Generalized intuitionistic fuzzy ideals of hemirings

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Abstract: In this paper we generalize the concept of quasi-coincident of an intuitionistic fuzzy point with an intuitionistic fuzzy set and define $(\varepsilon, \in \vee q_k)$-intuitionistic fuzzy ideals of hemirings and characterize different classes of hemirings by the properties of these ideals.

Keywords: Intuitionistic fuzzy sub-hemiring, intuitionistic fuzzy ideal, fully idemotent hemiring, regular hemiring.

1 introduction

Dedekind introduced the modern definition of the ideal of a ring in 1894 and observed that the family $Id(R)$ of all the ideals of a ring $R$ obeyed most of the rules that the ring $(R, +, \cdot)$ did, but $(Id(R), +, \cdot)$ was not a ring. In 1934, Vandiver [25] studied an algebraic system, which consists of a non-empty set $S$ with two binary operations ”$+$” and ”$.$” such that $S$ was semigroup under both the operations and $(S, +, \cdot)$ satisfies both the distributive laws but did no satisfy the cancellation law of addition. Vandiver named this system a ‘semiring’. Semirings are common generalization of rings and distributive lattices. A hemiring is a semiring in which ”$+$” is commutative and it has an absorbing element. Semirings (hemirings) appear in a natural manner in some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics (see for example [9,10,11,12,18,19] ).

Zadeh introduced the concept of fuzzy set in his definitive paper [26] of 1965. Many authors used this concept to generalize basic notions of algebra. In 1971, Rosen feld [22] laid the foundations of fuzzy algebra. He introduced the notions of fuzzy subgroup of a group. Ahsan et al. [3] initiated the study of fuzzy semirings. Murali [20] defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on fuzzy subset and Pu and Liu introduced the concept of quasicoincident of a fuzzy point with a fuzzy set in [21]. Bhakat and Das [5] used these ideas and defined $(\varepsilon, \in \vee q)$-fuzzy subgroup of a group which is a generalization of Rosenfeld’s fuzzy subgroup. Many researchers used these ideas to define $(\alpha, \beta)$-fuzzy substructures of algebraic structures (see [8,15,16,23] ).

Generalizing the concept of the quasi-coincident of a fuzzy point with a fuzzy subset, Jun [13] defined $(\varepsilon, \in \vee q_k)$-fuzzy subalgebra in BCK/BCI-algebras. In [24] Shabir et al. characterized semigroups by the properties of $(\varepsilon, \in \vee q_k)$-fuzzy ideals, quasi-ideal and bi-ideals. Jun et al. in [15] defined $(\varepsilon, \in \vee q_k)$-fuzzy ideals of hemirings. Asghar et al. [17], defined $(\varepsilon, \in \vee q_k)$-fuzzy bi-ideals in ordered semigroups.

On the other hand Atanassov [4] introduced the notion of intuitionistic fuzzy set which is a generalization of fuzzy set. Intuitionistic fuzzy hemirings are studied by Dudek in [7]. Coker and Demirici [6] introduced the notion of fuzzy point. In [14], Jun introduced the notion of $(\varphi, \psi)$-intuitionistic fuzzy subgroup of a intuitionistic group where

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\( \phi, \psi \in \{ \in, \cap, \infty \} \) and \( \phi \neq \in \cap \).

Generalizing the concept of quasi-coincident of an intuitionistic fuzzy point with an intuitionistic fuzzy set we define \( (\in, \in \cap \mathbb{Q}) \)-intuitionistic fuzzy ideals of hemirings and characterize different classes of hemirings by the properties of these ideals.

2 Preliminaries

A semiring is a set \( R \) together with two binary operations addition "+" and multiplication "." such that \((R, +)\) and \((R, \cdot)\) are semigroups, where both algebraic structures are connected by the ring like distributive laws:

\[
(a + b) \cdot c = a \cdot c + b \cdot c \quad \text{and} \quad a \cdot (b + c) = a \cdot b + a \cdot c
\]

for all \( a, b \) and \( c \in R \). An element \( 0 \in R \) is called a zero element of \( R \) if \( a + 0 = 0 + a = a \) and \( 0 \cdot a = a \cdot 0 = 0 \) for all \( a \in R \). A hemiring is a semiring with zero element, in which "+" is commutative. A hemiring \((R, +, \cdot)\) is called commutative if multiplication is commutative, that is \( ab = ba \) for all \( a, b \in R \). An element \( 1 \in R \) is called an identity element of \( R \) if \( a \cdot 1 = 1 \cdot a = a \) for all \( a \in R \). A non-empty subset \( I \) of a hemiring \( R \) is called a left (right) ideal of \( R \) if \( I \) is closed under addition and \( RI \subseteq I \) (\( IR \subseteq I \)). \( I \) is called a two-sided ideal or simply an ideal of \( R \) if \( I \) is both a left ideal and a right ideal of \( R \). A hemiring \( R \) is called regular if for each \( x \in R \) there exists \( a \in R \) such that \( x = axa \).

**Theorem 1.** [1] A hemiring \( R \) is regular if and only if \( A \cap B = AB \) for all right ideals \( A \) and left ideals \( B \) of \( R \). Generalizing the concept of regular hemirings, in [2] right weakly regular hemirings are defined as: A hemiring \( R \) is right weakly regular if for each \( x \in R \), we have \( x = xax \). If \( R \) is commutative then the concepts of regular and right weakly regular coincide. It is proved in [2].

**Theorem 2.** [2] The following conditions are equivalent for a hemiring \( R \) with 1.

1. \( R \) is right weakly regular.
2. \( A \cap B = AB \) for all right ideals \( A \) and two-sided ideals \( B \) of \( R \).
3. \( A^2 = A \) for every right ideal \( A \) of \( R \).
   - If \( R \) is commutative, then the above conditions are equivalent to
4. \( R \) is regular.

Let \( X \) be a non-empty fixed set. An intuitionistic fuzzy subset \( A \) of \( X \) is an object having the form

\[
A = \{ (x, \mu_A(x), \lambda_A(x) : x \in X) \}
\]

where the functions \( \mu_A : X \rightarrow [0, 1] \) and \( \lambda_A : X \rightarrow [0, 1] \) denote the degree of membership (namely \( \mu_A(x) \)) and the degree of nonmembership (namely \( \lambda_A(x) \)) of each element of \( x \in X \) to \( A \), respectively, and \( 0 \leq \mu_A(x) + \lambda_A(x) \leq 1 \) for all \( x \in X \). For the sake of simplicity, we use the symbol \( A = (\mu_A, \lambda_A) \) for the intuitionistic fuzzy subset (briefly, IFS) \( A = \{ (x, \mu_A(x), \lambda_A(x) : x \in X) \} \). If \( A = (\mu_A, \lambda_A) \) and \( B = (\mu_B, \lambda_B) \) are intuitionistic fuzzy subsets of \( X \), then

1. \( A \subseteq B \iff \mu_A(x) \leq \mu_B(x) \) and \( \lambda_A(x) \geq \lambda_B(x) \) \( \forall x \in X \)
2. \( A = B \iff A \subseteq B \text{ and } B \subseteq A \).
3. \( A = (\lambda_A, \mu_A) \). More generally if \( \{ A_i : i \in I \} \) is a family of intuitionistic fuzzy subset of \( X \), then by the union and intersection of this family we mean an intuitionistic fuzzy subsets
4. \( \bigcup_{i \in I} A_i = \left( \bigvee_{i \in I} \mu_{A_i}, \bigwedge_{i \in I} \lambda_{A_i} \right) \).
Let $a$ be a point in a non-empty set $X$. If $\alpha \in (0, 1]$ and $\beta \in [0, 1)$ are two real numbers such that $0 \leq \alpha + \beta \leq 1$ then $a(\alpha, \beta) = \langle x, a_\alpha, 1 - a_{1 - \beta} \rangle$ is called an intuitionistic fuzzy point(IFP) in $X$, where $\alpha$ and $\beta$ is the degree of membership and nonmembership of $a(\alpha, \beta)$ respectively and $a \in X$ is the support of $a(\alpha, \beta)$.

Let $a(\alpha, \beta)$ be an IFP in $X$, and $A = (\mu_A, \lambda_A)$ be an IFS in $X$. Then $a(\alpha, \beta)$ is said to belong to $A$, written $a(\alpha, \beta) \in A$, if $\mu_A(a) \geq \alpha$ and $\lambda_A(a) \leq \beta$ and quasi-coincident with $A$, written $a(\alpha, \beta)qA$, if $\mu_A(a) + \alpha > 1$, and $\lambda_A + \beta < 1$. $a(\alpha, \beta) \in qA$, means that $a(\alpha, \beta) \in A$ or $a(\alpha, \beta)qA$ and $a(\alpha, \beta) \in \wedge qA$, means that $a(\alpha, \beta) \in A$ and $a(\alpha, \beta)qA$ and $a(\alpha, \beta) \in qA$, means that $a(\alpha, \beta) \in \vee qA$ doesn’t hold.

Let $a(\alpha, \beta)$ be an IFP in $X$, and $A = (\mu_A, \lambda_A)$ be an IFS in $R$, Then for all $x, y \in R$ and $t \in (0, 1], s \in [0, 1)$, we define the following:

(i) $x(t, s)qA$ if $\mu_A(x) + t + k > 1$ and $\lambda_A(x) + s + k < 1$.
(ii) $x(t, s) \in \vee qA$ if $x(t, s) \in A$ or $x(t, s)qA$.
(iii) $x(t, s) \in \wedge qA$ if $x(t, s) \in A$ and $x(t, s)qA$.
(iv) $x(t, s) \in \vee qA$ means that $x(t, s) \in \vee qA$ doesn’t hold, where $k \in [0, 1)$.

3 $(\alpha, \beta)$-intuitionistic fuzzy ideals

Throughout the remaining paper $k \in [0, 1)$, $\alpha$ any one of $\in, q_k$, $\in \vee q_k$ and $\beta$ any one of $\in, q_k$, $\in \vee q_k$ unless otherwise specified.

