On some generalised $I$-convergent sequence spaces of double interval numbers

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Abstract: In this article we introduce and study some spaces of $I$-convergent sequences of double interval numbers with the help of a double sequence $F = (f_{i,j})$ of modulii and double bounded sequence $p = (p_{i,j})$ of positive real numbers. We study some topological and algebraic properties, prove the decomposition theorem and study some inclusion relations on these spaces.

Keywords: Double interval numbers, ideal, filter, double $I$-convergent sequence spaces, solid and monotone space, Banach space, modulus function.

1 Introduction

Recently, Chiao[4] introduced the sequences of interval numbers and defined usual convergence of sequences of interval numbers. Sengönül and Eryimaz[28] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete.

A set(closed interval) of real numbers $x$ such that $a \leq x \leq b$ is called an interval number.[4] A real interval can also be considered as a set. Thus, we can investigate some properties of interval numbers for instance, arithmetic properties or analysis properties. Let us denote the set of all real valued closed intervals by $\mathbb{I}$. Any element of $\mathbb{I}$ is called a closed interval and it is denoted by $\bar{A} = [x_l, x_r]$. $\mathbb{I}$ is a quasilinear space under the algebraic operations and partial order relation for $\mathbb{I}$ found in [28,31]. and any subspace of $\mathbb{I}$ is called quasilinear subspace.

The set of all interval numbers $\mathbb{I}$ is a complete metric space defined by

$$d(\bar{A}_1, \bar{A}_2) = \max |x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|.$$  \hspace{1cm} (1)

where $x_l$ and $x_r$ are the first and last point of $\bar{A}$ respectively.

Vakeel A. Khan and Mohd. Shafiq defined the transformation $f$ from $\mathbb{N}$ to $\mathbb{I}$ by $k \rightarrow f(k) = \bar{A}_k = (A_k)$. The function $f$ is called sequence of interval numbers, where $\bar{A}_k$ is the $k^{th}$ term of the sequence $(A_k)$. Let us denote the set of sequences of interval numbers with real terms by

$$\omega(\bar{A}) = \{ \bar{A}_k : A_k \in \mathbb{I} \}.$$  \hspace{1cm} (2)

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The following definitions were given by Sengönül and Eryimaz[28]. A sequence \( s^0 = (\tilde{A}_k) = ([x_{k1}, x_{k2}]) \) of interval numbers is said to be convergent to an interval number \( \tilde{A}_0 = [x_{01}, x_{02}] \) if for each \( \varepsilon > 0 \), there exists a positive integer \( n_0 \) such that \( d(\tilde{A}_k, \tilde{A}_0) < \varepsilon \), for all \( k \geq n_0 \) and we denote it as \( \lim \limits_{k \to \infty} \tilde{A}_k = \tilde{A}_0 \).

Thus, \( \lim \limits_{k \to \infty} \tilde{A}_k = \tilde{A}_0 \Leftrightarrow \lim \limits_{k \to \infty} x_{k1} = x_0 \) and \( \lim \limits_{k \to \infty} x_{k2} = x_0 \), and it is said to be Cauchy sequence of interval numbers if for each \( \varepsilon > 0 \), there exists a positive integer \( k_0 \) such that \( d(\tilde{A}_k, \tilde{A}_m) < \varepsilon \), whenever \( k, m \geq k_0 \). Ayhan Esi and B. Hazarika[1] defined a transformation \( f \) from \( \mathbb{N} \times \mathbb{N} \) to \( \mathbb{IR} \) by \( i, j \to f(i, j) = s^0, s^0 = (\tilde{A}_{i,j}) \). Then \( s^0 = (\tilde{A}_{i,j}) \) is called double sequence of interval numbers. The \( \tilde{A}_{i,j} \) is called the \( (i, j)^{th} \) term of double sequence of interval numbers \( s^0 = (\tilde{A}_{i,j}) \).

Let us denote the set of double sequence of interval numbers by

\[ z \Omega(s^0) = \{ s^0 = (\tilde{A}_{i,j}) : \tilde{A}_{i,j} \in \mathbb{IR} \} . \tag{3} \]

**Definition 1.** An interval valued double sequence \( s^0 = (\tilde{A}_{i,j}) \) is said to be convergent in the Pringsheim’s sense or \( P \)-convergent to an interval number \( \tilde{A}_0 \), if for every \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( d(\tilde{A}_{i,j}, \tilde{A}_0) < \varepsilon \), for \( i, j > N \) and we denote it by \( P - \lim \tilde{A}_{i,j} = \tilde{A}_0 \). The interval number \( \tilde{A}_0 \) is called the Pringsheim limit of \( s^0 = (\tilde{A}_{i,j}) \).

More exactly, we say that a double sequence of interval numbers \( s^0 = (\tilde{A}_{i,j}) \) converges to a finite interval number \( \tilde{A}_0 \) if \( \tilde{A}_{i,j} \) tends to \( \tilde{A}_0 \) as both \( i \) and \( j \) tend to infinity independently of each another. \( s^0 = (\tilde{A}_{i,j}) \) is said to be null if \( \tilde{A}_0 = 0 \).

**Definition 2.** An interval valued double sequence \( s^0 = (\tilde{A}_{i,j}) \) is bounded if there exists a positive number \( M \) such that \( d(\tilde{A}_{i,j}, \tilde{A}_0) \leq M \) for all \( i, j \in \mathbb{N} \).

**Definition 3.** An interval valued double sequence \( s^0 = (\tilde{A}_{i,j}) \) is said be Cauchy sequence if for each \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( d(\tilde{A}_{i,j}, \tilde{A}_{m,n}) < \varepsilon \) whenever \( i \geq m \geq N \) and \( j \geq n \geq N \).

Let \( p = (p_{i,j}) \) be a double sequence positive real numbers. If \( 0 < p_{i,j} \leq \sup \limits_{i,j} p_{i,j} = H < \infty \) and \( D = \max(1, 2H - 1) \), then for \( a_{i,j}, b_{i,j} \in \mathbb{IR} \) and for all \( i, j \in \mathbb{N} \) we have \( |a_{i,j} + b_{i,j}|^{p_{i,j}} \leq D(|a_{i,j}|^{p_{i,j}} + |b_{i,j}|^{p_{i,j}}) \).

Let us denote the space of all double convergent, double null and double bounded sequences of double interval numbers by \( z\ell\Omega(s^0) \), \( z\ell_0(s^0) \) and \( z\ell_{\infty}(s^0) \) respectively.

