Convergence of double singular integrals in weighted $L_\varphi$ spaces

Gumrah Uysal$^1$ and Ertan Ibikli$^2$

$^1$Department of Mathematics, Karabuk University, Karabuk, Turkey
$^2$Department of Mathematics, Ankara University, Ankara, Turkey

Received: 23 January 2016, Accepted: 9 March 2016
Published online: 22 June 2016.

Abstract: The paper is devoted to the study of pointwise approximation of functions $f \in L_\varphi (D)$ by double singular integral operators with radial kernels at $p-$generalized Lebesgue points. Here, $\varphi > 0$ is a weight function satisfying some sharp conditions and $L_\varphi (D)$ is the collection of all measurable and non-integrable functions for which $\left| \frac{f}{\varphi} \right|^p$ is integrable on $D$, where $D = (a, b; c, d)$ is an arbitrary bounded open, semi open or closed region or $D = \mathbb{R}^2$.

Keywords: $p-$generalized Lebesgue point, double singular integral, radial kernel, weighted pointwise approximation.

1 Introduction

In Taberski’s famous paper [18], the pointwise approximation of $2\pi-$ periodic Lebesgue integrable functions was investigated by the convolution type, linear singular integral operators of the form:

$$L_\lambda (f;x) = \int_{-\pi}^{\pi} f(t) K_\lambda (t-x) dt, \quad x \in (-\pi, \pi), \quad \lambda \in \Lambda \subset \mathbb{R}_0^+,$$

where $K_\lambda (t)$ is the kernel fulfilling appropriate conditions.

The papers [6] and [16], which are based on Taberski’s study [18], are devoted to the study of pointwise convergence of the operators of type (1) on some planar sets consist of characteristic points $(x_0, y_0)$ of various types. Besides, Bardaro [2] presented significant results about the rate of pointwise convergence of some classes of the linear singular integral operators. Distinctively, Esen [4, 5] obtained some approximation results concerning the pointwise convergence and the rate of pointwise convergence of non-convolution type linear singular integral operators at $p-$Lebesgue points. Moreover, Karsli and Ibikli [8] extended the results in the articles [18], [6] and [16] by considering the more general integral operators of the form:

$$T_\lambda (f;x) = \int_{a}^{b} f(t) K_\lambda (t-x) dt, \quad x \in (a, b), \quad \lambda \in \Lambda \subset \mathbb{R}_0^+,$$

where $f \in L_1 (a, b)$ and $(a, b)$ denotes an arbitrary interval in $\mathbb{R}$ such as $[a, b], (a, b), [a, b)$ or $(a, b]$. For some studies concerning approximation of functions by linear positive operators in several settings, the reader may see also e.g.
In [19], Taberski studied the pointwise approximation of functions $f \in L_1(Q)$ by the three parameter family of convolution type double singular integral operators of the form:

$$V_{\lambda}(f; x, y) = \int_{Q} f(t, s)K_{\lambda}(t-x, s-y)\,dsdt, \quad (x, y) \in Q,$$

where $\lambda \in \Lambda \subset \mathbb{R}_0^+$ and $Q$ denotes a given rectangle.

In [22], Yilmaz et al. investigated the pointwise convergence of double singular integral operators with radial kernels in the following setting:

$$L_{\lambda}(f; x, y) = \int_{D} f(t, s)H_{\lambda}(t-x, s-y)\,dsdt, \quad (x, y) \in D, \quad \lambda \in \Lambda,$$

where $D = (a, b; c, d)$ is an arbitrary bounded closed, semi-closed or open region or $D = \mathbb{R}^2$ and $f \in L_p(D)$ provided $1 \leq p < \infty$. Besides, the region $D$ was particularly chosen as $(-\pi, \pi; -\pi, \pi)$ in this article.

It was natural to consider pointwise approximation in weighted Lebesgue spaces, as well as Lebesgue spaces. Therefore, Alexits [1], Mamedov [11] and Esen [4] presented necessary conditions satisfied by kernel functions in order to obtain a desired convergence, separately. Also, Taberski [20] studied the weighted pointwise convergence of double singular integral operators of type (3) using two dimensional counterparts of the conditions obtained by Alexits [1]. Later on, some weighted pointwise approximation results for the operator of type (4) were obtained in [21] using two dimensional counterpart of the approximation method presented by Esen [4].