Definition 1. An IFS $A = (\mu_A, \lambda_A)$ of a hemiring $R$ is called an $(\alpha, \beta)$-intuitionistic fuzzy sub-hemiring of $R$, if $\forall x, y \in R$ and $t_1, t_2 \in (0, 1], s_1, s_2 \in [0, 1)$,

1. $x(t_1, s_1), y(t_2, s_2)A \Rightarrow (x + y)(\min(t_1, t_2), \max(s_1, s_2))A$,
2. $x(t_1, s_1), y(t_2, s_2)A \Rightarrow (xy)(\min(t_1, t_2), \max(s_1, s_2))A$.

Definition 2. An IFS $A = (\mu_A, \lambda_A)$ of a hemiring $R$ is called an $(\alpha, \beta)$-intuitionistic fuzzy left (right) ideal of $R$, if $\forall x, y \in R$ and $t_1, t_2 \in (0, 1], s_1, s_2 \in [0, 1)$,

1. $x(t_1, s_1), y(t_2, s_2)A \Rightarrow (x + y)(\min(t_1, t_2), \max(s_1, s_2))A$,
2. $y(t_1, s_1)A, x \in R \Rightarrow (xy)(t_1, s_1)A \Rightarrow (xy)(t_1, s_1)A$.

An IFS $A = (\mu_A, \lambda_A)$ of a hemiring $R$ is called an $(\alpha, \beta)$-intuitionistic fuzzy ideal of $R$, if it is both $(\alpha, \beta)$-intuitionistic fuzzy left ideal and $(\alpha, \beta)$-intuitionistic fuzzy right ideal of $R$.

Theorem 3. Let $A = (\mu_A, \lambda_A)$ be an $(\alpha, \beta)$-intuitionistic fuzzy ideal of $R$. Then the set

$$R_{(0,1)} = \{x \in R : \mu_A(x) > 0 \text{ and } \lambda_A(x) < 1\} \neq \emptyset$$

is an ideal of $R$. 

Proof. Let \( x, y \in R_{(0,1)} \). Then \( \mu_A(x) > 0 \) and \( \lambda_A(x) < 1 \), \( \mu_A(y) > 0 \) and \( \lambda_A(y) < 1 \). Assume that \( \mu_A(x + y) = 0 \) or \( \lambda_A(x + y) = 1 \). If \( \alpha \in \{ e, \in \ldots \} \), then, \( x(\mu_A(x), \lambda_A(x)) \alpha A \) and \( y(\mu_A(y), \lambda_A(y)) \alpha A \) but \( (x + y)(\min\{\mu_A(x), \mu_A(y)\}, \max\{\lambda_A(x), \lambda_A(y)\}) \) \( \beta A \) for every \( \beta \in \{ e, q_k, \in \ldots \} \), a contradiction. Also \( x(1,0)q_kA \) and \( y(1,0)q_kA \) but \( (x + y)(1,0) \) \( \beta A \) for every \( \beta \in \{ e, q_k, \in \ldots \} \), a contradiction. Hence \( \mu_A(x + y) = 0 \) and \( \lambda_A(x + y) < 1 \). Therefore, \( x + y \in R_{(0,1)} \).

Let \( x \in R_{(0,1)} \) and \( y \in R \). Then \( \mu_A(x) > 0 \) and \( \lambda_A(x) < 1 \). Suppose that \( \mu_A(xy) = 0 \) or \( \lambda_A(xy) = 1 \). If \( \alpha \in \{ e, \in \ldots \} \), then \( x(\mu_A(x), \lambda_A(x)) \alpha A \) but \( (xy)(\mu_A(x), \lambda_A(x)) \) \( \beta A \) for every \( \beta \in \{ e, q_k, \in \ldots \} \), a contradiction. Also \( x(1,0)q_kA \) and \( y(1,0)q_kA \) but \( (x + y)(1,0) \) \( \beta A \) for every \( \beta \in \{ e, q_k, \in \ldots \} \), a contradiction. Thus \( \mu_A(xy) > 0 \) and \( \lambda_A(xy) < 1 \). Therefore, \( xy \in R_{(0,1)} \). Similarly \( xy \in R_{(0,1)} \). This completes the proof.

Theorem 4. Let \( A = (\mu_A, \lambda_A) \) be an \((\alpha, \beta)\)-intuitionistic fuzzy sub-hemiring of \( R \). Then the set

\[
R_{(0,1)} = \{ x \in R : \mu_A(x) > 0 \text{ and } \lambda_A(x) < 1 \}
\]

is a sub-hemiring of \( R \).

Proof. Let \( x, y \in R_{(0,1)} \). Then \( \mu_A(x) > 0 \) and \( \lambda_A(x) < 1 \), \( \mu_A(y) > 0 \) and \( \lambda_A(y) < 1 \). Assume that \( \mu_A(x + y) = 0 \) or \( \lambda_A(x + y) = 1 \). If \( \alpha \in \{ e, \in \ldots \} \), then \( x(\mu_A(x), \lambda_A(x)) \alpha A \) and \( y(\mu_A(y), \lambda_A(y)) \alpha A \) but \( (x + y)(\min\{\mu_A(x), \mu_A(y)\}, \max\{\lambda_A(x), \lambda_A(y)\}) \) \( \beta A \) for every \( \beta \in \{ e, q_k, \in \ldots \} \), a contradiction. Also \( x(1,0)q_kA \) and \( y(1,0)q_kA \) but \( (x + y)(1,0) \) \( \beta A \) for every \( \beta \in \{ e, q_k, \in \ldots \} \), a contradiction. Thus \( \mu_A(x + y) > 0 \) and \( \lambda_A(x + y) < 1 \). Therefore, \( x + y \in R_{(0,1)} \).

4 \((\in, \in \ldots)\)-intuitionistic fuzzy ideals

Definition 3. An IFS \( A = (\mu_A, \lambda_A) \) of a hemiring \( R \) is called an \((\in, \in \ldots)\)-intuitionistic fuzzy sub-hemiring of \( R \), if \( \forall x, y \in R \) and \( t_1, t_2 \in [0,1] \), \( s_1, s_2 \in [0,1] \),

1a) \( x(t_1, s_1), y(t_2, s_2) \in A \Rightarrow (x + y)(\min(t_1, t_2), \max(s_1, s_2)) \in q_kA \).
2a) \( x(t_1, s_1), y(t_2, s_2) \in A \Rightarrow (xy)(\min(t_1, t_2), \max(s_1, s_2)) \in q_kA \).

Definition 4. An IFS \( A = (\mu_A, \lambda_A) \) of a hemiring \( R \) is called an \((\in, \in \ldots)\)-intuitionistic fuzzy left (right) ideal of \( R \), if \( \forall x, y \in R \) and \( t_1, t_2 \in [0,1] \), \( s_1, s_2 \in [0,1] \),

1a) \( x(t_1, s_1), y(t_2, s_2) \in A \Rightarrow (x + y)(\min(t_1, t_2), \max(s_1, s_2)) \in q_kA \).
3a) \( x(t_1, s_1), y(t_2, s_2) \in A \Rightarrow (xy)(\min(t_1, t_2), \max(s_1, s_2)) \in q_kA \).

An IFS \( A = (\mu_A, \lambda_A) \) of a hemiring \( R \) is called an \((\in, \in \ldots)\)-intuitionistic fuzzy ideal of \( R \), if it is both \((\in, \in \ldots)\)-intuitionistic fuzzy left ideal and \((\in, \in \ldots)\)-intuitionistic fuzzy right ideal of \( R \).

Theorem 5. Let \( A \) be an intuitionistic fuzzy subset of a hemiring \( R \). Then \((1a) \Rightarrow (1b), (2a) \Rightarrow (2b), (3a) \Rightarrow (3b), \)

where \( \forall x, y \in R \) and \( k \in [0,1] \).
(1b) $\mu_{A}(x+y) \geq \min \{ \mu_{A}(x), \mu_{A}(y), \frac{1}{2} \}$ and $\lambda_{A}(x+y) \leq \max \{ \lambda_{A}(x), \lambda_{A}(y), \frac{1}{2} \}$.

(2b) $\mu_{A}(xy) \geq \min \{ \mu_{A}(x), \mu_{A}(y), \frac{1}{2} \}$ and $\lambda_{A}(xy) \leq \max \{ \lambda_{A}(x), \lambda_{A}(y), \frac{1}{2} \}$.

(3b) $\mu_{A}(xy) \geq \min \{ \mu_{A}(y), \frac{1}{2} \}$ and $\lambda_{A}(xy) \leq \max \{ \lambda_{A}(y), \frac{1}{2} \}$.

Proof. (1a) $\Rightarrow$ (1b) Let $A$ be an intuitionistic fuzzy subset of a hemiring $R$, and (1a) holds. Suppose that (1b) doesn’t hold then there exist $x, y \in R$ such that $\mu_{A}(x+y) < \min \{ \mu_{A}(x), \mu_{A}(y), \frac{1}{2} \}$ or $\lambda_{A}(x+y) > \max \{ \lambda_{A}(x), \lambda_{A}(y), \frac{1}{2} \}$. So there exits three possible cases.

(i) $\mu_{A}(x+y) < \min \{ \mu_{A}(x), \mu_{A}(y), \frac{1}{2} \}$ and $\lambda_{A}(x+y) \leq \max \{ \lambda_{A}(x), \lambda_{A}(y), \frac{1}{2} \}$,

(ii) $\mu_{A}(x+y) \geq \min \{ \mu_{A}(x), \mu_{A}(y), \frac{1}{2} \}$ and $\lambda_{A}(x+y) > \max \{ \lambda_{A}(x), \lambda_{A}(y), \frac{1}{2} \}$,

(iii) $\mu_{A}(x+y) < \min \{ \mu_{A}(x), \mu_{A}(y), \frac{1}{2} \}$ and $\lambda_{A}(x+y) > \max \{ \lambda_{A}(x), \lambda_{A}(y), \frac{1}{2} \}$.