The spaces \( z\ell\Omega(s^0) \), \( z\ell_0(s^0) \) and \( z\ell_{\infty}(s^0) \) are complete metric spaces with the metric

\[ d(\tilde{A}_{i,j}, \tilde{B}_{i,j}) = \sup \limits_{i,j} \max \{ |x_{i,j,1} - y_{i,j,1}|, |x_{i,j,2} - y_{i,j,2}| \} \] \qquad \tag{4} \]

If we take \( \tilde{B}_{i,j} = 0 \) in (4), then the metric \( d \) reduces to

\[ d(\tilde{A}_{i,j}, 0) = \sup \limits_{i,j} \max \{ |x_{i,j,1}|, |x_{i,j,2}| \} \] \qquad \tag{5} \]

In this paper we assume that a norm \( ||\tilde{A}_{i,j}|| \) of the double sequence of interval numbers \( (\tilde{A}_{i,j}) \) is the distance from \( (\tilde{A}_{i,j}) \) to \( 0 \) and satisfies the following properties: For all \( \tilde{A}_{i,j}, \tilde{B}_{i,j} \in z\ell\Omega(s^0) \) and for all \( \alpha \in \mathbb{R} \),

\begin{align*}
(N1) \quad ||\tilde{A}_{i,j}||_{z\ell\Omega(s^0)} > 0, & \quad \forall \quad \tilde{A}_{i,j} \in z\ell\Omega(s^0) - \{0\}, \\
(N2) \quad ||\tilde{A}_{i,j}||_{z\ell\Omega(s^0)} = 0 \Leftrightarrow \tilde{A}_{i,j} = 0, \\
(N3) \quad ||\tilde{A}_{i,j} + \tilde{B}_{i,j}||_{z\ell\Omega(s^0)} \leq ||\tilde{A}_{i,j}||_{z\ell\Omega(s^0)} + ||\tilde{B}_{i,j}||_{z\ell\Omega(s^0)}
\end{align*}
The notion of $I$-convergence was initially introduced by Kostyrko, et. al.[15] as generalization of statistical convergence(See [6],[27]) which is based on the structure of the ideal $I$ of subsets of natural numbers $\mathbb{N}$. Kostyrko, et. al. gave some of basic properties of $I$-convergence and dealt with extremal $I$-limit points. Although an ideal is defined as a heredity and additive family of subsets of a non-empty arbitrary set $X$, here in our study it suffices to take $I$ as a family of subsets of $\mathbb{N}$, positive integers, i.e.$I \subset \mathbb{N}^2$, such that $A \cup B \subset I$ for each $A,B \subset I$, and each subset of an element of $I$ is an element of $I$.

A non-empty family of sets $\mathcal{F} \subset \mathbb{N}^2$ is a filter on $\mathbb{N}$ if and only if $\phi \notin \mathcal{F}, A \cap B \subset \mathcal{F}$, for each $A,B \subset \mathcal{F}$, and any superset of an element of $\mathcal{F}$ is an element of $\mathcal{F}$. An ideal $I$ is called non-trivial if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I$ is non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{\mathbb{N} - A : A \subset I\}$ is a filter in $\mathbb{N}$, called the filter associated with the ideal $I$. A non-trivial ideal $I$ is called admissible if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. A non-trivial ideal $I$ is maximal if there can not exist any non-trivial ideal $J \neq I$ containing $I$ as a subset. Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be $I$-convergent to a real number $\ell$ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \subset I$ for every $\varepsilon > 0$([15]). In this case we write $I - \lim x_k = \ell$. The notion of $I$-convergence double sequence was initially introduced by Tripathy and Tripathy(See[31]).

Let $I$ be an ideal of $\mathbb{N} \times \mathbb{N}$. Then a double sequence of interval numbers $\mathcal{A} = (\bar{A}_{i,j}) \subset 2\omega(\mathcal{A})$, $\mathcal{D}$ is said to be $I$-convergent to an interval number $\bar{A}_0$ if for every $\varepsilon > 0$,

\[
\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|\bar{A}_{i,j} - \bar{A}_0\| \geq \varepsilon\} \subset I.
\]

In this case we write $I - \lim \bar{A}_{i,j} = \bar{A}_0$. If $\bar{A}_0 = \bar{0}$. Then the sequence $\bar{A} = (\bar{A}_{i,j}) \subset 2\omega(\mathcal{A})$ is said to be $I$-null. In this case we write $I - \lim \bar{A}_{i,j} = \bar{0}$.

(ii) is said to be $I$-Cauchy, if for every $\varepsilon > 0$, there exist numbers $m = m(\varepsilon), n = n(\varepsilon)$ such that

\[
\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|\bar{A}_{i,j} - \bar{A}_{m,n}\| \geq \varepsilon\} \subset I,
\]

(iii) is said to be $I$-bounded, if there exists some $M > 0$ such that

\[
\{(i,j) \in \mathbb{N} \times \mathbb{N} : \|\bar{A}_{i,j}\| \geq M\} \subset I.
\]

**Definition 4.** A sequence space $2\lambda(\mathcal{A})$ of double sequence of interval numbers,

(i) is said be solid(normal), if $(\alpha_{i,j}\bar{A}_{i,j}) \subset 2\lambda(\mathcal{A})$, whenever $(\bar{A}_{i,j}) \subset 2\lambda(\mathcal{A})$ and for any double sequence $(\alpha_{i,j})$ of scalars with $|\alpha_{i,j}| \leq 1$, for all $(i,j) \in \mathbb{N} \times \mathbb{N}$,

(ii) is said be symmetric, if $(\bar{A}_{\pi(i,j)}) \subset 2\lambda(\mathcal{A})$, whenever $(\bar{A}_{i,j}) \subset 2\lambda(\mathcal{A})$ where $\pi$ is permutation on $\mathbb{N} \times \mathbb{N}$,

(iii) is said be sequence algebra, if $(\bar{A}_{i,j} + \bar{B}_{i,j}) = (\bar{A}_{i,j}, \bar{B}_{i,j}) \subset 2\lambda(\mathcal{A})$, whenever $(\bar{A}_{i,j}), (\bar{B}_{i,j}) \subset 2\lambda(\mathcal{A})$,

(iv) is said be convergence free, if $(\bar{B}_{i,j}) \subset 2\lambda(\mathcal{A})$ whenever $(\bar{A}_{i,j}) \subset 2\lambda(\mathcal{A})$ and $\bar{A}_{i,j} = \bar{0}$ implies $\bar{B}_{i,j} = \bar{0}$, for all $i,j$.

**Definition 5.** Let $K = \{(i_1, j_1) < (i_2, j_2) < (i_3, j_3) < \cdots \} \subset \mathbb{N} \times \mathbb{N}$. The $K$-step space of $2\lambda(\mathcal{A})$, is a sequence space

\[
2\mu_p^K(\mathcal{A}) = \{ (\bar{A}_{i,n,j}) \in 2\omega(\mathcal{A}) : (\bar{A}_{i,j}) \subset 2\lambda(\mathcal{A}) \}. 
\]
Definition 6. A canonical preimage of a double sequence of interval numbers \((\bar{A}_{i,j}) \in 2\mu_k^\lambda(\mathcal{J})\) is double sequence \((\bar{B}_{i,j}) \in 2\omega(\mathcal{J})\) defined by

\[
\bar{B}_{i,j} = \begin{cases} 
\bar{A}_{i,j}, & \text{if } (i,j) \in K, \\
0, & \text{otherwise}.
\end{cases}
\]

A canonical preimage of a step space \(2\mu_k^\lambda(\mathcal{J})\) is a set of canonical preimages of all elements in \(2\mu_k^\lambda(\mathcal{J})\). That is \(\mathcal{B}\) is the canonical preimage of \(2\mu_k^\lambda(\mathcal{J})\) if and only if \(\mathcal{B}\) is the canonical preimage of some \(\mathcal{J} \in 2\mu_k^\lambda(\mathcal{J})\).