This paper may be seen as a continuation and further generalization of [21]. In this paper, our main concern is to prove that the operators of type (4) converge to the function $f \in L_{p,q}(D)$ at $p-$generalized Lebesgue point of it as $(x, y, \lambda)$ tends to $(x_0, y_0, \lambda_0)$. Here, $\varphi > 0$ is a weight function satisfying some sharp conditions and $L_{p,q}(D)$ is the collection of all measurable and non-integrable functions for which $\left|\frac{t}{\varphi}\right|^p$ is integrable on $D$ ($1 \leq p < \infty$), where $D = (a, b; c, d)$ is open, semi open or closed bounded region or $D = \mathbb{R}^2$.

The paper is organized as follows: In Section 2, we give some preliminary concepts. In Section 3, the existence of the operators of type (4) is explored. In Section 4, main results are presented. In Section 5, the rate of pointwise convergence of the operators of type (4) is established.

### 2 Preliminaries

**Definition 1.** A function $H \in L_1(\mathbb{R}^2)$ is said to be radial, if there exists a function $K : \mathbb{R}_0^+ \to \mathbb{R}$ such that $H(t,s) = K(\sqrt{t^2+s^2})$ a.e. [3].

**Definition 2.** (Class $\Lambda_{\varphi}$) Let $H_{\lambda} : \mathbb{R}^2 \times \Lambda \to \mathbb{R}$ be a radial function i.e., there exists a function $K_{\lambda} : \mathbb{R}_0^+ \times \Lambda \to \mathbb{R}$ such that the following equality holds for $(t, s) \in \mathbb{R}^2$ a.e.: $H_{\lambda}(t,s) := K_{\lambda}(\sqrt{t^2+s^2})$, where $\Lambda$ is a given set of non-negative numbers with accumulation point $\lambda_0$. In addition, let $\lambda(t,s) = \sup_{(t,s) \in D} \frac{\varphi(t,s)}{\varphi(x_0,y_0)}$ for every $(t,s) \in D$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$. $H_{\lambda}(t,s)$ belongs to class $\Lambda_{\varphi}$, if the following conditions are satisfied:

(a) $H_{\lambda}(t,s) = K_{\lambda}(\sqrt{t^2+s^2})$ is non-negative and integrable as a function of $(t,s)$ on $\mathbb{R}^2$ for each fixed $\lambda \in \Lambda$.

(b) For fixed $(x_0,y_0) \in D$, $K_{\lambda}(\sqrt{x_0^2+y_0^2})$ tends to infinity as $\lambda$ tends to $\lambda_0$. 

© 2016 BISKA Bilisim Technology
Throughout this paper we suppose that the kernel function $H_\lambda (t,s)$ belongs to class $A_\varphi$.

**Remark.** For more information about the concept of bimonotonicity, the reader may see also e.g. [7].

### 3 Existence of the operators

Main results in this work are based on the following theorem.

**Theorem 1.** Let $1 \leq p < \infty$. If $f \in L_{p,\varphi} (D)$, then $L_\lambda (f;x,y)$ defines a continuous transformation acting on $L_{p,\varphi} (D)$.

**Proof.** Let $D = \langle a,b;c,d \rangle$ is an arbitrary bounded closed, semi-closed or open region and $1 < p < \infty$. By the linearity of the operator $L_\lambda (f;x,y)$, it is sufficient to show that the following expression:

$$
\|L_\lambda\|_\varphi = \sup_{f \neq 0} \frac{\|L_\lambda (f;x,y)\|_{L_{p,\varphi}(D)}}{\|f\|_{L_{p,\varphi}(D)}}
$$

remains uniformly bounded.