For the first case, there exist $t \in (0, 1]$ such that $\mu_{A}(x+y) < t < \min \{ \mu_{A}(x), \mu_{A}(y), \frac{1}{2} \}$. Now choose $s = 1 - t$, then clearly $x(t,s) \in A$ and $y(t,s) \in A$ but $(x+y)(t,s) \in \overline{qv_{4}A}$. Which is a contradiction. Second case is similar to this case.

Now consider case (iii), i.e $\mu_{A}(x+y) < \min \{ \mu_{A}(x), \mu_{A}(y), \frac{1}{2} \}$ and $\lambda_{A}(x+y) > \max \{ \lambda_{A}(x), \lambda_{A}(y), \frac{1}{2} \}$. Then there exist $t \in (0, 1]$ and $s \in [0, 1)$, such that $\mu_{A}(x+y) < t \leq \min \{ \mu_{A}(x), \mu_{A}(y), \frac{1}{2} \}$ and $\lambda_{A}(x+y) > s \geq \max \{ \lambda_{A}(x), \lambda_{A}(y), \frac{1}{2} \}$

$\Rightarrow x(t,s) \in A$ and $y(t,s) \in A$ but $(x+y)(t,s) \in \overline{qv_{4}A}$. Which is again a contradiction. So our supposition is wrong. Hence (1b) holds.

Similarly we can prove $(2a) \implies (2b), (3a) \implies (3b)$.

Definition 5. Let $A = (\mu_{A}, \lambda_{A})$ be an IFS of a hemiring $R$. Then $A$ is an $(\in, \in \vee q_{4})$-intuitionistic fuzzy sub-hemiring of $R$ if it satisfies the conditions (1b) and (2b).

Definition 6. Let $A = (\mu_{A}, \lambda_{A})$ be an IFS of a hemiring $R$. Then $A$ is an $(\in, \in \vee q_{4})$-intuitionistic fuzzy left ideal of $R$ if it satisfies the conditions (1b) and (3b).

Remark. Every $(\in, \in \vee q_{4})$-intuitionistic fuzzy left ideal (right ideal, sub-hemiring) $A = (\mu_{A}, \lambda_{A})$ of $R$ need not be an $(\in, \in \vee q_{4})$-intuitionistic fuzzy left ideal (right ideal, sub-hemiring) of $R$.

Example 1. Let $\mathbb{N}$ be the set of all non negative integers and $A = (\mu_{A}, \lambda_{A})$ be an IFS of $\mathbb{N}$ defined as follows:

$\mu_{A}(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0.5 & \text{if } 1 \leq x \leq 4 \\
0 & \text{if } x > 4 
\end{cases}$

$\lambda_{A}(x) = \begin{cases} 
0.5 & \text{if } 1 \leq x \leq 4 \\
0 & \text{if } x = 0 \\
0.4 & \text{if } 4 < x 
\end{cases}$

For all $x, y \in R$,

(1) $\mu_{A}(x+y) \geq \min \{ \mu_{A}(x), \mu_{A}(y), 0.4 \}$ and $\lambda_{A}(x+y) \leq \max \{ \lambda_{A}(x), \lambda_{A}(y), 0.4 \}$,

(2) $\mu_{A}(xy) \geq \min \{ \mu_{A}(y), 0.4 \}$ and $\lambda_{A}(xy) \leq \max \{ \lambda_{A}(y), 0.4 \}$,

(3) $\mu_{A}(xy) \geq \min \{ \mu_{A}(x), 0.4 \}$ and $\lambda_{A}(xy) \leq \max \{ \lambda_{A}(x), 0.4 \}$.

Thus $A = (\mu_{A}, \lambda_{A})$ is an $(\in, \in \vee q_{4})$-intuitionistic fuzzy ideal of $\mathbb{N}$. But $2(0.45, 0.55), 3(0.45, 0.55) \in A \implies (2.3)(0.45, 0.55) \in \overline{qv_{4}A}$. Thus $A = (\mu_{A}, \lambda_{A})$ is not an $(\in, \in \vee q_{4})$-intuitionistic fuzzy ideal of $\mathbb{N}$.

Definition 7. For any intuitionistic fuzzy set $A = (\mu_{A}, \lambda_{A})$ in $R$ and $t \in (0, 1]$, $s \in [0, 1)$ and $k \in [0, 1)$ we define $U_{(t,s)} = \{ x \in R : x(t,s) \in A \}$, $A_{(t,s)}_{k} = \{ x \in R : x(t,s)q_{4}A \}$ and $[A]_{(t,s)}_{k} = \{ x \in R : x(t,s) \in \overline{qv_{4}A} \}$.

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Obviously, \( [A]_{(t,s)} = A_{(t,s)} \cup U_{(t,s)} \), where \( U_{(t,s)} \), \( A_{(t,s)} \) and \( [A]_{(t,s)} \) are called \( \epsilon \)-level set, \( \eta \)-level set and in \( \forall \eta \)-level set of \( A = (\mu_A, \lambda_A) \), respectively.

**Lemma 1.** Every intuitionistic fuzzy subset \( A = (\mu_A, \lambda_A) \) of a hemiring \( R \) satisfies the following condition:

\[
t \in (0, \frac{1}{t+k}], s \in [\frac{1}{t+k}, 1) \iff [A]_{(t,s)} = U_{(t,s)}.
\]

**Proof.** Let \( t \in (0, \frac{1}{t+k}) \), and \( s \in [\frac{1}{t+k}, 1) \). It is clear that \( U_{(t,s)} \subseteq [A]_{(t,s)} \). Let \( x \in [A]_{(t,s)} \). If \( x \notin U_{(t,s)} \), then \( \mu_A(x) < t \), or \( \lambda_A(x) > s \) and so \( \mu_A(x) + t \leq 2t \leq 1 - k \), or \( \mu_A(x) > s \geq 1 - k \). This shows that \( x \notin U_{(t,s)} \). Therefore \( [A]_{(t,s)} \subseteq U_{(t,s)} \).

**Theorem 6.** If \( A = (\epsilon, \in \forall \eta) \)-intuitionistic fuzzy ideal of \( R \), then the set \( A_{(t,s)} \) is an ideal of \( R \) when it is non-empty for all \( t \in \left( \frac{1}{t+k}, 1 \right] \), \( s \in \left[ \frac{t+k}{1+k}, 1 \right) \).

**Proof.** Assume that \( A \) is an \((\epsilon, \in \forall \eta)\)-intuitionistic fuzzy ideal of \( R \), and let \( t \in \left( \frac{1}{t+k}, 1 \right] \), \( s \in \left[ \frac{1}{t+k}, 1 \right) \) be such that \( A_{(t,s)} \neq \phi \). Let \( x, y \in A_{(t,s)} \). Then \( \mu_A(x) + t + k > 1, \lambda_A(x) + s + k < 1 \) and \( \mu_A(y) + t + k > 1, \lambda_A(y) + s + k < 1 \). As \( \mu_A(x+y) \geq \min \{ \mu_A(x), \mu_A(y), \frac{1}{t+k} \} \), \( \lambda_A(x+y) \leq \max \{ \lambda_A(x), \lambda_A(y), \frac{1}{t+k} \} \). We have \( \mu_A(x+y) \geq \min \{ 1 - t - k, \frac{1}{t+k} \} \) and \( \lambda_A(x+y) \leq \max \{ 1 - s - k, \frac{1}{t+k} \} \). Since \( t \in \left( \frac{1}{t+k}, 1 \right] \), and \( s \in \left[ \frac{1}{t+k}, 1 \right) \), so \( 1 - t - k < \frac{1}{t+k} \) and \( 1 - s - k > \frac{1}{t+k} \), thus \( \mu_A(x+y) > 1 - t - k \) and \( \lambda_A(x+y) < 1 - s - k \). Hence \( x + y \in A_{(t,s)} \). Let \( x \in A_{(t,s)} \) and \( y \in R \). Then \( \mu_A(x) + t + k > 1, \lambda_A(x) + s + k < 1 \). Then \( \lambda_A(x) > 1 - t - k, \lambda_A(x) < 1 - s - k \). Since \( A = (\epsilon, \in \forall \eta)\)-intuitionistic fuzzy ideal of \( R \), we have \( \mu_A(xy) \geq \min \{ \mu_A(x), \frac{1}{t+k} \} \), \( \lambda_A(x+y) \leq \max \{ \lambda_A(x), \frac{1}{t+k} \} \). Implies that \( \mu_A(xy) \geq \min \{ 1 - t - k, \frac{1}{t+k} \} \), \( \lambda_A(xy) \leq \max \{ 1 - s - k, \frac{1}{t+k} \} \). Since \( t \in \left( \frac{1}{t+k}, 1 \right] \), and \( s \in \left[ \frac{1}{t+k}, 1 \right) \), so \( 1 - t - k < \frac{1}{t+k} \) and \( 1 - s - k > \frac{1}{t+k} \), thus \( \mu_A(xy) > 1 - t - k \) and \( \lambda_A(xy) < 1 - s - k \). This implies \( xy \in A_{(t,s)} \). Similarly \( xy \in A_{(t,s)} \), Hence \( A_{(t,s)} \) is an ideal of \( R \).

**Theorem 7.** For any intuitionistic fuzzy subset \( A \) of \( R \), the following are equivalent:

(i) \( A \) is an \((\epsilon, \in \forall \eta)\)-intuitionistic fuzzy ideal of \( R \).

(ii) For all \( t \in (0, \frac{1}{t+k}], s \in [\frac{1}{t+k}, 1) \), \( U_{(t,s)} \neq \phi \iff U_{(t,s)} \) is an ideal of \( R \).