Definition 7. A sequence space \(2\lambda(\mathcal{J})\) is said to be monotone if it contains the canonical preimage of its step space.

Definition 8. A function \(f : [0, \infty) \to [0, \infty)\) is called a modulus function if

- (i) \(f(t) = 0\) if and only if \(t = 0\),
- (ii) \(f(t + u) \leq f(t) + f(u)\) for all \(t, u \geq 0\),
- (iii) \(f\) is increasing,
- (iv) \(f\) is continuous from the right at zero.

A modulus function \(f\) is said to satisfy \(\triangle_2\)-condition for all values of \(u\) if there exists a constant \(K > 0\) such that \(f(Lu) \leq KLf(u)\) for all values of \(L > 1\). The idea of modulus function was introduced by Nakano in 1953, see[20], Nakano, 1953).

For any modulus function \(f\), we have the inequalities \(|f(x) - f(y)| \leq f(x - y)\) and \(f(nx) \leq nf(x)\), for all \(x, y \in [0, \infty)\).

Ruckle[21-23] used the idea of modulus function \(f\) to construct the sequence space

\[
X(f) = \{x = (x_k) : \sum_{k=1}^\infty f(|x_k|) < \infty\} = \{x = x_k : (f(|x_k|)) \in X\}.
\]  
(7)

After then, E. Kolk[12,13] gave an extension of \(X(f)\) by considering a sequence of moduli \(\mathcal{F} = (f_k)\) and defined the sequence space

\[
X(f) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.
\]  
(8)

Now we give an extension of \(X(f)\) by considering a double sequence of modulii \(\mathcal{F} = (f_{i,j})\) and define the sequence space

\[
2X(f) = \{x = (x_{i,j}) : (f_{i,j}(|x_{i,j}|)) \in X\}.
\]  
(9)

Mursaleen and Naman[18] introduced the notion of \(\lambda\)-convergent and \(\lambda\)-bounded sequences.

Vakeel A. Khan and Mohd. shafiq extended this concept to the sequence of interval numbers as follows: Let \(\lambda = (\lambda_k)_{k=1}^\infty\) be a strictly increasing sequence of positive real numbers tending to infinity. That is

\[
0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \lambda_k \to \infty \quad \text{as} \quad k \to \infty.
\]  
(10)

The sequence \(\mathcal{J} = (\bar{A}_k) \in \ell_\infty(\mathcal{J})\) is \(\lambda\)-convergent to an interval number \(\bar{A}_0\), called the \(\lambda\)-limit of \(\mathcal{J}\), if \(\Lambda_m(\mathcal{J}) \to \bar{A}_0\) as \(m \to \infty\), where

\[
\Lambda_m(\mathcal{J}) = \frac{1}{\lambda_m} \sum_{k=1}^m (\lambda_k - \lambda_{k-1})\bar{A}_k, \quad k \in \mathbb{N}.
\]

Any term with a negative subscript is equal to naught. For example \(\lambda_{-1} = 0\).
In particular, \(\mathcal{A} = (\hat{A}_k) \in \ell_\infty(\mathcal{A})\) is said to be \(\lambda\)-null, if \(\wedge_m(\mathcal{A}) \to 0\) as \(m \to \infty\).

The sequence \(\mathcal{A} = (\hat{A}_k) \in \ell_\infty(\mathcal{A})\) is \(\lambda\)-bounded if \(\sup_m \| \wedge_m(\mathcal{A}) \| < \infty\). It can be seen that if \(\lim_m \hat{A}_m = \hat{A}\) in the ordinary sense of convergence of interval numbers, then

\[
\lim_m \left( \frac{1}{\lambda_m} \left( \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) \| \hat{A}_k - \hat{A} \| \right) \right) = 0.
\] (11)

This implies that

\[
\lim_m \| \wedge_m(\mathcal{A}) - \hat{A} \| = \lim_m \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) (\hat{A}_k - \hat{A}) \| = 0,
\] (12)

which yields that

\[
\lim_m \wedge_m(\mathcal{A}) = \hat{A} \text{ and hence } \mathcal{A} = (\hat{A}_k) \in \ell_\infty(\mathcal{A}) \text{ is } \lambda\text{-convergent to } \hat{A}.
\]

On generalizing the above notation we introduce the concept of \(\lambda\)- convergence and \(\lambda\)-boundedness for double sequence of interval numbers.

Let \(\lambda = (\lambda_{i,j})\) be a strictly increasing double sequence of positive real numbers tending to infinity. That is,

\[
0 < \lambda_{i,j_0} < \lambda_{i,j_1} < \cdots < \lambda_{i,k,j} < \cdots \quad \lambda_{i,k,j} \to \infty \text{ as } i,k,j \to \infty.
\]

The double sequence \(\mathcal{A} = (\hat{A}_{i,j}) \in 2\ell_\infty(\mathcal{A})\) is said to be \(\lambda\)-convergent to an interval number \(\bar{A}_0\), called the \(\lambda\)-limit of \(\mathcal{A}\), if \(\wedge_{i,j}(\mathcal{A}) \to \bar{A}_0\), as \(i,j \to \infty\), where

\[
\wedge_{i,j}(\mathcal{A}) = \frac{1}{\lambda_{m,n}} \sum_{i=1}^{m} \sum_{j=1}^{n} (\lambda_{i,j} - \lambda_{i-1,j-1}) \bar{A}_{i,j}, \quad (i,j) \in \mathbb{N} \times \mathbb{N}.
\]

Here and in the sequel, we shall use \(\lambda_{-1,-1} = 0\).

In particular, \(\mathcal{A} = (\hat{A}_{i,j}) \in 2\ell_\infty(\mathcal{A})\) is said to be \(\lambda\)-null, if \(\wedge_{i,j}(\mathcal{A}) \to 0\), as \(i,j \to \infty\).