We define a function such that

$$
g(t,s) = \begin{cases} 
f(t,s), & (t,s) \in D, \\
0, & (t,s) \in \mathbb{R}^2 \setminus D. \end{cases}
$$

The expression $\left( \int_D \left| \frac{f(x,y)}{\varphi(x,y)} \right|^p \, dx \right)^{\frac{1}{p}}$ defines the norm in the space $L_{p,\varphi} (D)$; see, for example, [20]. The following equality is obtained for the norm of the operator of type (4) i.e.:

$$
\|L_\lambda (f;x,y)\|_{L_{p,\varphi}(D)} = \|L_\lambda (g;x,y)\|_{L_{p,\varphi}(D)}
$$

$$
= \left( \int_D \left| \frac{1}{\varphi(x,y)} \right|^p \left| \int g(t,s)K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \, dstd \right|^p \, dx \right)^{\frac{1}{p}}.
$$

By using the generalized Minkowski inequality (see, e.g., [17]) and condition (f) of class $A_\varphi$, we obtain

$$
\|L_\lambda (f;x,y)\|_{L_{p,\varphi}(D)} \leq M \|f\|_{L_{p,\varphi}(D)}.
$$

Note that the proof of the case $p = 1$ is quite similar to the above one. In addition, one may prove the assertion for the case $D = \mathbb{R}^2$ analogous to the above proof. Thus the proof is completed.
4 Pointwise convergence

**Theorem 2.** Suppose that \( D = \langle a, b; c, d \rangle \) is an arbitrary bounded closed, semi-closed or open region. If \((x_0, y_0) \in D\) is a common \( p \)-generalized Lebesgue point of the functions \( f \in L_{p, \varphi}(D) \) and \( \varphi \in L_p(D) \), then

\[
\lim_{(x, y) \to (x_0, y_0)} L_\lambda(f; x, y) = f(x_0, y_0)
\]

under the conditions

\[
\frac{\partial K_\lambda}{\partial t}\left(\sqrt{(t-x)^2 + (s-y)^2}\right) \times \frac{\partial \varphi(t, s)}{\partial t} > 0, \text{ for each fixed } (x, y) \in D, \quad (6)
\]

\[
\frac{\partial K_\lambda}{\partial s}\left(\sqrt{(t-x)^2 + (s-y)^2}\right) \times \frac{\partial \varphi(t, s)}{\partial s} > 0, \text{ for each fixed } (x, y) \in D, \quad (7)
\]

and

\[
\frac{\partial^2 K_\lambda}{\partial t \partial s}\left(\sqrt{(t-x)^2 + (s-y)^2}\right) \times \frac{\partial^2 \varphi(t, s)}{\partial t \partial s} > 0, \text{ for each fixed } (x, y) \in D \quad (8)
\]

providing that first and second order (mixed) partial derivatives of \( K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2}\right) \) and \( \varphi(t, s) \) w.r.t. \((t, s)\) exist a.e. on \( \mathbb{R}^2 \), on any set \( Z \) on which the function

\[
\int_{x_0-\delta x_0-\delta}^{x_0+\delta y_0+\delta} \int_{y_0-\delta y_0-\delta} f(t, s) K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2}\right) d((x_0-t)^{(\alpha+1)}(s-y_0)^{(\alpha+1)}) \]

is the Lebesgue-Stieltjes measure with respect to \((x_0-t)^{(\alpha+1)}(s-y_0)^{(\alpha+1)}\), is bounded as \((x, y, \lambda)\) tends to \((x_0, y_0, \lambda_0)\).

**Proof.** Let \((x_0, y_0) \in D\) be a common \( p \)-generalized Lebesgue point of the functions \( f \in L_{p, \varphi}(D) \) and \( \varphi \in L_p(D) \). Let \(|x-x_0| < \frac{\delta}{2} \) and \(|y-y_0| < \frac{\delta}{2}\) for a given \( \delta > 0 \). The proof will be given for the case \( 0 < x_0 - x < \frac{\delta}{2} \) and \( 0 < y_0 - y < \frac{\delta}{2}\) for all \( \delta > 0 \) satisfying \( x_0 + \delta < b, x_0 - \delta > a, y_0 + \delta < d \) and \( y_0 - \delta > c \). For the remaining cases, the proof follows a similar line. Also, for the simplicity, the proof will be stated for the case \( 1 < p < \infty \). The proof of the case \( p = 1 \) may be given using similar strategy.

