CONTINUED FRACTION EXPANSIONS OF SOME FUNCTIONS OF POSITIVE DEFINITE MATRICES

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Abstract: In this paper we recall some results of matrix functions with real coefficients. The aim of this paper is to provide some properties and results of continued fractions with matrix arguments. Then we give continued fractions expansions of some inverse of hyperbolic and circular functions \( \arcsin(A), \arcsinh(A), \arccos A \) and \( \arccosh(A) \) where \( A \) is a positive definite matrix.

Keywords: Continued fraction expansion, positive definite matrix, function of matrices

1. Introduction and motivation

Over the last two hundred years, the theory of continued fractions has been a topic of extensive study. The basic idea of this theory over real numbers is to give an approximation of various real numbers by the rational ones. One of the main reasons why continued fractions are so useful in computation is that they often provide representation for transcendental functions that are much more generally valid than the classical representation by, say, the power series. Further; in the convergent case, the continued fractions expansions have the advantage that they converge more rapidly than other numerical algorithms.

Recently, the extension of continued fractions theory from real numbers to the matrix case has seen several developments and interesting applications (see [5], [7], [11]). The real case is relatively well studied in the literature. However, in contrast to the theoretical importance, one can find in mathematical literature only a few results on the continued fractions with matrices arguments. The main difficulty arises from the fact that the algebra of square matrices is not commutative.

For simplicity and clearness, we restrict ourselves to positive definite matrices, but our results can be, without special difficulties, projected to the case of positive definite operators from an infinite dimensional Hilbert space into itself.

2. Preliminary and notations

Matrix functions play a widespread role in science and engineering, with applications areas ranging from nuclear magnetic resonance [2]. So for any scalar polynomial \( p(z) = \sum_{i=0}^{k} a_i z^i \) gives rise to a matrix polynomial with scalar coefficients by simply substituting \( A^i \) for \( z^i \):

\[ P(A) = \sum_{i=0}^{k} a_i A^i \]

More generally, for function \( f \) with a series representation on an open disk containing the eigenvalues of \( A \), we are able to define the matrix function \( f(A) \) via the Taylor series for \( f \) [4].

Alternatively, given a function \( f(z) \) that is analytic inside on a closed contour \( \Gamma \) which encircles the eigenvalues of \( A \), \( f(A) \) can be defined, by analogy with Cauchy's integral theorem, by
\[
f(A) = \frac{1}{2\pi i} \int_C f(z)(zI - A)^{-1} dz
\]

The definition is known as the matrix version of Cauchy's integral theorem. We now mention an important result of matrix functions.

**Lemma 2.1** (i) If two matrices \( A \in M_m \) and \( B \in M_m \) are similar, with \( A = XBX^{-1} \), then the matrices \( f(A) \) and \( f(B) \) are also similar, with \( f(A) = Xf(B)X^{-1} \).

(ii) If \( B \in M_m \) is a block diagonal matrix \( f(B) = \text{diag}(f(B_1), f(B_2), \ldots, f(B_r)) \).

**Proof.** For \( A = XBX^{-1} \) we have \( A^k = ZB^kZ^{-1} \). Hence for every polynomial \( p \) it follows that \( p(A) = Xp(B)X^{-1} \).

Therefore if either one of \( p(A) \) or \( p(B) \) equals zero then so does the other, implying that \( A \) and \( B \) share the same minimal polynomial. From definition there exists an interpolating polynomial \( r(Z) \) such that \( f(A) = r(A), \quad f(B) = r(B) \) and since for every polynomial we have \( p(A) = Zp(B)Z^{-1} \), the result follows.

(ii) We deduce it from (i).

Let \( A \in M_m \), \( A \) is said to be positive semidefinite (resp. positive definite) if \( A \) is symmetric and \( \forall x \in \mathbb{R}^m, <Ax,x > \geq 0 \) (resp. \( \forall x \in \mathbb{R}^m, x \neq 0 <Ax,x > > 0 \)) where \(<,>\) denotes the standard scalar product of \( \mathbb{R}^m \).

We observe that positive semidefiniteness induces a partial ordering on the space of symmetric matrices: if \( A \) and \( B \) are two symmetric matrices, we write \( A \leq B \) if \( B - A \) is positive semidefinite.

Henceforth, whenever we say that \( A \in M_m \) is positive semidefinite (or positive definite), it will be assumed that \( A \) is symmetric.