The double sequence \(\mathcal{A} = (\hat{A}_{i,j}) \in 2\ell_\infty(\mathcal{A})\) is \(\lambda\)-bounded, if \(\sup_{i,j} \| \wedge_{i,j}(\mathcal{A}) \| < \infty\). It can be seen that if \(\lim_{i,j} \hat{A}_{i,j} = \bar{A}\) in the Pringsheim’s sense of convergence of double interval numbers, then

\[
\lim_{i,j} \left( \frac{1}{\lambda_{m,n}} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} (\lambda_{i,j} - \lambda_{i-1,j-1}) \| \hat{A}_{i,j} - \bar{A} \| \right) \right) = 0
\] (13)

This implies that

\[
\lim_{i,j} \| \wedge_{i,j}(\mathcal{A}) - \bar{A} \| = \lim_{i,j} \frac{1}{\lambda_{m,n}} \sum_{i=1}^{m} \sum_{j=1}^{n} (\lambda_{i,j} - \lambda_{i-1,j-1}) (\hat{A}_{i,j} - \bar{A}) \| = 0
\] (14)

which yields that \(\lim_{i,j} \wedge_{i,j}(\mathcal{A}) = \bar{A}\) and hence \(\mathcal{A} = (\hat{A}_{i,j}) \in 2\ell_\infty(\mathcal{A})\) is \(\lambda\)-convergent to \(\bar{A}\).

Let us denote the classes of double \(I\)-convergent, double \(I\)-null, double bounded \(I\)-convergent and double bounded \(I\)-null sequences of double interval numbers by \(2\mathcal{C}(\mathcal{A})\), \(2\mathcal{C}_0(\mathcal{A})\), \(2\mathcal{A}_I(\mathcal{A})\) and \(2\mathcal{A}_0(\mathcal{A})\), respectively.

Now we give some important lemmas.
Lemma 1. Every solid space is monotone.

Lemma 2. Let \( K \in \mathcal{F}(I) \) and \( M \subseteq \mathbb{N} \). If \( M \notin I \), then \( M \cap K \notin I \) where \( \mathcal{F}(I) \subseteq 2^\mathbb{N} \) filter on \( \mathbb{N} \).

Lemma 3. If \( I \subseteq 2^\mathbb{N} \) and \( M \subseteq \mathbb{N} \). If \( M \notin I \), then \( M \cap N \notin I \).

Definition 9. [30] Let \( \mathcal{X} \) be the space of interval numbers. A function \( g : \mathcal{X} \rightarrow \mathbb{R} \) is called a paranorm on \( \mathcal{X} \), if for all \( A, B \in \mathcal{X}, (P_1) g(A) = 0, \) \( (P_2) g(A) \geq 0, \) \( (P_3) g(-A) = g(A), \) \( (P_4) g(A + B) \leq g(A) + g(B), \) \( (P_5) \) if \( \lambda_n \) is a sequence of scalars with \( \lambda_n \rightarrow \lambda \) \( (n \to \infty) \) and \( (A_n), A_0 \in \mathcal{X} \) with \( g(A_n) \rightarrow g(A_0)(n \to \infty) \) then \( g(\lambda_n A_n - \lambda A_0) \rightarrow 0 \) \( (n \to \infty). \)

In this article, we introduce and study the following classes of double sequences:

Let \( I \) be an ideal of \( \mathbb{N} \times \mathbb{N} \) and \( (p_{i,j}) \) be a double bounded sequence positive real numbers.

\[
2\mathcal{C}^I(\mathcal{A}, \land, \mathcal{F}, p) = \{ \mathcal{A} = (\bar{A}, j) \in 2\ell^\infty(\mathcal{A}) : \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \land, j(\mathcal{A}) - \bar{A} \|) \geq \varepsilon \} \in I, \text{ for some } \bar{A} \},
\]
\[
2\mathcal{C}_0^I(\mathcal{A}, \land, \mathcal{F}, p) = \{ \mathcal{A} = (\bar{A}, j) \in 2\ell^\infty(\mathcal{A}) : \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \land, j(\mathcal{A}) \|) \geq \varepsilon \} \in I \}
\]
\[
2\mathcal{L}^I(\mathcal{A}, \land, \mathcal{F}, p) = \{ \mathcal{A} = (\bar{A}, j) \in 2\ell^\infty(\mathcal{A}) : \exists K > 0 \text{ s.t. } \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \land, j(\mathcal{A}) \|) \geq K \} \in I \}
\]
\[
2\mathcal{L}_0(\mathcal{A}, \land, \mathcal{F}, p) = \{ \mathcal{A} = (\bar{A}, j) \in 2\ell^\infty(\mathcal{A}) : \sup_{i,j} f_{i,j}(\| \land, j(\mathcal{A}) \|) < \varepsilon \}
\]

We also denote
\[
2\mathcal{M}^I(\mathcal{A}, \land, \mathcal{F}, p) = 2\ell^\infty(\mathcal{A}, \land, \mathcal{F}, p) \cap 2\mathcal{C}^I(\mathcal{A}, \land, \mathcal{F}, p),
\]
and
\[
2\mathcal{M}_0^I(\mathcal{A}, \land, \mathcal{F}, p) = 2\ell^\infty(\mathcal{A}, \land, \mathcal{F}, p) \cap 2\mathcal{C}_0^I(\mathcal{A}, \land, \mathcal{F}, p),
\]
where \( \mathcal{F} = (f_{i,j}) \) is a double sequence of moduli and \( \mathcal{A} = (\bar{A}, j) \in 2\ell^\infty(\mathcal{A}) \subset 2\ell(\mathcal{A}) \) is a double bounded sequence of interval numbers. If we take \( p = (p_{i,j}) = 1 \) for all \( (i, j) \in \mathbb{N} \times \mathbb{N} \), we have

\[
2\mathcal{C}^I(\mathcal{A}, \land, \mathcal{F}) = \{ \mathcal{A} = (\bar{A}, j) \in 2\ell^\infty(\mathcal{A}) : \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \land, j(\mathcal{A}) - \bar{A} \|) \geq \varepsilon \} \in I, \text{ for some } \bar{A} \},
\]
\[
2\mathcal{C}_0^I(\mathcal{A}, \land, \mathcal{F}) = \{ \mathcal{A} = (\bar{A}, j) \in 2\ell^\infty(\mathcal{A}) : \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \land, j(\mathcal{A}) \|) \geq \varepsilon \} \in I \}
\]
\[
2\mathcal{L}^I(\mathcal{A}, \land, \mathcal{F}) = \{ \mathcal{A} = (\bar{A}, j) \in 2\ell^\infty(\mathcal{A}) : \exists K > 0 \text{ s.t. } \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \land, j(\mathcal{A}) \|) \geq K \} \in I \}
\]
\[
2\mathcal{L}_0(\mathcal{A}, \land, \mathcal{F}) = \{ \mathcal{A} = (\bar{A}, j) \in 2\ell^\infty(\mathcal{A}) : \sup_{i,j} f_{i,j}(\| \land, j(\mathcal{A}) \|) < \varepsilon \}
\]

2 Main results

Theorem 1. Let \( \mathcal{F} = (f_{i,j}) \) be a double sequence of modulus functions and \( p = (p_{i,j}) \) be the double bounded sequence of positive real numbers. Then the classes of sequences \( 2\mathcal{M}^I(\mathcal{A}, \land, \mathcal{F}, p) \) and \( 2\mathcal{M}_0^I(\mathcal{A}, \land, \mathcal{F}, p) \) are paranormed spaces, paranormed by

\[
g(\mathcal{A}) = g((\bar{A}, j)) = \sup_{i,j} f_{i,j}(\| \land, j(\mathcal{A}, j) \|) \cdot \frac{1}{p_{i,j}},
\]

where \( M = \max\{1, \sup_{i,j} p_{i,j}\}. \)

Proof. Let \( \mathcal{A} = (\bar{A}, j), \mathcal{B} = (\bar{B}, j) \in 2\mathcal{M}^I(\mathcal{A}, \land, \mathcal{F}, p). \)

(P1) It is clear that \( g(\mathcal{A}) = 0, \) if \( \bar{A} = \bar{0}. \)
(P2) It is also obvious that $g(\mathcal{A}) \geq 0$.