According to the definition of \( p \)-generalized Lebesgue point given in [22], for all given \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( h, k \) satisfying \( 0 < h, k \leq \delta \), the inequality

\[
\int_{x_0-\delta x_0-\delta}^{x_0+\delta y_0+\delta} \int_{y_0-\delta y_0-\delta} f(t, s) \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p ds dt < \epsilon^{p/(\alpha+1)}
\]

holds.
Set $I(x, y, \lambda) := |L_\lambda (f, x, y) - f(x_0, y_0)|$. We may easily write

$$I(x, y, \lambda) = \left| \int\int_D f(t, s)H_\lambda (t-x, s-y) \, ds \, dt - f(x_0, y_0) \right| = \left| \int\int_D f(t, s)K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \, ds \, dt - f(x_0, y_0) \right|$$

$$\leq \int\int_D \frac{f(t, s) - f(x_0, y_0)}{\varphi(t, s)} \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \, ds \, dt$$

$$+ \int\int_D \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \left| \int\int_D \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \, ds \, dt - \varphi(x_0, y_0) \right|.$$

Since whenever $m, n$ being positive numbers the inequality $(m+n)^p \leq 2^p (m^p + n^p)$ holds (see, e.g., [15]), we have

$$[I(x, y, \lambda)]^p \leq 2^p \left( \int\int_{D \setminus B_\delta} \left| \frac{f(t, s) - f(x_0, y_0)}{\varphi(t, s)} \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \right| \, ds \, dt \right)^p$$

$$+ 2^p \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \left| \int\int_D \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \, ds \, dt - \varphi(x_0, y_0) \right|^p$$

$$= I_1 + I_2,$$

where $B_\delta := \left\{ (t, s) : (t-x_0)^2 + (s-y_0)^2 \leq \delta^2, \ (x_0, y_0) \in D \right\}$.

From Theorem 4.1 in [22], we see that $I_2 \to 0$ as $(x, y, \lambda)$ tends to $(x_0, y_0, \lambda_0)$. Now, applying Hölder’s inequality (for Hölder’s inequality, see [15]) to the integral $I_1$, and then using the inequality given as $(m+n)^p \leq 2^p (m^p + n^p)$, we obtain

$$I_1 \leq c(x, y, \lambda) 2^{2p} \left\{ K_\lambda \left( \frac{\delta}{\sqrt{2}} \right) \sup_{D \setminus B_\delta} \varphi(t, s) \left\{ \|f\|_{L^p, \varphi(D)}^p \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right\} \, ds \, dt \right\}$$

$$+ c(x, y, \lambda) 2^p \int\int_{B_\delta} \left| \frac{f(t, s) - f(x_0, y_0)}{\varphi(t, s)} \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \right| \, ds \, dt$$

$$= c(x, y, \lambda) \left\{ 2^{2p} I_{11} + 2^p I_{21} \right\},$$

where $c(x, y, \lambda) := \left( \int\int_D \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \, ds \, dt \right)^{\frac{p}{2}}$.

From Theorem 4.1 in [22], $c(x, y, \lambda) \to [\varphi(x_0, y_0)]^\frac{p}{2}$ as $(x, y, \lambda)$ tends to $(x_0, y_0, \lambda_0)$, and by condition $(e)$ of class $A_\varphi$, $I_{11} \to 0$ as $(x, y, \lambda)$ tends to $(x_0, y_0, \lambda_0)$.