**Proof.** Let \( A \) be an \((\epsilon, \in \forall \eta)\)-intuitionistic fuzzy ideal of \( R \) and \( x, y \in U_{(t,s)} \) for some \( t \in (0, \frac{1}{t+k}], s \in [\frac{1}{t+k}, 1) \). Then \( \mu_A(x+y) \geq \min \{ \mu_A(x), \mu_A(y), \frac{1}{t+k} \} \geq \min \{ t, \frac{1}{t+k} \} \) and \( \lambda_A(x+y) \leq \max \{ \lambda_A(x), \lambda_A(y), \frac{1}{t+k} \} \leq \max \{ s, \frac{1}{t+k} \} = s \), which implies \( x + y \in U_{(t,s)} \). Now, if \( x \in U_{(t,s)} \) and \( y \in R \) then \( \mu_A(xy) \geq \min \{ \mu_A(x), \frac{1}{t+k} \} \geq \min \{ t, \frac{1}{t+k} \} = t \) and \( \lambda_A(xy) \leq \max \{ \lambda_A(x), \frac{1}{t+k} \} \leq \max \{ s, \frac{1}{t+k} \} = s \), which implies \( xy \in U_{(t,s)} \). Similarly \( xy \in U_{(t,s)} \). This shows that \( U_{(t,s)} \) is an ideal of \( R \).

Conversely, assume that for every \( t \in (0, \frac{1}{t+k}], s \in [\frac{1}{t+k}, 1) \), each non-empty \( U_{(t,s)} \) is an ideal of \( R \). Suppose \( A \) is not an \((\epsilon, \in \forall \eta)\)-intuitionistic fuzzy ideal of \( R \), then there exist \( x, y \in R \) such that one of the following three cases is true.

(i) \( \mu_A(x+y) < \min \{ \mu_A(x), \mu_A(y), \frac{1}{t+k} \} \) and \( \lambda_A(x+y) \leq \max \{ \lambda_A(x), \lambda_A(y), \frac{1}{t+k} \} \).

(ii) \( \mu_A(x+y) \geq \min \{ \mu_A(x), \mu_A(y), \frac{1}{t+k} \} \) and \( \lambda_A(x+y) > \max \{ \lambda_A(x), \lambda_A(y), \frac{1}{t+k} \} \).

(iii) \( \mu_A(x+y) < \min \{ \mu_A(x), \mu_A(y), \frac{1}{t+k} \} \) and \( \lambda_A(x+y) > \max \{ \lambda_A(x), \lambda_A(y), \frac{1}{t+k} \} \).

For the first case, \( t \in (0, \frac{1}{t+k}] \) such that \( \mu_A(x+y) \leq t \leq \min \{ \mu_A(x), \mu_A(y), \frac{1}{t+k} \} \). Now choose \( s = 1 - t \), then clearly \( x + y \in U_{(t,s)} \) but \( x + y \notin U_{(t,s)} \). Which is a contradiction. Case (ii) is similar to the case (i).

Now consider case (iii), then there exist \( t \in (0, \frac{1}{t+k}], s \in [\frac{1}{t+k}, 1) \), such that \( \mu_A(x+y) < t \leq \min \{ \mu_A(x), \mu_A(y), \frac{1}{t+k} \} \) and \( \lambda_A(x+y) > s \geq \max \{ \lambda_A(x), \lambda_A(y), \frac{1}{t+k} \} \). This is a contradiction. So our supposition is wrong, hence \( \mu_A(x+y) \geq \min \{ \mu_A(x), \mu_A(y), \frac{1}{t+k} \} \) and \( \lambda_A(x+y) \leq \max \{ \lambda_A(x), \lambda_A(y), \frac{1}{t+k} \} \) for all
theorem 10.

definition 9.

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\[ x, y \in R. \]

In a similar way we can show that \( \mu_A(xy) \geq \min \{ \mu_A(x), \frac{1-k}{2} \} \) and \( \lambda_A(xy) \leq \max \{ \lambda_A(x), \frac{1-k}{2} \} \), \( \mu_A(xy) \geq \min \{ \mu_A(y), \frac{1-k}{2} \} \) and \( \lambda_A(xy) \leq \max \{ \lambda_A(y), \frac{1-k}{2} \} \) for all \( x, y \in R. \)

**Theorem 8.** Let \( \{ A_i : i \in I \} \) be a family of \((\varepsilon, \in \vee q_k)^*\)-intuitionistic fuzzy sub-hemiring of \( R \). Then \( A = \cap_{i \in I} A_i \) is an \((\varepsilon, \in \vee q_k)^*\)-intuitionistic fuzzy sub-hemiring of \( R \).

**Proof.** Straightforward.

**Theorem 9.** Let \( \{ A_i : i \in I \} \) be a family of \((\varepsilon, \in \vee q_k)^*\)-intuitionistic fuzzy left (right) ideals of \( R \). Then \( A = \cap_{i \in I} A_i \) is an \((\varepsilon, \in \vee q_k)^*\)-intuitionistic fuzzy left (right) ideal of \( R \).

**Proof.** Straightforward.

## 5 Regular and idempotent hemirings

**Definition 8.** Let \( A \) and \( B \) be two intuitionistic fuzzy subsets of a hemiring \( R \), then \( A \cdot_k B \) is defined as, \( A \cdot_k B = \langle \mu_{A \cdot_k B}, \lambda_{A \cdot_k B} \rangle \) where

\[
\begin{align*}
(\mu_{A \cdot_k B})(x) &= \min \left\{ \sum_{i=1}^{\infty} y_i z_i \left[ \bigwedge_{1 \leq p \leq I} \mu_A(y_i) \land \mu_B(z_i) \right] \right\} \bigwedge \frac{1-k}{2} \\
&= 0 \text{ if } x \text{ cannot be expressed as } x = \sum_{i=1}^{\infty} y_i z_i \\
(\lambda_{A \cdot_k B})(x) &= \max \left\{ \sum_{i=1}^{\infty} y_i z_i \left[ \bigvee_{1 \leq p \leq I} \lambda_A(y_i) \lor \lambda_B(z_i) \right] \right\} \bigvee \frac{1-k}{2} \\
&= 1 \text{ if } x \text{ cannot be expressed as } x = \sum_{i=1}^{\infty} y_i z_i
\end{align*}
\]

where \( x \in R. \)

**Definition 9.** let \( A \) and \( B \) an intuitionistic fuzzy subsets of \( R \). We define the intuitionistic fuzzy subsets \( A_k, A \cap_k B, A \cup_k B \) and \( A \cdot_k B \) of \( R \) as follows:

\[
A_k = \left( \mu_A \land \frac{1-k}{2}, \lambda_A \lor \frac{1-k}{2} \right),
\]

\[
A \cap_k B = (A \cap B)_k = (\mu_A \land_k B, \lambda_A \lor_k B),
\]

\[
A \cup_k B = (A \cup B)_k = (\mu_A \lor_k B, \lambda_A \land_k B).
\]

**Theorem 10.** Let \( A \) be an \((\varepsilon, \in \vee q_k)^*\)-intuitionistic fuzzy sub-hemiring of \( R \). Then \( A_k \) is an \((\varepsilon, \in \vee q_k)^*\)-intuitionistic fuzzy sub-hemiring of \( R \).

**Proof.** Suppose \( A \) is an \((\varepsilon, \in \vee q_k)^*\)-intuitionistic fuzzy sub-hemiring of \( R \) and \( x, y \in R. \) Then

\[
(\mu_A \land \frac{1-k}{2})(x+y) = \mu_A(x+y) \land \frac{1-k}{2}
\]

\[
\geq \left( \min \left\{ \mu_A(x), \mu_A(y), \frac{1-k}{2} \right\} \right) \land \frac{1-k}{2}
\]

\[
= \min \left\{ \mu_A(x) \land \frac{1-k}{2}, \mu_A(y) \land \frac{1-k}{2} \right\}
\]

\[
= \min \left\{ (\mu_A \land \frac{1-k}{2})(x), (\mu_A \land \frac{1-k}{2})(y) \land \frac{1-k}{2} \right\}.
\]
and
\[
(\lambda_A \vee \frac{1-k}{2})(x+y) = \lambda_A(x+y) \vee \frac{1-k}{2} \\
\leq (\max\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\}) \vee \frac{1-k}{2}.
\]
\[
= \max\{\lambda_A(x) \vee \frac{1-k}{2}, \lambda_A(y) \vee \frac{1-k}{2}, \frac{1-k}{2}\}.
\]
\[
= \max\{\lambda_A \left(\frac{1-k}{2}\right)(x), (\lambda_A(y) \vee \frac{1-k}{2})(y), \frac{1-k}{2}\}.
\]
Similarly we can show that
\[
(\mu_A \wedge \frac{1-k}{2})(xy) \geq \min\{\mu_A(x), \mu_A(y), \frac{1-k}{2}\}
\]
and
\[
(\lambda_A \vee \frac{1-k}{2})(xy) \leq \max\{\lambda_A \left(\frac{1-k}{2}\right)(x), (\lambda_A(y) \vee \frac{1-k}{2})(y), \frac{1-k}{2}\}.
\]
This shows that $A_k = A \cap \frac{1-k}{2}$ is an $(\epsilon, \in \vee \eta_k)^*$-intuitionistic fuzzy sub-hemiring of $R$.

**Theorem 11.** Let $A$ be an $(\epsilon, \in \vee \eta_k)^*$-intuitionistic fuzzy ideal of $R$. Then $A_k$ is an $(\epsilon, \in \vee \eta_k)^*$-intuitionistic fuzzy ideal of $R$.

**Proof.** This proof is similar to the proof of the theorem 10.

**Remark.** Let $A$ and $B$ be intuitionistic fuzzy subsets of $R$. Then the following hold.

(i) $A \cap B = (A \cap B_k)$.

(ii) $A \cup B = (A_k \cup B_k)$.

(iii) $A \cdot B = (A_k \cdot B_k)$.