(P3) $g(\mathcal{A}) = g(-\mathcal{A})$ is obvious.

(P4) Since $\frac{p_i}{p_i} \leq 1$ and $M > 1$, using Minkowski’s inequality, we have

$$g(\mathcal{A} + \mathcal{B}) = g(\tilde{A}_{i,j} + \tilde{B}_{i,j}) = \sup_{i,j} f_{i,j}(\| \tilde{A}_{i,j} + \tilde{B}_{i,j} \|^{\frac{p_i}{p_i}})$$

$$= \sup_{i,j} f_{i,j}(\| \tilde{A}_{i,j} + \tilde{B}_{i,j} \|^{\frac{p_i}{p_i}})$$

$$\leq \sup_{i,j} f_{i,j}(\| \tilde{A}_{i,j} \|^{\frac{p_i}{p_i}}) + \sup_{i,j} f_{i,j}(\| \tilde{B}_{i,j} \|^{\frac{p_i}{p_i}})$$

$$= g(\mathcal{A}) + g(\mathcal{B}).$$

Thus $g(\mathcal{A} + \mathcal{B}) \leq g(\mathcal{A}) + g(\mathcal{B})$, for all $\mathcal{A}, \mathcal{B} \in \mathcal{M}^1(\mathcal{A}, \mathcal{B}, \mathcal{F}, p)$.

(P5) Let $(\tilde{A}_{i,j})$ be a double sequence of scalars with $(\tilde{A}_{i,j}) \rightarrow \lambda$ (i, j → ∞) and $(\tilde{A}_{i,j}), \tilde{A}_0 \in \mathcal{M}^1(\mathcal{A}, \mathcal{B}, \mathcal{F}, p)$ with $g(\tilde{A}_{i,j}) \rightarrow g(\tilde{A}_0), (i, j \rightarrow \infty)$. Note that $g(\lambda\mathcal{A}) \leq \max\{1, |\lambda| \} g(\mathcal{A})$. Then, since the inequality $g(\tilde{A}_{i,j}) \leq g(\tilde{A}_{i,j} - \tilde{A}_0) + g(\tilde{A}_0)$ holds by subadditivity of $g$, the sequence $(g(\tilde{A}_{i,j}))$ is bounded.

Therefore

$$|g(\lambda_{i,j}\tilde{A}_{i,j}) - g(\lambda\tilde{A}_0)| = |g(\lambda_{i,j}\tilde{A}_{i,j}) - g(\lambda\tilde{A}_{i,j}) + g(\lambda\tilde{A}_{i,j}) - g(\lambda\tilde{A}_0)|$$

$$\leq |\lambda_{i,j} - \lambda| \frac{p_i}{p_i} |g(\lambda_{i,j}\tilde{A}_{i,j})| + |\lambda| \frac{p_i}{p_i} |g(\lambda_{i,j} - \lambda\tilde{A}_0)| \rightarrow 0, \text{ as } (i, j \rightarrow \infty).$$

That is to say that scalar multiplication is continuous.

(P6) Since each $f_{i,j}, (i, j) \in \mathbb{N} \times \mathbb{N}$ is an increasing function, it is clear that $g(\mathcal{A}) \leq g(\mathcal{B})$, if $\mathcal{A} \subseteq \mathcal{B}$.

Hence $\mathcal{M}^1(\mathcal{A}, \mathcal{B}, \mathcal{F}, p)$ is a paranormed space. For $\mathcal{M}^1_0(\mathcal{A}, \mathcal{B}, \mathcal{F}, p)$ the result is similar.

**Theorem 2.** The set $\mathcal{M}^1(\mathcal{A}, \mathcal{B}, \mathcal{F}, p)$ is a closed subspace of $\ell_1(\mathcal{A}, \mathcal{B}, \mathcal{F}, p)$.

**Proof.** Let $\mathcal{A}^{(n)} = (\tilde{A}_{i,j})^{(n)}$ be a Cauchy sequence in $\mathcal{M}^1(\mathcal{A}, \mathcal{B}, \mathcal{F}, p)$ such that $\tilde{A}_{i,j}^{(n)} \rightarrow \tilde{A}_0$. We show that $\tilde{A} \in \mathcal{M}^1(\mathcal{A}, \mathcal{B}, \mathcal{F}, p)$. Since $\mathcal{A}^{(n)} = (\tilde{A}_{i,j})^{(n)} \in \mathcal{M}^1(\mathcal{A}, \mathcal{B}, \mathcal{F}, p)$. Then, there exists $\tilde{A}_n$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \tilde{A}_{i,j}^{(n)} - \tilde{A}_n \|^{\frac{p_i}{p_i}}) \geq \varepsilon \} \in I.$$
Then \(B_{\alpha}^\prime \), \(B_{\beta}^\prime \), \(B_{\gamma}^\prime \) \(\in I\). Let \(B^\prime = B_{\alpha}^\prime \cup B_{\beta}^\prime \cup B_{\gamma}^\prime \), where \(B = \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\|\bar{A}_q - \bar{A}_n\|)^{p_{ij}} < \epsilon\}\). Then \(B^\prime \in I\). We choose \((i_0, j_0) \in B^\prime\). Then for each \(n \geq i_0, q \geq j_0\), we have

\[
\{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\|\bar{A}_q - \bar{A}_n\|)^{p_{ij}} < \epsilon\} \\
\supseteq \bigl\{((i, j) \in \mathbb{N} \times \mathbb{N} f_{i,j}(\|\bar{A}_q - \bar{A}_n\|)^{p_{ij}} < \left(\frac{\epsilon}{3}\right)^M\bigr\} \\
\cap \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\|\bar{A}_q - \bar{A}_n\|)^{p_{ij}} < \left(\frac{\epsilon}{3}\right)^M\} \\
\cap \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\|\bar{A}_q - \bar{A}_n\|)^{p_{ij}} < \left(\frac{\epsilon}{3}\right)^M\}.
\]

Then, \((\bar{A}_n)\) is a Cauchy sequence of interval numbers, so there exists some interval number \(\bar{A}_0\) such that \(\bar{A}_n \rightarrow \bar{A}_0\) as \(n \rightarrow \infty\).