The following inequality holds for the integral $I_{21}$:

$$I_{21} \leq \left\{ \int\int_{x_0 - \delta}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0 + \delta} \int\int_{x_0 - \delta}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0 + \delta} \left| \frac{f(t, s) - f(x_0, y_0)}{\varphi(t, s)} \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \right| \, ds \, dt \right\}$$

$$+ \left\{ \int\int_{x_0 - \delta}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0 + \delta} \int\int_{x_0 - \delta}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0 + \delta} \left| \frac{f(t, s) - f(x_0, y_0)}{\varphi(t, s)} \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \right| \, ds \, dt \right\}$$

$$= I_{211} + I_{212} + I_{213} + I_{214}.$$
Since

\[ I_{21} \leq I_{211} + I_{212} + I_{213} + I_{214}, \]

it is sufficient to show that the terms on the right hand side of the last inequality tends to zero as \((x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)\) on \(Z\). Let us consider the integral \(I_{211}\).

For this aim, we define the new function as follows:

\[ F(t, s) := \int_{t}^{x_0} \int_{s}^{y_0} \left| \frac{f(u, v)}{\varphi(u, v)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p du \cdot \varphi(t, s)K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) ds. \]

From (9), for all \(t\) and \(s\) satisfying \(0 < x_0 - t \leq \delta\) and \(0 < y_0 - s \leq \delta\) we have

\[ |F(t, s)| \leq e^p (x_0 - t)^{(\alpha+1)} (y_0 - s)^{(\alpha+1)}. \tag{10} \]

Now, we can concentrate the integral \(I_{211}\). From Theorem 2.5 in [19], we have the following equality:

\[
I_{211} = \int_{x_0 - \delta y_0 - \delta}^{x_0} \int_{y_0}^{y_0} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s)K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) ds dt
\]

\[ = (LS) \int_{x_0 - \delta y_0 - \delta}^{x_0} \int_{y_0}^{y_0} \varphi(t, s)K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) dF(t, s), \]

where \((LS)\) denotes Lebesgue-Stieltjes integral.

Two-dimensional integration by parts (see Theorem 2.2, p.100 in [19]) gives us

\[
\int_{x_0 - \delta y_0 - \delta}^{x_0} \int_{y_0}^{y_0} \varphi(t, s)K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) dF(t, s)
\]

\[ = \int_{x_0 - \delta y_0 - \delta}^{x_0} \int_{y_0}^{y_0} F(t, s) d \left[ \varphi(t, s)K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \right]
\]

\[ + \int_{x_0 - \delta y_0}^{x_0} F(t, y_0 - \delta) dt \left[ \varphi(t, y_0 - \delta)K_\lambda \left( \sqrt{(t-x)^2 + (y_0 - y - \delta)^2} \right) \right]
\]

\[ + \int_{y_0 - \delta}^{y_0} F(x_0 - \delta, s) ds \left[ \varphi(x_0 - \delta, s)K_\lambda \left( \sqrt{(x_0 - x - \delta)^2 + (s-y)^2} \right) \right]
\]

\[ + F(x_0 - \delta, y_0 - \delta) \varphi(x_0 - \delta, y_0 - \delta)K_\lambda \left( \sqrt{(x_0 - x - \delta)^2 + (y_0 - y - \delta)^2} \right) . \]
From (10), we can write
\[
|I_{211}| \leq \epsilon^p \int_{x_0 - \delta y_0 - \delta}^{x_0} \int_{y_0 - \delta - \delta}^{y_0} (x_0 - t)^{(a+1)}(y_0 - s)^{(a+1)} \left| d(t, s)K_\lambda \left( \sqrt{(s - x)^2 + (t - y)^2} \right) \right|
+ \epsilon^p \delta^{(a+1)} \int_{x_0 - \delta}^{x_0} \left| d(t, y_0 - \delta)K_\lambda \left( \sqrt{(t - x)^2 + (y_0 - y)^2} \right) \right|
+ \epsilon^p \delta^{(a+1)} \int_{y_0 - \delta}^{y_0} \left| d(t, x_0 - \delta)K_\lambda \left( \sqrt{(x_0 - x - \delta)^2 + (s - y)^2} \right) \right|
+ \epsilon^p \delta^{2(a+1)} \phi(x_0 - \delta, y_0 - \delta)K_\lambda \left( \sqrt{(x_0 - \delta - x)^2 + (y_0 - \delta - y)^2} \right).
\]