**Proof.** Let $x \in R$.

1. $(\mu_A \wedge \frac{1-k}{2})(x) = (\mu_A \wedge \frac{1-k}{2})(x) \wedge 1 = \mu_A(x) \wedge \mu_B(x) \wedge \frac{1-k}{2} = (\mu_A(x) \wedge \frac{1-k}{2}) \wedge (\mu_B(x) \wedge \frac{1-k}{2})$

\[
= \mu_A(x) \wedge (\mu_B(x) \wedge \frac{1-k}{2})
\]
and
\[
(\lambda_A \vee \frac{1-k}{2})(x) = (\lambda_A \vee \frac{1-k}{2})(x) \vee 1 = \lambda_A(x) \vee \lambda_B(x) \vee \frac{1-k}{2} = (\lambda_A(x) \vee \frac{1-k}{2}) \vee (\lambda_B(x) \vee \frac{1-k}{2})
\]
\[
= \lambda_A(x) \vee (\lambda_B(x) \vee \frac{1-k}{2})
\]
Hence (1) holds. Similarly we can prove (2).

3. If $x$ is not expressible as $x = \sum_{i=1}^p y_i z_i$ where $y_i, z_i \in R$, then $(\mu_A \cdot \mu_B)(x) = 0$.

Thus $(\mu_A \cdot \mu_B)(x) = (\mu_A \cdot \mu_B)(x) \wedge \frac{1-k}{2} = 0$. As $x$ is not expressible as $x = \sum_{i=1}^p y_i z_i$ so $(\mu_A \wedge \frac{1-k}{2})(x) = 0 \Longrightarrow \mu_A \wedge \mu_B = \mu_A \cdot \mu_B$ and (\lambda_A \wedge \lambda_B)(x) = 1, thus $(\lambda_A \cdot \lambda_B)(x) = (\lambda_A \cdot \lambda_B)(x) \vee \frac{1-k}{2} = 1$ as $x$ is not expressible as $x = \sum_{i=1}^p y_i z_i$ so $(\lambda_A \cdot \lambda_B)(x) = 1 \Longrightarrow \lambda_A \wedge \lambda_B = \lambda_A \cdot \lambda_B$. Hence (3) holds.

**Theorem 12.** If $A$ and $B$ are $(\epsilon, \in \vee \eta_k)^*$-intuitionistic fuzzy ideals of $R$ then $A_k \cdot B$ is an $(\epsilon, \in \vee \eta_k)^*$-intuitionistic fuzzy ideal of $R$.

**Proof.** Let $x, y \in R$ be such that $x = \sum_{i=1}^p a_i b_i$, and $y = \sum_{j=1}^q a'_j b'_j$. Then
\[
(\mu_A \cdot \mu_B)(x) = \bigvee_{x=\sum_{i=1}^p a_i b_i} \left[ \bigwedge_{1 \leq i \leq p} [\mu_A(a_i) \wedge \mu_B(b_i)] \right] \wedge \frac{1-k}{2}.
\]
Thus
\[
(\mu_A \cdot \mu_B)(x') = \bigvee_{x' = \sum_{i=1}^{p} a' \beta_i} \left[ \bigwedge_{i \leq j \leq p} [\mu_A(a'_i) \land \mu_B(b'_j)] \right] \land \frac{1-k}{2}.
\]

Thus
\[
(\mu_A \cdot \mu_B)(x) \land (\mu_A \cdot \mu_B)(x') \land \frac{1-k}{2} = \left\{ \begin{array}{l}
\bigvee_{x = \sum_{i=1}^{p} a \beta_i} \left[ \bigwedge_{i \leq j \leq p} [\mu_A(a_i) \land \mu_B(b_j)] \right] \land \frac{1-k}{2} \bigvee_{x' = \sum_{i=1}^{p} a' \beta_i} \left[ \bigwedge_{i \leq j \leq p} [\mu_A(a'_i) \land \mu_B(b'_j)] \right] \land \frac{1-k}{2}
\end{array} \right\}
\]
\[
= \left\{ \bigvee_{x = \sum_{i=1}^{p} a \beta_i} \left[ \bigwedge_{i \leq j \leq p} [\mu_A(a_i) \land \mu_B(b_j)] \right] \land \frac{1-k}{2} \bigvee_{x' = \sum_{i=1}^{p} a' \beta_i} \left[ \bigwedge_{i \leq j \leq p} [\mu_A(a'_i) \land \mu_B(b'_j)] \right] \land \frac{1-k}{2} \right\}
\]
\[
= \left( \mu_A \cdot \mu_B \right)(x \lor x')
\]

and
\[
(\lambda_A \cdot \lambda_B)(x) = \left[ \bigwedge_{x = \sum_{i=1}^{p} a \beta_i} \left[ \bigvee_{i \leq j \leq p} [\lambda_A(a_i) \lor \lambda_B(b_j)] \right] \lor \frac{1-k}{2} \right],
\]
\[
(\lambda_A \cdot \lambda_B)(x') = \left[ \bigwedge_{x' = \sum_{i=1}^{p} a' \beta_i} \left[ \bigvee_{i \leq j \leq p} [\lambda_A(a'_i) \lor \lambda_B(b'_j)] \right] \lor \frac{1-k}{2} \right].
\]

Thus
\[
(\lambda_A \cdot \lambda_B)(x) \lor (\lambda_A \cdot \lambda_B)(x') \lor \frac{1-k}{2} = \left\{ \begin{array}{l}
\bigwedge_{x = \sum_{i=1}^{p} a \beta_i} \left[ \bigvee_{i \leq j \leq p} [\lambda_A(a_i) \lor \lambda_B(b_j)] \right] \lor \frac{1-k}{2} \bigvee_{x' = \sum_{i=1}^{p} a' \beta_i} \left[ \bigwedge_{i \leq j \leq p} [\lambda_A(a'_i) \lor \lambda_B(b'_j)] \right] \lor \frac{1-k}{2}
\end{array} \right\}
\]
\[
= \left\{ \bigwedge_{x = \sum_{i=1}^{p} a \beta_i} \left[ \bigvee_{i \leq j \leq p} [\lambda_A(a_i) \lor \lambda_B(b_j)] \right] \lor \frac{1-k}{2} \bigvee_{x' = \sum_{i=1}^{p} a' \beta_i} \left[ \bigwedge_{i \leq j \leq p} [\lambda_A(a'_i) \lor \lambda_B(b'_j)] \right] \lor \frac{1-k}{2} \right\}
\]
\[
\geq \left[ \bigwedge_{x = \sum_{i=1}^{p} a \beta_i} \left[ \bigvee_{i \leq j \leq p} [\lambda_A(a''_i) \lor \lambda_B(b''_j)] \right] \lor \frac{1-k}{2} \right]
\]
\[
= (\lambda_A \cdot \lambda_B)(x \lor x')
\]
\[
\implies \{ (\lambda_A \cdot \lambda_B)(x) \lor (\lambda_A \cdot \lambda_B)(x') \lor \frac{1-k}{2} \} \geq (\lambda_A \cdot \lambda_B)(x \lor x'). \text{ Also, } (\mu_A \cdot \mu_B)(x) \land \frac{1-k}{2}
\]
\[
= \left[ \bigvee_{x = \sum_{i=1}^{p} a \beta_i} \left[ \bigwedge_{i \leq j \leq p} [\mu_A(a_i) \land \mu_B(b_j)] \right] \lor \frac{1-k}{2} \right] \frac{1-k}{2}
\]
\[
= \left[ \bigvee_{x = \sum_{i=1}^{p} a \beta_i} \left[ \bigwedge_{i \leq j \leq p} [\mu_A(a_i) \land \mu_B(b_j)] \frac{1-k}{2} \right] \right] \lor \frac{1-k}{2}
\]
\[
= \left[ \bigvee_{x = \sum_{i=1}^{p} a \beta_i} \left[ \bigwedge_{i \leq j \leq p} [\mu_A(a_i) \land \mu_B(b_j)] \frac{1-k}{2} \right] \right] \lor \frac{1-k}{2}
\]
\[
\leq \left[ \bigvee_{x = \sum_{i=1}^{p} a \beta_i} \left[ \bigwedge_{i \leq j \leq p} [\mu_A(a_i) \land \mu_B(b_j)] \right] \right] \lor \frac{1-k}{2}
\]
Similarly we can prove

\[ \forall x \in \Sigma_{j=1}^{n}, \exists i \in [1, q] : \left( A \lor \lambda_B \right)(x) = \min \left\{ \mu_A(x) \lor \mu_B(x), \frac{1-k}{2} \right\} \]

Thus \( (\mu_A \land \mu_B)(x) \leq (\mu_A \lor \mu_B)(x) \).

Similarly we can prove \( (\lambda_A \land \lambda_B)(x) \leq \{ (\lambda_A \lor \lambda_B)(x) \lor \frac{1-k}{2} \} \Rightarrow A \land B \) is an \((\epsilon, \in \cap q)\)-intuitionistic fuzzy right ideal of \( R \). On the same line it can be proved that \( \{ (\mu_A \land \mu_B)(x) \lor \frac{1-k}{2} \} \leq (\mu_A \lor \mu_B)(x) \) and \( (\lambda_A \land \lambda_B)(x) \lor \frac{1-k}{2} \). Thus \( A \land B \) is an \((\epsilon, \in \cap q)\)-intuitionistic fuzzy right ideal of \( R \).

**Theorem 13.** If \( A \) and \( B \) are \((\epsilon, \in \cap q)\)-intuitionistic fuzzy left(right) ideals of \( R \), then so is \( A \lor \land B \).