For any matrices \( A, B \in M_m \) with \( B \) invertible, we write \( A_B = B^{-1}A \). It is easy to verify that for any invertible matrix \( X \) we have

\[
A_B = XA_BX^{-1} = AX
\]

**Definition 2.2** Let \( (A_n) n \geq 0 \) and \( (B_n) n \geq 0 \) be two sequences of matrices in \( M_m \). We define the sequences \( (P_n) n \geq -1 \) and \( (Q_n) n \geq -1 \) by

\[
\begin{align*}
P_{-1} &= 1, \quad P_0 = A_0 \\
Q_{-1} &= 0, \quad Q_0 = I
\end{align*}
\]

and

\[
\begin{align*}
P_n &= A_nP_{n-1} + B_nP_{n-2} \\
Q_n &= A_nQ_{n-1} + B_nQ_{n-2}
\end{align*}
\]

\( n \geq 1 \) \hspace{1cm} (2.1)

The matrix \( P_n/Q_n \) is called the \( n^{th} \) convergent of \( K(B_n/A_n) \), the fraction \( \frac{B_n}{A_n} \) is called its \( n^{th} \) partial quotient. The proof of the next proposition is elementary and we left it to the reader.
Proposition 2.3. For any two matrices C and D with C invertible, we have
\[ C \left[ \frac{B_k}{A_k} \right]_{k=1}^{n} = \left[ C A_0 D; \frac{B_1 D}{A_1 C^{-1}}; \frac{B_2 C^{-1}}{A_2}; \frac{B_k}{A_k} \right]_{k=3}^{n} \]

Definition 2.4. Let \((A_n), (B_n), (C_n)\) and \((D_n)\) be four sequences of matrices. We say that the continued fractions \(K(B_n/A_n)\) and \(K(D_n/C_n)\) are equivalent if we have \(F_n = G_n\) for all \(n \geq 1\), where \(F_n\) and \(G_n\) are the \(n^{th}\) convergents of \(K(B_n/A_n)\) and \(K(D_n/C_n)\) respectively.

In order to simplify the statements on some partial quotients of continued fractions with matrices arguments, we need the following proposition which is an example of equivalent continued fractions.

Proposition 2.5. (see [10]) Let \[ A_0; \frac{b_n}{a_n} \] be a given continued fraction.
Then
\[ \frac{P_n}{Q_n} = \left[ A_0; \frac{b_k}{A_k} \right]_{k=1}^{n} = \left[ A_0; \frac{X_k b_k X_k^{-1}}{X_k A_k X_k^{-1}} \right]_{k=1}^{n}, \]
where \(X_{-1} = X_0 = 1\) and \(X_1, X_2, ..., X_n\) are arbitrary invertible matrices.

We also recall the following proposition in real case.

Proposition 2.6. Let \((r_n)\) be a non-zero sequence of real numbers. The following continued fractions
\[ \left[ a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, ... \right] \quad \text{and} \quad \left[ a_0; \frac{r_1 b_1}{r_1 a_1}, \frac{r_2 b_2}{r_2 a_2}, ... \right] \]
are equivalent.

We end this section by introducing some topological notions of continued fractions with matrix arguments. We provide \(M_m\) with the topology induced by the following classical norm:
\[ \forall A \in M_m, \|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} \|Ax\| \]

Definition 2.7. Let \(\{A_n\}\) be a sequence of matrices in \(M_m\). We say that \(\{A_n\}\) converges in \(M_m\) if there exists a matrix \(A \in M_m\) such that \(\|A_n - A\|\) tends to 0 when \(n\) tends to \(+\infty\). In this case we write \(\lim_{n \to +\infty} A_n = A\).

The continued fraction \(A_0; b_n/A_k\) is said to be convergent in \(M_m\) if the sequence \(\{P_n/Q_n\}_n = \{Q_n^{-1} P_n\}_n\) converges in \(M_m\) in the sense that there exists a matrix \(F \in M_m\) such that \(\lim_{n \to +\infty} \|F_n - F\| = 0\)

3. Man Result

Let \(A \in M_m\) be a positive definite matrix. Our aim in this section is to give a continued fraction expansion of arcsin, arcsinh(A), arccos(A) and arccosh(A). For simplicity, we start with the real case and we begin by recalling Laguerre's continued fraction in the following lemma.

Lemma 3.11 (see [3]) Let \(x\) be a real number such that \(0 < x < 1\). Then there holds
\[
\frac{\arcsin(x)}{(1 - x^2)^{1/2}} = \left[ x, \frac{-2(k - 2)(2k)x^2}{-4k}, \frac{-(2k - 2)(2k)x^2}{-(4k + 1)} \right]_{k=1}^{+\infty}.
\]

Now we establish a main theorem which is a matrix version of the previous lemma.

Theorem 3.2. Let \(A \in M_m\) be a positive definite matrix such that \(\|A\| < 1\). Then a continued fraction expansion of \(\arcsin(A)\) is
arcsin (A) = \left[ 0; \frac{A(I - A^2)^{1/2}}{I}, \frac{(2k - 1)(2k)A^2}{-(4k - 1)I}, \frac{(2k - 1)(2k)A^2}{-(4k + 1)I} \right]_{k=1}^{+\infty} \tag{3.2}

Since arcsinh(A) = -i\arcsin(A), by vertu of proposition 2.6 and the previous theorem, we have the next result.