From (6) – (8), we see that the monotonicity properties of
\[
K_\lambda \left( \sqrt{(t - x)^2 + (s - y)^2} \right) \quad \text{and} \quad \phi(t, s)K_\lambda \left( \sqrt{(t - x)^2 + (s - y)^2} \right)
\]
coincide for each fixed \((x, y) \in D\). Applying two-dimensional integration by parts to the last inequality, we get the following inequality:
\[
|I_{211}| \leq \epsilon^p \int_{x_0 - \delta y_0 - \delta}^{x_0} \int_{y_0 - \delta - \delta}^{y_0} \phi(t, s)K_\lambda \left( \sqrt{(t - x)^2 + (s - y)^2} \right) d(t, s) \left( (x_0 - t)^{(a+1)}(y_0 - s)^{(a+1)} \right).
\]

For the integrals \(I_{212}, I_{213}, \text{and} I_{214}\) the proof is similar to the above one. Thus we obtain following inequalities:
\[
|I_{212}| \leq -\epsilon^p \int_{x_0 - \delta y_0}^{x_0} \int_{y_0 - \delta}^{y_0} \phi(t, s)K_\lambda \left( \sqrt{(t - x)^2 + (s - y)^2} \right) d(t - x_0)^{(a+1)}(y_0 - s)^{(a+1)},
\]
\[
|I_{213}| \leq -\epsilon^p \int_{x_0 - \delta y_0}^{x_0} \int_{y_0 - \delta}^{y_0} \phi(t, s)K_\lambda \left( \sqrt{(t - x)^2 + (s - y)^2} \right) d(x_0 - t)^{(a+1)}(s - y_0)^{(a+1)},
\]
\[
|I_{214}| \leq \epsilon^p \int_{x_0}^{x_0 + \delta y_0 + \delta} \int_{y_0}^{y_0 + \delta} \phi(t, s)K_\lambda \left( \sqrt{(t - x)^2 + (s - y)^2} \right) d(t - x_0)^{(a+1)}(s - y_0)^{(a+1)}.
\]

Collecting the estimates \(I_{212}, I_{213}, \text{and} I_{214},\) we have
\[
|I_{21}| \leq \epsilon^p \int_{x_0 - \delta y_0 - \delta}^{x_0 + \delta y_0 + \delta} \int_{y_0 - \delta - \delta}^{y_0 + \delta} \phi(t, s)K_\lambda \left( \sqrt{(t - x)^2 + (s - y)^2} \right) d(t - x_0)^{(a+1)}(s - y_0)^{(a+1)}.
\]

Therefore, if the points \((x, y, \lambda)\) are sufficiently close to \((x_0, y_0, \lambda_0)\), we have
\[
I_{21} \leq \epsilon^p C,
\]
In this section, two theorems concerning the rate of pointwise convergence of the operators of type (4) will be given.

Theorem 4. Suppose that $D = \mathbb{R}^2$, and the hypotheses (6)-(8) of Theorem 2 are satisfied. If $(x_0, y_0) \in \mathbb{R}^2$ is a common $p-$generalized Lebesgue point of the functions $f \in L_{p,\mathcal{F}}(\mathbb{R}^2)$ and $\varphi \in L_p(\mathbb{R}^2)$, then

$$
\lim_{(x, y, \lambda) \to (x_0, y_0, \lambda_0)} L_\lambda (f; x, y) = f(x_0, y_0)
$$

on any set $Z$ on which the function

$$
\int_{x_0 - \delta_0}^{x_0 + \delta_0} \int_{y_0 - \delta_0}^{y_0 + \delta_0} \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \, dt \left| (x_0 - t)^{(\alpha + 1)} (s - y_0)^{(\alpha + 1)} \right| = o \left( (\Delta (x, y, \lambda, \delta))^\alpha \right)
$$

where $d \left| (x_0 - t)^{(\alpha + 1)} (s - y_0)^{(\alpha + 1)} \right|$ is the Lebesgue-Stieltjes measure with respect to $(x_0 - t)^{(\alpha + 1)} (s - y_0)^{(\alpha + 1)}$, is bounded as $(x, y, \lambda)$ tends to $(x_0, y_0, \lambda_0)$.