**Proof.** We only consider the case of \((\epsilon, \in \cap q)\)-intuitionistic fuzzy left ideals.

Let \( x, y \in R \). Then

\[
(\mu_A \land \mu_B)(x + y) = \min \left\{ \mu_A(x + y), \mu_B(x + y), \frac{1-k}{2} \right\}
\]

\[
\geq \min \left\{ \min \{ \mu_A(x), \mu_A(y), \frac{1-k}{2} \}, \min \{ \mu_B(x), \mu_B(y), \frac{1-k}{2} \} \right\}
\]

\[
= \min \left\{ \min \{ \mu_A(x), \mu_B(x), \frac{1-k}{2} \}, \min \{ \mu_A(y), \mu_B(y), \frac{1-k}{2} \} \right\}
\]

\[
= \min \left\{ \mu_A \land \mu_B(x), \mu_A \land \mu_B(y), \frac{1-k}{2} \right\}
\]

and

\[
(\lambda_A \lor \lambda_B)(x + y) = \max \left\{ \lambda_A(x + y), \lambda_B(x + y), \frac{1-k}{2} \right\}
\]

\[
\leq \max \left\{ \max \{ \lambda_A(x), \lambda_A(y), \frac{1-k}{2} \}, \max \{ \lambda_B(x), \lambda_B(y), \frac{1-k}{2} \} \right\}
\]

\[
= \max \left\{ \max \{ \lambda_A(x), \lambda_B(x), \frac{1-k}{2} \}, \max \{ \lambda_A(y), \lambda_B(y), \frac{1-k}{2} \} \right\}
\]

\[
= \max \left\{ \lambda_A \lor \lambda_B(x), \lambda_A \lor \lambda_B(y), \frac{1-k}{2} \right\}.
\]

Now

\[
(\mu_A \lor \mu_B)(x, y) = \min \left\{ \mu_A(x, y), \mu_B(x, y), \frac{1-k}{2} \right\}
\]

\[
\geq \min \left\{ \min \{ \mu_A(y), \frac{1-k}{2} \}, \min \{ \mu_B(y), \frac{1-k}{2} \}, \frac{1-k}{2} \right\}
\]

\[
= \min \left\{ \min \{ \mu_A(y), \mu_B(y), \frac{1-k}{2} \}, \frac{1-k}{2} \right\} = \min \left\{ \mu_A \lor \mu_B(y), \frac{1-k}{2} \right\}
\]
Thus $A \cap B$ is an $(\epsilon, \varphi_{q_k})^{*}$-intuitionistic fuzzy left ideal of $R$.

**Theorem 14.** If $A$ is an $(\epsilon, \varphi_{q_k})^{*}$-intuitionistic fuzzy right ideal, and $B$ is an $(\epsilon, \varphi_{q_k})^{*}$-intuitionistic fuzzy left ideal of $R$, then $A^{*}B \subseteq A \cap B$.

**Proof.** Let $A$ and $B$ be $(\epsilon, \varphi_{q_k})^{*}$-intuitionistic fuzzy right and left ideals of $R$ respectively. For any $x \in R$,

$$
(\mu_{A^{*}B})(x) = \bigvee_{x \in \sum_{p} n_{b_i}} \left( \bigwedge_{1 \leq i \leq p} [\mu_{A}(a_i) \wedge \mu_{B}(b_i)] \right) \wedge \frac{1-k}{2}
$$

$$
= \bigvee_{x \in \sum_{p} n_{b_i}} \left( \bigwedge_{1 \leq i \leq p} [\mu_{A}(a_i) \wedge \frac{1-k}{2}] \wedge [\mu_{B}(b_i) \wedge \frac{1-k}{2}] \right) \wedge \frac{1-k}{2}
$$

$$
\leq \bigvee_{x \in \sum_{p} n_{b_i}} \left( \bigwedge_{1 \leq i \leq p} [\mu_{A}(a_i) \wedge \mu_{B}(a_ib_i)] \right) \wedge \frac{1-k}{2}
$$

$$
\leq \bigvee_{x \in \sum_{p} n_{b_i}} [\mu_{A}(x) \wedge \mu_{B}(x)] \wedge \frac{1-k}{2} = (\mu_{A} \wedge \mu_{B})(x),
$$

and

$$
(\lambda_{A^{*}B})(x) = \bigwedge_{x \in \sum_{p} n_{b_i}} \left( \bigvee_{1 \leq i \leq p} [\lambda_{A}(a_i) \vee \lambda_{B}(b_i)] \right) \lor \frac{1-k}{2}
$$

$$
= \bigwedge_{x \in \sum_{p} n_{b_i}} \left( \bigvee_{1 \leq i \leq p} [\lambda_{A}(a_i) \vee \frac{1-k}{2}] \lor [\lambda_{B}(b_i) \vee \frac{1-k}{2}] \right) \lor \frac{1-k}{2}
$$

$$
\geq \bigwedge_{x \in \sum_{p} n_{b_i}} \left( \bigvee_{1 \leq i \leq p} [\lambda_{A}(a_ib_i) \vee \lambda_{B}(a_ib_i)] \right) \lor \frac{1-k}{2}
$$

$$
= \bigwedge_{x \in \sum_{p} n_{b_i}} \left( \bigvee_{1 \leq i \leq p} \lambda_{A}(a_ib_i) \right) \lor \left( \bigvee_{1 \leq i \leq p} \lambda_{B}(a_ib_i) \right) \lor \frac{1-k}{2}
$$

$$
\geq \bigwedge_{x \in \sum_{p} n_{b_i}} [\lambda_{A}(x) \vee \lambda_{B}(x)] \lor \frac{1-k}{2} = (\lambda_{A} \vee \lambda_{B})(x).
$$

Thus $A^{*}B \subseteq A \cap B$. 

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Definition 10. Let $A$ and $B$ be $(\varepsilon, \in \vee q_k)$-intuitionistic fuzzy ideals of $R$. The intuitionistic fuzzy subset $A +_k B$ of $R$ is defined by

$$A +_k B = (\mu_A +_k \mu_B, \lambda_A +_k \lambda_B)$$

where

$$(\mu_A +_k \mu_B)(x) = \bigvee_{x = y + z} [\mu_A(y) \wedge \mu_B(z)] \wedge \frac{1-k}{2},$$

$$(\lambda_A +_k \lambda_B)(x) = \bigwedge_{x = y + z} [\lambda_A(y) \vee \lambda_B(z)] \vee \frac{1-k}{2} \text{ for } x \in R.$$

Proposition 1. For $(\varepsilon, \in \vee q_k)$-intuitionistic fuzzy ideals $A$ and $B$ of $R$, $A +_k B$ is an $(\varepsilon, \in \vee q_k)$-intuitionistic fuzzy ideal of $R$.

Proof. For any $x, x' \in R$,

$$(\mu_A +_k \mu_B)(x) \wedge (\mu_A +_k \mu_B)(x') \wedge \frac{1-k}{2} = \left[ \bigvee_{x = y + z} [\mu_A(y) \wedge \mu_B(z)] \wedge \frac{1-k}{2} \right] \wedge \left[ \bigvee_{x' = y' + z'} [\mu_A(y') \wedge \mu_B(z')] \wedge \frac{1-k}{2} \right] \wedge \frac{1-k}{2}$$

$$= \bigvee_{x = y + z} \bigvee_{x' = y' + z'} \left[ [\mu_A(y) \wedge \mu_B(z)] \wedge \frac{1-k}{2} \right] \wedge \left[ [\mu_A(y') \wedge \mu_B(z')] \wedge \frac{1-k}{2} \right] \wedge \frac{1-k}{2} \leq \bigvee_{x = y + z} \bigvee_{x' = y' + z'} [\mu_A(y + y') \wedge \mu_B(z + z')] \wedge \frac{1-k}{2} \leq (\mu_A +_k \mu_B)(x + x'),$$

and

$$(\lambda_A +_k \lambda_B)(x) \vee (\lambda_A +_k \lambda_B)(x') \vee \frac{1-k}{2} = \left[ \bigwedge_{x = y + z} [\lambda_A(y) \vee \lambda_B(z)] \vee \frac{1-k}{2} \right] \wedge \left[ \bigwedge_{x' = y' + z'} [\lambda_A(y') \vee \lambda_B(z')] \vee \frac{1-k}{2} \right] \wedge \frac{1-k}{2}$$

$$= \bigwedge_{x = y + z} \bigwedge_{x' = y' + z'} \left[ [\lambda_A(y) \vee \lambda_B(z)] \vee \frac{1-k}{2} \right] \wedge \left[ [\lambda_A(y') \vee \lambda_B(z')] \vee \frac{1-k}{2} \right] \wedge \frac{1-k}{2} \geq \bigwedge_{x = y + z} \bigwedge_{x' = y' + z'} [\lambda_A(y + y') \vee \lambda_B(z + z')] \vee \frac{1-k}{2} \geq (\lambda_A +_k \lambda_B)(x + x').$$
Again,
\[
(\mu_A + k \mu_B)(x) \wedge \frac{1 - k}{2} = \left\lfloor \frac{[\mu_A(y) \wedge \mu_B(z)] \wedge \frac{1 - k}{2}}{2} \right\rfloor \wedge \frac{1 - k}{2}
\]
\[
= \left\lfloor \frac{[(\mu_A(y) \wedge \frac{1 - k}{2}) \wedge (\mu_B(z) \wedge \frac{1 - k}{2})] \wedge \frac{1 - k}{2}}{2} \right\rfloor \wedge \frac{1 - k}{2}
\]
\[
\leq \left\lfloor \frac{[\mu_A(ya) \wedge \mu_B(za)] \wedge \frac{1 - k}{2}}{2} \right\rfloor \wedge \frac{1 - k}{2}
\]
\[
\leq \left\lfloor \frac{[\mu_A(y') \wedge \mu_B(z')]}{2} \wedge \frac{1 - k}{2} \right\rfloor = (\mu_A + k \mu_B)(xa),
\]
and
\[
(\lambda_A + k \lambda_B)(x) \vee \frac{1 - k}{2} = \left\lceil \frac{[\lambda_A(y) \vee \lambda_B(z)] \vee \frac{1 - k}{2}}{2} \right\rceil \vee \frac{1 - k}{2}
\]
\[
= \left\lceil \frac{[(\lambda_A(y) \vee \frac{1 - k}{2}) \vee (\lambda_B(z) \vee \frac{1 - k}{2})] \vee \frac{1 - k}{2}}{2} \right\rceil \vee \frac{1 - k}{2}
\]
\[
\geq \left\lceil \frac{(\lambda_A(ya) \vee \lambda_B(za)) \vee \frac{1 - k}{2}}{2} \right\rceil \vee \frac{1 - k}{2}
\]
\[
\geq \left\lceil \frac{(\lambda_A(y') \vee \lambda_B(z')) \vee \frac{1 - k}{2}}{2} \right\rceil = (\lambda_A + k \lambda_B)(xa).
\]
Similarly we can prove
\[
(\mu_A + k \mu_B)(x) \wedge \frac{1 - k}{2} \leq (\mu_A + k \mu_B)(ax) \quad \text{and} \quad (\lambda_A + k \lambda_B)(x) \vee \frac{1 - k}{2} \geq (\lambda_A + k \lambda_B)(ax).
\]
Hence \( A + k B \) is an \((\mathcal{E}, \vee \mathcal{Q})\)-intuitionistic fuzzy ideal of \( R \).

