon \epsilon K
\]

where

\[
K = \sup \left\{ \int_{x_0 - \delta y_0 - \delta}^{x_0 + \delta y_0 + \delta} \int_{y_0 - \delta}^{y_0 + \delta} K_\lambda (t, s, x, y) \left| \left| \mu_1 (|t - x_0|) \right| \right|_{x} \left| \left| \mu_2 (|s - y_0|) \right| \right|_{s} dsdt : (x, y, \lambda) \in Z \right\}.
\]

Thus, the proof is finished.

The following theorem gives a pointwise approximation of the integral operators type (3) to the function \( f \) at \( \mu \)-generalized Lebesgue point of \( f \in L_1 (\mathbb{R}^2) \) whenever \( D = \mathbb{R}^2 \).
Theorem 2. Suppose that the hypothesis of Theorem 1 is satisfied for \( \Omega = \mathbb{R}^2 \). If \( (x_0,y_0) \) is a \( \mu \)-generalized Lebesgue point of function \( f \in L_1(\mathbb{R}^2) \) then

\[
\lim_{(x,y) \to (x_0,y_0,\lambda_0)} L_{\lambda}(f;x,y) = f(x_0,y_0).
\]

Proof. Using the same strategy as in Theorem 1 we obtain

\[
I_\lambda(x,y) := |L_{\lambda}(f;x,y) - f(x_0,y_0)|
\leq \iint_{\mathbb{R}^2} |f(t,s) - f(x_0,y_0)| |K_\lambda(t,s;x,y)| \ dsdt + |f(x_0,y_0)| \iint_{\mathbb{R}^2} |K_\lambda(t,s;x,y)| \ dsdt - 1
= I + II
\]

\[
(I + II)^p \leq 2^p \iint_{\mathbb{R}^2} |f(t,s) - f(x_0,y_0)|^p |K_\lambda(t,s;x,y)| \ dsdt
\times \left( \iint_{\mathbb{R}^2} |K_\lambda(t,s;x,y)| \ dsdt \right)^{\frac{p}{q}} + 2^p |f(x_0,y_0)|^p \iint_{\mathbb{R}^2} |K_\lambda(t,s;x,y)| \ dsdt - 1
= 2^p (I_1 + I_2)
\]

Observe that by condition (b) of class A, the integral \( I^* \) tends to 1 as \( (x,y,\lambda) \to (x_0,y_0,\lambda_0) \).

\[
I_1 = \int_{\mathbb{R}^2 \setminus N_\delta(x_0,y_0)} \int_{\mathbb{R}^2 \setminus N_\delta(x_0,y_0)} |f(t,s) - f(x_0,y_0)|^p |K_\lambda(t,s;x,y)| \ dsdt
= I_{11} + I_{12}.
\]

\[
I_{11} = \int_{\mathbb{R}^2 \setminus N_\delta(x_0,y_0)} \int_{\mathbb{R}^2 \setminus N_\delta(x_0,y_0)} |f(t,s) - f(x_0,y_0)|^p |K_\lambda(t,s;x,y)| \ dsdt
\leq \int_{\mathbb{R}^2 \setminus N_\delta(x_0,y_0)} 2^p \left[ |f(t,s)|^p + f(x_0,y_0)|^p \right] |K_\lambda(t,s;x,y)| \ dsdt
= \sup_{\mathbb{R}^2 \setminus N_\delta(x_0,y_0)} K_\lambda(t,s;x,y) 2^p \left[ \|f\|_{L^1(\mathbb{R}^2)}^p + |f(x_0,y_0)|^p \right] \iint_{\mathbb{R}^2 \setminus N_\delta(x_0,y_0)} |K_\lambda(t,s;x,y)| \ dsdt
\leq \sup_{\mathbb{R}^2 \setminus N_\delta(x_0,y_0)} K_\lambda(t,s;x,y) 2^p \left[ \|f\|_{L^1(\mathbb{R}^2)}^p + |f(x_0,y_0)|^p \right] \iint_{\mathbb{R}^2 \setminus N_\delta(x_0,y_0)} |K_\lambda(t,s;x,y)| \ dsdt
\]

Hence by condition (d) and (c) of class A, \( I_{11} \to 0 \) as \( \lambda \to \lambda_0 \). Using the same operations for the integral \( I_{12} \) as in Theorem 1, we have the following inequality:

\[
I_{12} = \int_{N_\delta(x_0,y_0)} \int_{x_0 + \delta_0 + \delta} |f(t,s) - f(x_0,y_0)|^p |K_\lambda(t,s;x,y)| \ dsdt
\leq \epsilon \int_{x_0 - \delta_0 - \delta} \int_{x_0 + \delta_0 + \delta} |K_\lambda(t,s;x,y)| \left[ \left[ \mu_1 \left( |t - x_0| \right) \right] \left[ \mu_2 \left( |s - y_0| \right) \right] \right] \ dsdt.
\]
Therefore, if the points \((x, y, \lambda) \in Z\) are sufficiently near to \((x_0, y_0, \lambda_0)\), we have

\[
I_{12} \leq \varepsilon K
\]

where

\[
K = \sup \left\{ \int_{x_0 - \delta_0 - \delta}^{x_0 + \delta + \delta} \int_{y_0 - \delta_0 - \delta}^{y_0 + \delta + \delta} \left| K_\lambda(t, s; x, y) \right| \left| \mu_1(|t - x_0|) \right| \left| \mu_2(|s - y_0|) \right| dtsdt : (x, y, \lambda) \in Z \right\}.
\]

Thus, the proof is finished.