Proof. The proof of this theorem is quite similar to the proof of Theorem 2, and it is omitted.

5 Rate of convergence

In this section, two theorems concerning the rate of pointwise convergence of the operators of type (4) will be given.

Theorem 4. Suppose that the hypotheses of Theorem 2 are satisfied. Let

$$
\Delta (x, y, \lambda, \delta) = \int_{x_0 - \delta_0}^{x_0 + \delta_0} \int_{y_0 - \delta_0}^{y_0 + \delta_0} |x_0 - t| |s - y_0| \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \, ds \, dt,
$$

where $0 < \delta \leq \delta_0$ for a fixed (and finite!) positive number $\delta_0$, and the following conditions are satisfied:

(i) $\Delta (x, y, \lambda, \delta) \to 0$ as $(x, y, \lambda)$ tends to $(x_0, y_0, \lambda_0)$ for some $\delta > 0$.

(ii) $\forall 0 > \alpha$ and $\forall \alpha \in (0, 1)$, we have $K_\lambda (\xi) = o((\Delta (x, y, \lambda, \delta))^\alpha)$ as $(x, y, \lambda)$ tends to $(x_0, y_0, \lambda_0)$.

(iii) $\forall \alpha \in (0, 1)$, $\left| \int_D \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \, ds \, dt - \varphi(x_0, y_0) \right| = o((\Delta (x, y, \lambda, \delta))^\alpha)$ as $(x, y, \lambda)$ tends to $(x_0, y_0, \lambda_0)$.

Then, at each common $p-$generalized Lebesgue point of $f \in L_{p,\mathcal{F}}(D)$ and $\varphi \in L_p(D)$ we have

$$
|L_\lambda (f; x, y) - f(x_0, y_0)| = o \left( (\Delta (x, y, \lambda, \delta))^\alpha \right)
$$

as $(x, y, \lambda)$ tends to $(x_0, y_0, \lambda_0)$.
Proof. By the hypotheses of Theorem 2, the following inequality holds:

\[
|L_h (f;x,y) - f (x_0,y_0)|^p \leq 2^{2p}K_h \left( \frac{\delta}{\sqrt{2}} \right) \sup_{D,B_h} \varphi(t,s) \|f\|_{L_p,q(t)}^p \left( \int_D \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) dsdt \right)^{\frac{p}{q}} \\
+ 2^{2p}K_h \left( \frac{\delta}{\sqrt{2}} \right) \int_{x_0+\delta y_0-\delta}^{x_0+\delta y_0+\delta} |s - y_0| \left( \int_D \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) dsdt \right)^{\frac{p}{q}} \\
+ 2^p (\alpha + 1)^2 \epsilon^p \int_{x_0-\delta y_0-\delta}^{x_0-\delta y_0+\delta} |x_0 - t| |s - y_0|^\alpha \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) dsdt \\
\times \left( \int_D \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) dsdt \right)^{\frac{p}{q}} \\
+ 2^p \left| \int_{x_0+\delta y_0-\delta}^{x_0+\delta y_0+\delta} \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) dsdt - \varphi (x_0,y_0) \right| ^p.
\]

Set

\[
A(x,y,\lambda,\delta) := \int_{x_0-\delta y_0-\delta}^{x_0+\delta y_0+\delta} |x_0 - t| |s - y_0|^\alpha \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) dsdt.
\]

Now, we can apply the proof method used by Mamedov [12] to the remaining part. Since 0 < \alpha < 1, we have \( \frac{1}{\alpha} > 1 \), and the conjugate of \( \frac{1}{\alpha} \) is \( (1-\alpha) \). Applying H"{o}lder’s inequality to the term \( A(x,y,\lambda,\delta) \), we have

\[
A(x,y,\lambda,\delta) = \int_{x_0-\delta y_0-\delta}^{x_0+\delta y_0+\delta} |x_0 - t| |s - y_0|^\alpha \left( \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \right)^{(1-\alpha)} dsdt
\]

\[
\times \left( \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \right)^{(1-\alpha)} dsdt
\]

\[
\leq \left\{ \int_{x_0-\delta y_0-\delta}^{x_0+\delta y_0+\delta} |x_0 - t| |s - y_0|^\alpha \left( \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \right)^{(1-\alpha)} dsdt \right\} \left\{ \int_{x_0-\delta y_0-\delta}^{x_0+\delta y_0+\delta} \left( \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) \right)^{(1-\alpha)} dsdt \right\}^{\frac{1}{1-\alpha}}
\]

\[
\leq (\Delta (x,y,\lambda,\delta))^{\alpha} \times \left\{ \int_D \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) dsdt \right\}^{(1-\alpha)}
\]