**Definition 11.** [18] If \( S \subseteq R \), then intuitionistic characteristic function of \( S \) is denoted by \( C_S = (\chi_S, \chi_S^c) \) and is defined by
\[
\chi_S(x) = \begin{cases} 
1 & \text{if } x \in S \\
0 & \text{if } x \not\in S
\end{cases}
\]
and \( \chi_S^c(x) = \begin{cases} 
0 & \text{if } x \in S \\
1 & \text{if } x \not\in S
\end{cases} \)
In particular, we let \( T = (\chi_R, \chi_R^c) \) be the intuitionistic fuzzy set in \( R \).

**Lemma 2.** A non-empty subset \( L \) of a hemiring \( R \) is a left ideal of \( R \) if and only if the intuitionistic characteristic function \( C_L = (\chi_L, \chi_L^c) \) is an \((\mathcal{E}, \vee \mathcal{Q})\)-intuitionistic fuzzy left ideal of \( R \).

**Proof.** Let \( L \) be a left ideal of \( R \), then obviously \( C_L \) is an \((\mathcal{E}, \vee \mathcal{Q})\)-intuitionistic fuzzy left ideal of \( R \).

Conversely assume that \( C_L \) is an \((\mathcal{E}, \vee \mathcal{Q})\)-intuitionistic fuzzy left ideal of \( R \). Let \( x, y \in L \). Then \( \chi_L(x) = 1 \), \( \chi_L^c(x) = 0 \), and \( \chi_L(y) = 1 \), \( \chi_L^c(y) = 0 \) so \( x(1,0), y(1,0) \in C_L \). Since \( C_L \) is an \((\mathcal{E}, \vee \mathcal{Q})\)-intuitionistic fuzzy left ideal, so
\[
(\chi_L)(x+y) \geq \min \{ \chi_L(x), \chi_L(y), \frac{1+k}{2} \}
\]
and
\[
(\chi_L^c)(x+y) \leq \max \{ \chi_L^c(x), \chi_L^c(y), \frac{1+k}{2} \}
\]
i.e. \( (\chi_L)(x+y) = 1 \) and \( (\chi_L^c)(x+y) = 0 \). Thus \( x+y \in L \).

Let \( y \in L \) and \( x \in R \). Then \( \chi_L(y) = 1 \), and \( \chi_L(y) = 0 \) so \( y(1,0) \in C_L \). Since \( C_L \) is an \((\mathcal{E}, \vee \mathcal{Q})\)-intuitionistic fuzzy left
ideal, so \((\chi_L)(xy) \geq \min \{\chi_L(y), \frac{1-k}{2}\}\) and \((\chi_R)(xy) \leq \max \{\chi_L(y), \frac{1-k}{2}\}\). i.e. \((\chi_L)(xy) = 1\) and \((\chi_R)(xy) = 0\). Hence \(xy \in L\). Thus \(L\) is a left ideal of \(R\).

**Lemma 3.** A non-empty subset \(L\) of a hemiring \(R\) is a left ideal of \(R\) if and only if the intuitionistic fuzzy set \((C_L)_k = (\chi_L \wedge \frac{1-k}{2}, \chi_L \vee \frac{1-k}{2})\) is an \((\epsilon, \in \vee q_k)^*\)-intuitionistic fuzzy left ideal of \(R\).

**Proof.** Straightforward.

**Lemma 4.** Let \(A\) and \(B\) be non-empty subsets of a hemiring \(R\). Then the following hold:

1. \(C_A \cap_k C_B = (C_{A \cap B})_k\)
2. \(C_A \cdot_k C_B = (C_{A \cdot B})_k\)

**Proof.** Straightforward.

**Theorem 15.** For a hemiring \(R\), the following conditions are equivalent:

(i) \(R\) is hemiregular.

(ii) \(A \cap_k B = A \cdot_k B\) for every \((\epsilon, \in \vee q_k)^*\)-intuitionistic fuzzy right ideal \(A\) and every \((\epsilon, \in \vee q_k)^*\)-intuitionistic fuzzy left ideal \(B\) of \(R\).

**Proof.** Let \(A\) be an \((\epsilon, \in \vee q_k)^*\)-intuitionistic fuzzy right ideal and \(B\) be an \((\epsilon, \in \vee q_k)^*\)-intuitionistic fuzzy left ideal of \(R\) and \(x \in R\). Then there exists \(a \in R\), such that \(x = xa\). Now

\[
(\mu_{A \cdot_k B})(x) = \bigvee_{x = \sum_{i=1}^{m} y_i z_i} \left[ \bigwedge_{1 \leq i \leq p} [\mu_A(y_i) \wedge \mu_B(z_i)] \right] \wedge \frac{1-k}{2} \geq \left[ \mu_A(xa) \wedge \mu_B(x) \wedge \frac{1-k}{2} \right] \geq \left[ \mu_A(x) \wedge \mu_B(x) \wedge \frac{1-k}{2} \right] = (\mu_A \wedge_k \mu_B)(x)
\]

and

\[
(\lambda_A \cdot_k \lambda_B)(x) = \bigwedge_{x = \sum_{i=1}^{m} y_i z_i} \left[ \bigvee_{1 \leq i \leq p} [\lambda_A(y_i) \vee \lambda_B(z_i)] \right] \vee \frac{1-k}{2} \leq \left[ \lambda_A(xa) \vee \lambda_B(x) \vee \frac{1-k}{2} \right] \leq \left[ \lambda_A(x) \vee \lambda_B(x) \vee \frac{1-k}{2} \right] = (\lambda_A \wedge_k \lambda_B)(x).
\]

Thus \(A \cap_k B \subseteq A \cdot_k B\).

By Theorem 14 \(A \cdot_k B \subseteq A \cap_k B\). Hence \(A \cdot_k B = A \cap_k B\).

\((ii) \implies (i)\) Let \(A\) and \(B\) be right ideal and left ideal of \(R\) respectively. Then \(C_A\) is an \((\epsilon, \in \vee q_k)^*\)-intuitionistic fuzzy right ideal and \(C_B\) is an \((\epsilon, \in \vee q_k)^*\)-intuitionistic fuzzy left ideal of \(R\), by assumption

\[C_A \cdot_k C_B = C_A \cap_k C_B \implies (C_A \cdot C_B)_k = (C_A \cap C_B)_k \implies (C_{AB})_k = (C_{A \cap B})_k \implies AB = AB = A \cap B.\]

Thus by Theorem 1 \(R\) is regular.

**Theorem 16.** The following assertions for a hemiring \(R\) with identity are equivalent:

1. \(R\) is fully idempotent.
(2) Each \((\varepsilon, \in \vee q_k)^*\)-intuitionistic fuzzy ideal of \(R\) is idempotent. (an \((\varepsilon, \in \vee q_k)^*\)-intuitionistic fuzzy ideal \(A\) of \(R\) is called idempotent if \(A \cdot_1 A = A_k\).)

(3) For each pair of \((\varepsilon, \in \vee q_k)^*\)-intuitionistic fuzzy ideals \(A\) and \(B\) of \(R\), \(A \cap_k B = A \cdot_1 B\).

(4) If \(R\) is assumed to be commutative, then the above assertions are equivalent to \(R\) is regular.

Proof. (1) \(\implies\) (2). Let \(A = (\mu_A, \lambda_A)\) be an \((\varepsilon, \in \vee q_k)^*\)-intuitionistic fuzzy ideal of \(R\). For any \(x \in R\), by Theorem 14 \(A \cdot_1 A \subseteq A_k\).