(2) Let \(0 < \delta < 1\) be given. Then, we show that, if \(U = \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\|\bar{A}_q - \bar{A}_0\|)^{p_{ij}} < \delta\}\), then \(U^\prime \in I\). Since \(\mathbb{A}^{(n)}(A_{ij}) \rightarrow A\), there exists \(q_0 \in \mathbb{N}\) such that

\[
P = \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\|\bar{A}_q - \bar{A}_0\|)^{p_{ij}} < \left(\frac{\delta}{3D}\right)^M\}
\]

implies \(P^\prime \in I\), where \(D = \max\{1, 2^{n-1}\}, H = \sup_{i,j} p_{ij} \geq 0\). The number \(q_0\) can be chosen that together with (23), we have

\[
Q = \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\|\bar{A}_q - \bar{A}_0\|)^{p_{ij}} < \left(\frac{\delta}{3D}\right)^M\} \text{ such that } Q^\prime \in I.
\]

Since \(\{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\|\bar{A}_q - \bar{A}_0\|)^{p_{ij}} < \left(\frac{\delta}{3D}\right)^M\}\) \(\supseteq \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\|\bar{A}_q - \bar{A}_0\|)^{p_{ij}} < \left(\frac{\delta}{3D}\right)^M\}\), we have the following result.

Theorem 3. The spaces \(Z_{\alpha}^4, Z_{\beta}^4, Z_{\gamma}^4\) and \(Z_{\alpha}^0, Z_{\beta}^0, Z_{\gamma}^0\) are nowhere dense subsets of \(Z_{\alpha}^\omega, Z_{\beta}^\omega, Z_{\gamma}^\omega\).

Theorem 4. The spaces \(Z_{\alpha}^0, Z_{\beta}^0, Z_{\gamma}^0\) and \(Z_{\alpha}^0, Z_{\beta}^0, Z_{\gamma}^0\) are both solid and monotone.

Proof. We shall prove the result for \(Z_{\alpha}^0, Z_{\beta}^0, Z_{\gamma}^0\). For \(Z_{\alpha}^0, Z_{\beta}^0, Z_{\gamma}^0\), the result follows similarly. For, let \(A^\prime = (A_{ij}) \in Z_{\alpha}^0, Z_{\beta}^0, Z_{\gamma}^0\) and \(\alpha_{ij}\) be sequence of scalars with \(|\alpha_{ij}| \leq 1\), for all \((i, j) \in \mathbb{N} \times \mathbb{N}\). Since \(|\alpha_{ij}| \leq \max\{1, |\alpha_{ij}|^G\} \leq 1\), for all \((i, j) \in \mathbb{N} \times \mathbb{N}\), we have

\[
\|\alpha_{ij}|^{p_{ij}}\| \leq \max\{1, |\alpha_{ij}|^G\} \leq 1\).
\]

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\( f_i,j(\| \alpha_i,j \land_i,j (\tilde{A}_{i,j}) \|)_{P_i,j} \leq f_i,j(\| \land_i,j (\tilde{A}_{i,j}) \|)_{P_i,j}, \text{ for all } (i,j) \in \mathbb{N} \times \mathbb{N}, \)

which further implies that

\[
\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_i,j(\| \land_i,j (\tilde{A}_{i,j}) \|)_{P_i,j} \geq \varepsilon \} \supseteq \{(i,j) \in \mathbb{N} \times \mathbb{N} : f_i,j(\| \alpha_i,j \land_i,j (\tilde{A}_{i,j}) \|)_{P_i,j} \geq \varepsilon \}.
\]

Thus, \( \alpha_i,j(\tilde{A}_{i,j}) \in 2C^I(\mathcal{A}, \land, \mathcal{F}, p) \). Therefore, the space \( 2C^I(\mathcal{A}, \land, \mathcal{F}, p) \) is solid and hence by Lemma 1.1 it is monotone.

**Theorem 5.** Let \( G = \sup_{i,j} p_{i,j} < \infty \) and \( I \) be an admissible ideal. Then the following are equivalent.

(a) \( \mathcal{A} = (\tilde{A}_{i,j}) \in 2C^I(\mathcal{A}, \land, \mathcal{F}, p) \);
(b) there exists \( \mathcal{B} = (B_{i,j}) \in 2C(\mathcal{A}, \land, \mathcal{F}, p) \) such that \( \tilde{A}_{i,j} = B_{i,j} \) for a.a. \((i,j) \) \( r.I \);
(c) there exists \( \mathcal{B} = (B_{i,j}) \in 2C(\mathcal{A}, \land, \mathcal{F}, p) \) and \( \mathcal{C} = (\tilde{C}_{i,j}) \in 2C^0(\mathcal{A}, \land, \mathcal{F}, p) \) such that

\[
\tilde{A}_{i,j} = B_{i,j} + \tilde{C}_{i,j} \text{ for all } (i,j) \in \mathbb{N} \times \mathbb{N}
\]

and

\[
\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_i,j(\| \land_i,j (\tilde{A}_{i,j}) - \tilde{A} \|)_{P_i,j} \geq \varepsilon \} \in I;
\]

(d) there exists a subset \( K = \{ (i_1,j_1) < (i_2,j_2) < \cdots \} \) of \( \mathbb{N} \times \mathbb{N} \) such that \( K \in \mathcal{F}(I) \) and \( \lim_{n \to \infty} f_{i,j}(\| \land_i,j (\tilde{A})_{i,n,j,n} \|)_{P_{i,n,j,n}} = 0 \).

**Proof.** (a) implies (b). Let \( \mathcal{A} = (\tilde{A}_{i,j}) \in 2C^I(\mathcal{A}, \land, \mathcal{F}, p) \). Then, there exists interval number \( \tilde{A} \) such that the set

\[
\{(i,j) \in \mathbb{N} \times \mathbb{N} : f_i,j(\| \land_i,j (\tilde{A}_{i,j}) - \tilde{A} \|)_{P_i,j} \geq \varepsilon \} \in I.
\]

Let \((m_t,n_t)\) be an increasing double sequence with

\[
(m_t,n_t) \in \mathbb{N} \times \mathbb{N} \text{ such that } \{(i,j) \leq (m_t,n_t) : f_i,j(\| \land_i,j (\tilde{A}_{i,j}) - \tilde{A} \|)_{P_i,j} \geq t^{-1} \} \in I.
\]

Define a sequence \( \mathcal{B} = (B_{i,j}) \) as \( B_{i,j} = \tilde{A}_{i,j} \) for all \( (i,j) \leq (m_t,n_t) \). For \((m_t,n_t) < (i,j) \leq (m_{t+1},n_{t+1}) \), \( t \in \mathbb{N} \),

\[
B_{i,j} = \begin{cases} 
\tilde{A}_{i,j}, & \text{if } f_i,j(\| \land_i,j (\tilde{A}_{i,j}) - \tilde{A} \|)_{P_i,j} < t^{-1}, \\
\tilde{A}, & \text{otherwise}
\end{cases}
\]

Then, \( \mathcal{B} = (B_{i,j}) \in 2C(\mathcal{A}, \land, \mathcal{F}, p) \) and from the inclusion

\[
\{(i,j) \leq (m_t,n_t) : \tilde{A}_{i,j} \neq B_{i,j} \} \subseteq \{(i,j) \leq (m_t,n_t) : f_i,j(\| \land_i,j (\tilde{A}_{i,j}) - \tilde{A} \|)_{P_i,j} \geq \varepsilon \} \in I.
\]

We get \( \tilde{A}_{i,j} = B_{i,j} \) for a.a. \((i,j) \) \( r.I \).