Since \( \int_D \varphi (t,s) K_h \left( \sqrt{(t-x)^2 + (s-y)^2} \right) dsdt \to \varphi (x_0,y_0) \) as \( (x,y,\lambda) \) tends to \( (x_0,y_0,\lambda_0) \), the rest of the proof is clear by the conditions (i)-(iii). Thus the proof is completed.
Theorem 5. Suppose that the hypotheses of Theorem 3 are satisfied. Let

\[ \Delta(x, y, \lambda, \delta) = \int_{x_0-\delta y_0}^{x_0+\delta y_0+ \delta} - \int_{x_0-\delta y_0}^{x_0+\delta y_0+ \delta} \int_{y_0}^{y_0+\delta} |x_0-t| |s-y_0| \varphi(t, s) K_\lambda \left( \sqrt{(t-x)^2 + (s-y)^2} \right) dsdt, \]

where \( 0 < \delta \leq \delta_0 \) for a fixed (and finite!) positive number \( \delta_0 \), and the following conditions are satisfied:

(i) \( \Delta(x, y, \lambda, \delta) \to 0 \) as \( (x, y, \lambda) \) tends to \( (x_0, y_0, \lambda_0) \) for some \( \delta > 0 \).

(ii) \( \forall \xi > 0 \) and \( \forall \alpha \in (0, 1) \), we have \( K_\lambda(\xi) = o\left((\Delta(x, y, \lambda, \delta))^\alpha\right) \) as \( (x, y, \lambda) \) tends to \( (x_0, y_0, \lambda_0) \).

(iii) \( \forall \xi > 0 \) and \( \forall \alpha \in (0, 1) \), we have \( \int_{\xi \leq \sqrt{s^2 + t^2}} K_\lambda \left( \sqrt{s^2 + t^2} \right) dsdt = o\left(\left((\Delta(x, y, \lambda, \delta))^\alpha\right)\right) \) as \( (x, y, \lambda) \) tends to \( (x_0, y_0, \lambda_0) \).

(iv) \( \forall \alpha \in (0, 1) \), \( \int_{\mathbb{R}^2} \varphi(t, s) K_\lambda \left( \sqrt{s^2 + t^2} \right) dsdt = o\left(\left((\Delta(x, y, \lambda, \delta))^\alpha\right)\right) \) as \( (x, y, \lambda) \) tends to \( (x_0, y_0, \lambda_0) \).

Then, at each common \( p \)-generalized Lebesgue point of \( f \in L_{p, \phi}(\mathbb{R}^2) \) and \( \varphi \in L_p(\mathbb{R}^2) \) we have

\[ |L_\lambda(f; x, y) - f(x_0, y_0)| = o\left(\left((\Delta(x, y, \lambda, \delta))^\phi\right)\right) \]

as \( (x, y, \lambda) \) tends to \( (x_0, y_0, \lambda_0) \).

Proof. The proof of this theorem is analogous to the proof of Theorem 4, and it is omitted.

6 Conclusion

In this paper, the weighted pointwise convergence of the convolution type double singular integral operators depending on three parameters is investigated. Since the approximation results and the character of the kernel functions are related, a special class of kernel functions has been defined. Therefore, the main results are presented as Theorem 1 and Theorem 2. By using main results, the rate of pointwise convergence of the indicated type operators is computed.

Acknowledgments

The reported study was partially supported by Ankara University within the scientific research project No.15L0430010 during the preparation stage. The authors would like to thank Ankara University.

References