5 Rate of convergence

In this section, we give a theorem concerning the rate of pointwise convergence.

**Theorem 3.** Suppose that the hypothesis of Theorem 1 is satisfied. Let

\[
\Delta(\lambda, \delta, x, y) = \int_{x_0 - \delta_0 - \delta}^{x_0 + \delta + \delta} \int_{y_0 - \delta_0 - \delta}^{y_0 + \delta + \delta} \left| K_\lambda(t, s; x, y) \right| \left| \mu_1(|t - x_0|) \right| \left| \mu_2(|s - y_0|) \right| dtsdt
\]

for \(0 < \delta < \delta_0\) and the following assumptions are satisfied:

(i) \(\Delta(\lambda, \delta, x, y) \to 0\) as \((x, y, \lambda) \to (x_0, y_0, \lambda_0)\) for some \(\delta > 0\).

(ii) For any fixed \((x, y) \in \Psi\) there exists a point \((x_0, y_0) \in \Omega\) such that

\[
\int_{\mathbb{R}^2 \setminus \mathbb{N}} K_\lambda(t, s; x, y) dtsdt = o(\Delta(\lambda, \delta, x, y)), \forall N \in \tilde{N}(x_0, y_0)
\]

as \(\lambda \to \lambda_0\).

(iii) For any fixed \((x, y) \in \Psi\) there exists a point \((x_0, y_0) \in \Omega\) such that

\[
\sup_{\mathbb{R}^2 \setminus \mathbb{N}} K_\lambda(t, s; x, y) = o(\Delta(\lambda, \delta, x, y)) \quad , \delta > 0,
\]

as \(\lambda \to \lambda_0\). Then at each \(\mu\)-generalized Lebesgue point of \(f \in L_1(\Omega)\) and we have as \((x, y, \lambda) \to (x_0, y_0, \lambda_0)\)

\[
|L_\lambda(f; x, y) - f(x_0, y_0)|^p = o(\Delta(\lambda, \delta, x, y))
\]

**Proof.** The result is clear by Theorem 1 and Theorem 2.

For the one-dimensional counterpart of the following kernel, we refer the reader to see [13].

**Example 1.** We define \(K_\lambda(t, s, x, y) = \begin{cases} \lambda^2 |xy|, & \text{if } (t, s) \in [0^+ x] \times [0^+ y], \ xy \neq 0, \\ 0, & \text{if } (t, s) \in \mathbb{R}^2 \setminus [0^+ x] \times [0^+ y], \end{cases}\) and consider the function

\[
f(t, s) = \frac{1}{(1 + r^2)(1 + s^2)}, \quad (t, s) \in \mathbb{R}^2.
\]

Now first, we compute \(L_\lambda\) and then we have the following:

\[
L_\lambda(f; x, y) = \int_{[0^+ x] \times [0^+ y]} \frac{1}{(1 + r^2)(1 + s^2)} \lambda^2 |xy| dtsdt = \frac{\lambda^2}{|xy|} \arctan \left( \frac{y}{x} \right) \arctan \left( \frac{y}{x} \right), \ xy \neq 0.
\]

We give the graph of \(f(t, s) = \frac{1}{(1 + r^2)(1 + s^2)}, \ (t, s) \in \mathbb{R}^2\) as shown in Figure 1:
Fig. 1: Graph of $f(t, s) = \frac{1}{(1+t^2)(1+s^2)}$

Now we give the two graph of $L_\lambda(f; x, y)$ as $(x, y, \lambda) \to (0, 0, \infty)$. In the first graph we choose $\lambda = 10$ in the second one $\lambda = 100$.

Graph of $L_\lambda(f; x, y) = \frac{\lambda^2}{xy} \arctan\left(\frac{x}{\lambda}\right) \arctan\left(\frac{y}{\lambda}\right), xy \neq 0$ when $\lambda = 10, x \in (0.01, 1)$ and $y \in (0.01, 1)$.

Fig. 2: Graph of $L_\lambda(f; x, y)$ when $\lambda = 10$

Graph of $L_\lambda(f; x, y) = \frac{\lambda^2}{xy} \arctan\left(\frac{x}{\lambda}\right) \arctan\left(\frac{y}{\lambda}\right), xy \neq 0$ when $\lambda = 100, x \in (0.01, 1)$ and $y \in (0.01, 1)$
Fig. 3: Graph of $L_{\lambda}(f;x,y)$ when $\lambda = 100$

Figures are generated by Wolfram Mathematica 7, we refer the reader to see [20].

References

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