(b) implies (c). For \( \mathcal{A} = (\tilde{A}_{i,j}) \in 2C^I(\mathcal{A}, \land, \mathcal{F}, p) \), there exists \( \mathcal{B} = (B_{i,j}) \in 2C(\mathcal{A}, \land, \mathcal{F}, p) \) such that \( \tilde{A}_{i,j} = B_{i,j} \), for a.a.(i,j) \( r.I \). Let \( K = \{ (i,j) \in \mathbb{N} \times \mathbb{N} : \tilde{A}_{i,j} \neq B_{i,j} \} \) then \( K \in \mathcal{F}(I) \) Define \( \mathcal{C} = (\tilde{C}_{i,j}) \) as follows:

\[
\tilde{C}_{i,j} = \begin{cases} 
\tilde{A}_{i,j} - B_{i,j}, & \text{if } (i,j) \in K, \\
0, & \text{if } (i,j) \notin K
\end{cases}
\]

Then, \( \mathcal{C} = (\tilde{C}_{i,j}) \in 2C^0(\mathcal{A}, \land, \mathcal{F}, p) \) and \( \mathcal{B} = (B_{i,j}) \in 2C(\mathcal{A}, \land, \mathcal{F}, p) \).

(c) implies (d). Suppose (c) holds. Let \( \varepsilon > 0 \) be given. Let

\[
P_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : f_i,j(\| \land_i,j (\tilde{C}_{i,j}) \|)_{P_i,j} \geq \varepsilon \} \in I
\]
and
\[ K = \mathcal{P}_I = \{(i_1, j_1) < (i_2, j_2) < (i_3, j_3) < \cdots \} \in \mathcal{F}(I). \]

Then, we have \( \lim_{n \to \infty} f_{i,j}(\| \land_{ij}(\bar{A}_{\epsilon, jn}) - \bar{A} \|)^{P_{in, jn}} = 0. \)

(d) implies (a). Let \( K = \{(i_1, j_1) < (i_2, j_2) < (i_3, j_3) < \cdots \} \in \mathcal{F}(I) \) and
\[ \lim_{n \to \infty} f_{i,j}(\| \land_{ij}(\bar{A}_{\epsilon, jn}) - \bar{A} \|)^{P_{in, jn}} = 0. \]

Then for any \( \epsilon > 0 \), and Lemma 1.2, we have
\[ \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \land_{ij}(\bar{A}_{i,j}) - \bar{A} \|)^{P_{ij}} \geq \epsilon \} \subseteq K^c \cup \{(i, j) \in K : f_{i,j}(\| \land_{ij}(\bar{A}_{i,j}) - \bar{A} \|)^{P_{ij}} \geq \epsilon \}. \]

Thus, \( \bar{\omega} = (\bar{A}_{i,j}) \in 2C^I(\mathcal{O}, \land, \mathcal{F}, p). \)

**Theorem 6.** Let \( \mathcal{F} = (f_{i,j}) \) and \( \mathcal{G} = (g_{i,j}) \) be two sequences of modulus functions and for each \( (i, j) \in \mathbb{N} \times \mathbb{N}, (f_{i,j}) \) and \( (g_{i,j}) \) satisfying \( \Delta_2 \)-condition and \( p = (p_{i,j}) \in 2\mathbb{F}_0 \) be a bounded sequence of positive real numbers. Then

(a) \( 2\mathcal{X}(\bar{\omega}, \land, \mathcal{F}, p) \subseteq 2\mathcal{X}(\mathcal{G}, \land, \mathcal{F} \circ \mathcal{G}, p) \),

(b) \( 2\mathcal{X}(\bar{\omega}, \land, \mathcal{F}, p) \cap 2\mathcal{X}(\mathcal{G}, \land, \mathcal{F} \circ \mathcal{G}, p) \subseteq 2\mathcal{X}(\mathcal{G}, \land, \mathcal{F} + \mathcal{G}, p) \) for \( 2\mathcal{X} = 2C^I, 2C^I_0, 2\mathcal{M}_I^1 \) and \( 2\mathcal{M}_{\epsilon_0}^I \).

**Proof.** (a) Let \( \bar{\omega} = (\bar{A}_{i,j}) \in 2C^I_0(\mathcal{O}, \land, \mathcal{G}, p) \) be any arbitrary element. Then, the set
\[ \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\| \land_{ij}(\bar{A}_{i,j}) \|)^{P_{ij}} \geq \epsilon \} \in I. \]

Let \( \epsilon > 0 \) and choose \( \delta > 0 \) such that \( f_{i,j}(t) < \epsilon, 0 \leq t < \delta \). Let us denote
\[ \bar{B}_{i,j} = g_{i,j}(\| \land_{ij}(\bar{A}_{i,j}) \|)^{P_{ij}} \]

and consider
\[ \lim_{i,j} f_{i,j}(\bar{B}_{i,j}) = \lim_{\bar{B}_{i,j} \leq \delta (i,j) \in \mathbb{N} \times \mathbb{N}} f_{i,j}(\bar{B}_{i,j}) + \lim_{\bar{B}_{i,j} > \delta (i,j) \in \mathbb{N} \times \mathbb{N}} f_{i,j}(\bar{B}_{i,j}). \]

Now, since \( f_{i,j} \) for each \( (i, j) \in \mathbb{N} \times \mathbb{N} \) is modulus function, we have
\[ \lim_{\bar{B}_{i,j} \leq \delta (i,j) \in \mathbb{N} \times \mathbb{N}} f_{i,j}(\bar{B}_{i,j}) \leq f_{i,j}(2) \lim_{\bar{B}_{i,j} \leq \delta (i,j) \in \mathbb{N} \times \mathbb{N}} (\bar{B}_{i,j}). \]

For \( \bar{B}_{i,j} > \delta \), we have \( \bar{B}_{i,j} < \frac{\bar{B}_{i,j}}{\delta} < 1 + \frac{\bar{B}_{i,j}}{\delta} \). Now, since each \( f_{i,j} \) is non-decreasing and modulus, it follows that
\[ f_{i,j}(\bar{B}_{i,j}) < f_{i,j}(1 + \frac{\bar{B}_{i,j}}{\delta}) < \frac{1}{2} f_{i,j}(2) + \frac{1}{2} f_{i,j}(\frac{2\bar{B}_{i,j}}{\delta}). \]