Since each ideal of \(R\) is idempotent, therefore, \((x) = (x)^2\) for each \(x \in R\). Since \(x \in (x)\) it follows that \(x \in (x)^2 = RxRRxR\). Hence \(x = \sum_{i=1}^{q} a_i x a_i^t b_i^t x b_i^t\) and \(q \in N\). Now,

\[
\left(\mu_A \land \frac{1 - k}{2}\right) (x) = \mu_A (x) \land \mu_A (x) \land \frac{1 - k}{2} = \left[\mu_A (x) \land \frac{1 - k}{2}\right] \land \left[\mu_A (x) \land \frac{1 - k}{2}\right] \land \frac{1 - k}{2} \\
\leq \mu_A (a_i x a_i^t) \land \mu_A (b_i x b_i^t) \land \frac{1 - k}{2}, (1 \leq i \leq q).
\]

Therefore,

\[
\left(\mu_A \land \frac{1 - k}{2}\right) (x) \leq \bigvee_{1 \leq i \leq q} \left[\mu_A (a_i x a_i^t) \land \mu_A (b_i x b_i^t)\right] \land \frac{1 - k}{2} \\
\leq \bigvee_{x = \sum_{i=1}^{q} a_i x a_i^t b_i^t x b_i^t} \left[\bigwedge_{1 \leq i \leq q} \left[\mu_A (a_i x a_i^t) \land \mu_A (b_i x b_i^t)\right]\right] \land \frac{1 - k}{2} \\
\leq \bigvee_{x = \sum_{i=1}^{q} a_i x a_i^t b_i^t x b_i^t} \left[\bigwedge_{1 \leq j \leq r} \left[\mu_A (a_j) \land \mu_A (b_j)\right]\right] \land \frac{1 - k}{2} = (\mu_A \cdot_1 \mu_A) (x)
\]

and

\[
\left(\lambda_A \lor \frac{1 - k}{2}\right) (x) = \lambda_A (x) \lor \lambda_A (x) \lor \frac{1 - k}{2} \\
= \left[\lambda_A (x) \lor \frac{1 - k}{2}\right] \lor \left[\lambda_A (x) \lor \frac{1 - k}{2}\right] \lor \frac{1 - k}{2} \\
\geq \lambda_A (a_i x a_i^t) \lor \lambda_A (b_i x b_i^t) \lor \frac{1 - k}{2}, (1 \leq i \leq q).
\]

Therefore,

\[
\left(\lambda_A \lor \frac{1 - k}{2}\right) (x) \geq \bigvee_{1 \leq i \leq q} \left[\lambda_A (a_i x a_i^t) \lor \lambda_A (b_i x b_i^t)\right] \lor \frac{1 - k}{2} \\
\geq \bigwedge_{x = \sum_{i=1}^{q} a_i x a_i^t b_i^t x b_i^t} \left[\bigvee_{1 \leq i \leq q} \left[\lambda_A (a_i x a_i^t) \lor \lambda_A (b_i x b_i^t)\right]\right] \lor \frac{1 - k}{2} \\
\geq \bigwedge_{x = \sum_{i=1}^{q} a_i x a_i^t b_i^t x b_i^t} \left[\bigvee_{1 \leq j \leq r} \left[\lambda_A (a_j) \lor \lambda_A (b_j)\right]\right] \lor \frac{1 - k}{2} = (\lambda_A \cdot_1 \lambda_A) (x).
\]

Thus \(A \cdot_1 A = A_k\).

(2) \(\implies\) (1). Let \(I\) be an ideal of \(R\). Then \(C_I\), the intuitionistic characteristic function of \(I\), is an \((\varepsilon, \in \vee q_k)^*\)-intuitionistic
fuzzy ideal of $R$. Hence $C_1 \cdot_k C_1 = (C_1 \cdot C_1)_k = (C_1^2)_k = (C_1)_k$. It follows that $I^2 = I$.

(1) $\implies$ (3). Let $A$ and $B$ be $(\in, \in \cup q_k)^*$-intuitionistic fuzzy ideals of $R$.

By Theorem 14 $A \cdot_k B \subseteq A \cap_k B$. Again since $R$ is fully idempotent, $(x) = (x)^2$, for any $x \in R$. Hence, as argued in the first part of the proof of this theorem, we have

$$\mu_A \wedge_k \mu_B)(x) = (\mu_A)(x) \wedge (\mu_B)(x) \wedge \frac{1-k}{2} \leq \bigvee_{x=\Sigma_{i=1}^r a_i b_i} \left[ \bigwedge_{1 \leq i \leq r} [\mu_A(a_i) \wedge \mu_B(b_i)] \right] \wedge \frac{1-k}{2} = (\mu_A \wedge_k \mu_B)(x)$$

and

$$\lambda_A \vee_k \lambda_B)(x) = \lambda_A(x) \vee \lambda_B(x) \vee \frac{1-k}{2} \geq \bigwedge_{x=\Sigma_{i=1}^r a_i b_i} \left[ \bigvee_{1 \leq i \leq r} [\lambda_A(a_i) \vee \lambda_B(b_i)] \right] \vee \frac{1-k}{2} = (\lambda_A \wedge_k \lambda_B)(x).$$

Thus $A \cdot_k B = A \cap_k B$.

(3) $\implies$ (1). Let $A$ and $B$ be any pair of $(\in, \in \cup q_k)^*$-intuitionistic fuzzy ideals of $R$. We have $A \cdot_k B = A \cap_k B$. Take $A = B$. Thus $A \cdot_k A = A \cap_k A = A_k$, where $A$ is any $(\in, \in \cup q_k)^*$-intuitionistic fuzzy ideal of $R$. Hence, (3) $\implies$ (2). Since we already proved that (1) and (2) are equivalent, hence (3) $\implies$ (1) and so (1) $\iff$ (3). This establishes (1) $\iff$ (2) $\iff$ (3). Finally, if $A$ is commutative then it is easy to verify that (1) $\iff$ (4).

**Theorem 17.** For a hemiring $R$ with 1, the following conditions are equivalent.

1. $R$ is right weakly regular hemiring.
2. All $(\in, \in \cup q_k)^*$-intuitionistic fuzzy right ideals of $R$ are idempotent.
3. $A \cdot_k B = A \cap_k B$ for $(\in, \in \cup q_k)^*$-intuitionistic fuzzy right ideal $A$ and all $(\in, \in \cup q_k)^*$-intuitionistic fuzzy two-sided ideals $B$ of $R$.

**Proof.** (1) $\implies$ (2) Let $A$ be an $(\in, \in \cup q_k)^*$-intuitionistic fuzzy right ideal of $R$. Then we have $A \cdot_k A \subseteq A_k$.

For the reverse inclusion, let $x \in R$. Since $R$ is right weakly regular, so there exist $a_i, b_i \in R$ such that $x = \sum_{i=1}^q a_i x b_i$. Now we have

$$\left( \mu_A \wedge \frac{1-k}{2} \right)(x) = \mu_A(x) \wedge \mu_A(x) \wedge \frac{1-k}{2}$$

$$= \left[ \mu_A(x) \wedge \frac{1-k}{2} \right] \wedge \left[ \mu_A(x) \wedge \frac{1-k}{2} \right] \wedge \frac{1-k}{2}$$

$$\leq \mu_A(x a_i) \wedge \mu_A(x b_i) \wedge \frac{1-k}{2}, 
(1 \leq i \leq q).$$

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Therefore,

\[
\left( \mu_A \wedge \frac{1-k}{2} \right)(x) \leq \bigwedge_{1 \leq i \leq q} [\mu_A(xa_i) \wedge \mu_A(xb_i)] \wedge \frac{1-k}{2}
\]
\[
\leq \bigvee_{x = \sum_{j=1}^{m} a_j b_j} \left[ \bigwedge_{1 \leq i \leq q} [\mu_A(xa_i) \wedge \mu_A(xb_i)] \right] \wedge \frac{1-k}{2}
\]
\[
\leq \bigvee_{x = \sum_{j=1}^{m} a_j b_j} \left[ \bigwedge_{1 \leq i \leq q} [\mu_A(a_j) \wedge \mu_A(b_j)] \right] \wedge \frac{1-k}{2} = (\mu_A \cdot \mu_A)(x).
\]

and

\[
\left( \lambda_A \vee \frac{1-k}{2} \right)(x) = \lambda_A(x) \vee \lambda_A(x) \vee \frac{1-k}{2}
\]
\[
= \left[ \lambda_A(x) \vee \frac{1-k}{2} \right] \vee \left[ \lambda_A(x) \vee \frac{1-k}{2} \right] \vee \frac{1-k}{2}
\]
\[
\geq \lambda_A(xa_i) \vee \lambda_A(xb_i) \vee \frac{1-k}{2}, \quad (1 \leq i \leq q).
\]

Therefore,

\[
\left( \lambda_A \vee \frac{1-k}{2} \right)(x) \geq \bigvee_{1 \leq i \leq q} [\lambda_A(xa_i) \vee \lambda_A(xb_i)] \vee \frac{1-k}{2}
\]
\[
\geq \bigwedge_{x = \sum_{j=1}^{m} a_j b_j} \left[ \bigvee_{1 \leq i \leq q} [\lambda_A(a_j) \vee \lambda_A(b_j)] \right] \vee \frac{1-k}{2}
\]
\[
\geq \bigwedge_{x = \sum_{j=1}^{m} a_j b_j} \left[ \bigvee_{1 \leq i \leq q} [\lambda_A(a_j) \vee \lambda_A(b_j)] \right] \vee \frac{1-k}{2} = (\lambda_A \cdot \lambda_A)(x).
\]

Thus \(A \cdot_k A = A_k\)

\((2) \implies (3)\) Let \(A\) and \(B\) be \((\epsilon, \in \cup q_k)^*\)-intuitionistic fuzzy right ideal and \((\epsilon, \in \cup q_k)^*\)-intuitionistic fuzzy two-sided ideal of \(R\) respectively. Then \(A \cap_k B\) is an \((\epsilon, \in \cup q_k)^*\)-intuitionistic fuzzy right ideal of \(R\). By Theorem 14, \(A \cap_k B \subseteq A \cap_k B\). By hypothesis,

\[(A \cap_k B) = (A \cap_k B)_{\cdot_k} (A \cap_k B) \subseteq A \cap_k B\]

Hence \(A \cdot_k B = A \cap_k B\).

\((3) \implies (1)\) Let \(B\) be a right ideal of \(R\) and \(A\) be two sided-ideal of \(R\). Then the intuitionistic characteristic function \(C_A\) and \(C_B\) are \((\epsilon, \in \cup q_k)^*\)-intuitionistic fuzzy two-sided ideal and \((\epsilon, \in \cup q_k)^*\)-intuitionistic fuzzy right ideal of \(R\), respectively. Hence by hypothesis

\[C_B \cdot_k C_A = C_B \cap_k C_A \implies (C_{B\cdot_k A})_{\cdot_k} = (C_{A\cap_k B})_{\cdot_k} \implies B \cdot A = B \cap A\]

Thus by Theorem 2, \(R\) is right weakly regular hemiring.
References