Again, since each \( f_{i,j}, (i, j) \in \mathbb{N} \times \mathbb{N} \) satisfies \( \Delta_2 \)-condition, we have
\[ f_{i,j}(\bar{B}_{i,j}) < \frac{1}{2} f_{i,j}(2) + \frac{1}{2} f_{i,j}(\frac{2\bar{B}_{i,j}}{\delta}). \]

Thus, \( f_{i,j}(\bar{B}_{i,j}) < \frac{K(\bar{B}_{i,j})}{\delta} f_{i,j}(2). \) Hence
\[ \lim_{\bar{B}_{i,j} > \delta (i,j) \in \mathbb{N} \times \mathbb{N}} f_{i,j}(\bar{B}_{i,j}) \leq \max\{1, (K \delta^{-1} f_{i,j}(2))^H\} \lim_{\bar{B}_{i,j} \geq \delta (i,j) \in \mathbb{N} \times \mathbb{N}} (\bar{B}_{i,j}), \quad H = \max\{1, \sup_{i,j} p_{i,j}\}. \]
Therefore, from (26), (27) and (28), we have $\mathcal{A} = (\bar{A}_{i,j}) \in 2C^0_0(\mathcal{A}, \land, \mathcal{F} \circ \mathcal{G}, p)$. Thus,

$$2C^0_0(\mathcal{A}, \land, \mathcal{F} \circ \mathcal{G}, p) \subseteq 2C^0_0(\mathcal{A}, \land, \mathcal{F}, p).$$

Hence,

$$2\chi(\mathcal{A}, \land, \mathcal{F} \circ \mathcal{G}, p) \subseteq 2\chi(\mathcal{A}, \land, \mathcal{F}, p), \text{ for } 2\chi = 2C^0_0.$$

For $2\chi = 2C^l_0$, $2\mathcal{M}^l_0$ and $2\mathcal{M}^l_{c_0}$ the inclusions can be established similarly.

(b) Let $\mathcal{A} = (\bar{A}_{i,j}) \in 2C^l_0(\mathcal{A}, \land, \mathcal{F}, p) \cap 2C^0_0(\mathcal{A}, \land, \mathcal{G}, p)$. Let $\varepsilon > 0$ be given. Then, the sets

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\|\land_{i,j}(\bar{A}_{i,j})\|^{p_{i,j}}) \geq \varepsilon\} \in I \quad (29)$$

and

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : g_{i,j}(\|\land_{i,j}(\bar{A}_{i,j})\|^{p_{i,j}}) \geq \varepsilon\} \in I \quad (30)$$

Therefore, from (29) and (30), we have

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : \mathcal{F} + \mathcal{G}(\|\land_{i,j}(\bar{A}_{i,j})\|^{p_{i,j}}) \geq \varepsilon\} \in I.$$

Thus, $\mathcal{A} = (\bar{A}_{i,j}) \in 2C^0_0(\mathcal{A}, \land, \mathcal{F} + \mathcal{G}, p)$. Hence,

$$2C^0_0(\mathcal{A}, \land, \mathcal{F} + \mathcal{G}, p) \cap 2C^l_0(\mathcal{A}, \land, \mathcal{G}, p) \subseteq 2C^l_0(\mathcal{A}, \land, \mathcal{F}, p).$$

For $2\chi = 2C^l_0$, $2\mathcal{M}^l_0$ and $2\mathcal{M}^l_{c_0}$ the inclusions are similar. For $g_{i,j}(x) = x$ and $f_{i,j}(x) = f(x), \forall x \in [0, \infty)$, we have the following corollary.

**Corollary 1.** $2\chi(\mathcal{A}, \land, \mathcal{F}, p) \subseteq 2\chi(\mathcal{A}, \land, \mathcal{F} + \mathcal{G}, p), \text{ for } 2\chi = 2C^l_0, 2\mathcal{M}^l_0 \text{ and } 2\mathcal{M}^l_{c_0}.$

**Theorem 7.** Let $\mathcal{F} = (f_{i,j})$ be a double sequence of modulus function. Then the inclusions

$$2C^0_0(\mathcal{A}, \land, \mathcal{F}, p) \subset 2C^l(\mathcal{A}, \land, \mathcal{F}, p) \subset 2\ell_{\infty}(\mathcal{A}, \land, \mathcal{F}, p)$$

hold.

**Proof.** Let $\mathcal{A} = (\bar{A}_{i,j}) \in 2C^l(\mathcal{A}, \land, \mathcal{F}, p)$ be an arbitrary element. Then there exists some double interval number $\bar{A}$ such that the set

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{i,j}(\|\land_{i,j}(\bar{A}_{i,j}) - \bar{A}\|^{p_{i,j}}) \geq \varepsilon\} \in I. \text{ Since each } f_{i,j}$$

is modulus, we have

$$f_{i,j}(\|\land_{i,j}(\bar{A}_{i,j})\|^{p_{i,j}}) = f_{i,j}(\|\land_{i,j}(\bar{A}_{i,j}) - \bar{A}\|^{p_{i,j}}) \leq f_{i,j}(\|\land_{i,j}(\bar{A}_{i,j}) - \bar{A}\|^{p_{i,j}} + f_{i,j}(\|\bar{A}\|)^{p_{i,j}}).$$

Taking the supremum over $(i, j)$ on both sides, we get

$$\mathcal{A} = (\bar{A}_{i,j}) \in 2\ell_{\infty}(\mathcal{A}, \land, \mathcal{F}, p).$$
The inclusion
\[
Z_{C^0}(A, \wedge, F, p) \subset Z_{C^1}(A, \wedge, F, p)
\]
is obvious. Hence
\[
Z_{C^0}(A, \wedge, F, p) \subset Z_{C^1}(A, \wedge, F, p) \subset Z_{C^2}(A, \wedge, F, p).
\]

**Theorem 8.** The spaces \(Z_{C^0}(A, \wedge, F, p)\) and \(Z_{C^1}(A, \wedge, F, p)\) are sequence algebra.

**Proof.** Let \(A = (A_{ij}), A = (B_{ij}) \in Z_{C^0}(A, \wedge, F, p)\), then the sets
\[
\{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{ij}([\wedge_{ij} (A_{ij})])^{p_{ij}} \geq \varepsilon \} \in I
\]
and
\[
\{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{ij}([\wedge_{ij} (B_{ij})])^{p_{ij}} \geq \varepsilon \} \in I
\]
Therefore, from (31) and (32), we have
\[
\{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{ij}([\wedge_{ij} (A_{ij}B_{ij})])^{p_{ij}} \geq \varepsilon \} \in I.
\]
Thus, \(A, A \in Z_{C^0}(A, \wedge, F, p)\). Hence \(Z_{C^0}(A, \wedge, F, p)\) is a sequence algebra. Similarly, we can prove that \(Z_{C^1}(A, \wedge, F, p)\) is a sequence algebra.

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**References**