**Corollary 3.3.** A continued fraction expansion of arcsinh(A) is given by

\[ \text{arcsinh}(A) = \left[ 0; \frac{A(I - A^2)^{1/2}}{I}, \frac{(2k - 1)(2k)A^2}{-(4k - 1)I}, \frac{(2k - 1)(2k)A^2}{-(4k + 1)I} \right]_{k=1}^{+\infty} \tag{3.3} \]

**Proof.** Let \( A \in \mathbb{M}_m \) be a positive definite matrix. Then there exists an invertible matrix X such that \( A = XD X^{-1} \), where \( D = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_m) \) and \( \lambda_i > 0 \).

As the function \( z \to \arcsin(z) \) is analytic in the open half-plane \( \{ z \in \mathbb{C} / \Re(z) > 0 \} \), then

\[ \arcsin(A) = X(\arcsin(D))X^{-1} = X \text{diag}(\arcsin(\lambda_1), \arcsin(\lambda_2), ..., \arcsin(\lambda_m))X^{-1}. \]

Let us define the sequences \( \{ P_n \} \) and \( \{ Q_n \} \) by:

\[
\begin{align*}
\{ P_{-1} = I, P_0 = 0, P_1 = D \\
Q_{-1} = 0, Q_0 = I, Q_1 = I \\
\end{align*}
\]

and for \( k \geq 1, \)

\[
\begin{align*}
P_{2k} &= -(4k - 1)P_{2k-1} + (2k - 1)(2k)D^2P_{2k-2} \\
Q_{2k} &= (4k - 1)Q_{2k-1} + (2k - 1)(2k)D^2Q_{2k-2} \\
P_{2k+1} &= -(4k - 1)P_{2k+1} + (2k - 1)(2k)D^2P_{2k-1} \\
Q_{2k+1} &= (4k - 1)Q_{2k+1} + (2k - 1)(2k)D^2Q_{2k-1} \\
\end{align*}
\]

We see that \( P_n \) and \( Q_n \) are diagonal matrices, by setting \( p_n = \text{diag}(p_{n1}, p_{n2}, ..., p_{nm}) \) and \( q_n = \text{diag}(q_{n1}, q_{n2}, ..., q_{nm}) \), we obtain for each \( i, 1 \leq i \leq m \)

\[
\begin{align*}
p_{-1}^i = 1, p_0^i = 0, p_1^i = \lambda_i \\
q_{-1}^i = 0, q_0^i = 1, q_1^i = 1 \\
\end{align*}
\]

and for \( k \geq 1, \)

\[
\begin{align*}
p_{2k}^i &= -(4k - 1)p_{2k-1}^i + (2k - 1)(2k)\lambda_i^2p_{2k-2}^i \\
q_{2k}^i &= -(4k - 1)q_{2k-1}^i + (2k - 1)(2k)\lambda_i^2q_{2k-2}^i \\
p_{2k+1}^i &= -(4k - 1)p_{2k+1}^i + (2k - 1)(2k)\lambda_i^2p_{2k-1}^i \\
q_{2k+1}^i &= -(4k - 1)q_{2k+1}^i + (2k - 1)(2k)\lambda_i^2q_{2k-1}^i \\
\end{align*}
\]

By lemma 3.1, the convergent \( p_n^i/q_n^i \) converges to \((1 - \lambda_i^2)^{-1/2} \arcsin \lambda_i\). It follows that \( P_n/Q_n \) converges to \((I - D^2) - 1/2 \arcsin D\), so that

\[ \frac{\arcsin(D)}{(I - D^2)^{1/2}} = \left[ 0; \frac{D}{I}, \frac{2D^2}{-3I}, \frac{2D^2}{-5I}, \frac{-(2k - 2)(2k)D^2}{-(4k - 1)I}, \frac{-(2k - 2)(2k)D^2}{-(4k + 1)I} \right]_{k=2}^{+\infty} \]

Then, we multiply the continued fraction \( \frac{\arcsin(D)}{(I - D^2)^{1/2}} \) by \((I - D^2)^{1/2}\) in the left to obtain
arcsin(D) = \left[ 0; \frac{D}{(1 - D)^{1/2}}, \frac{2D^2(1 - D)^{1/2}}{-3I}, \frac{2D^2(2k - 1)(2k)D^2}{-(4k - 1)I}, \frac{-(2k - 1)(2k)D^2}{-(4k + 1)I} \right]_{k=2}^{+\infty}

by proposition 2.3, we get

X(\arcsin(D))X^{-1} = \left[ 0; \frac{DX^{-1}}{(1 - D)^{-1/2}X^{-1}}, \frac{2D^2(1 - D)^{1/2}X^{-1}}{-3I}, \frac{2D^2(2k - 1)(2k)D^2}{-(4k - 1)I}, \frac{-(2k - 1)(2k)D^2}{-(4k + 1)I} \right]_{k=2}^{+\infty}

Let us define the sequence \((X_n)_{n \geq 1}\) by

\[
\begin{align*}
X_0 &= I \\
X_n &= X(I - D^{1/2}), \text{ for } n \geq 1.
\end{align*}
\]

Then

\[
\begin{align*}
X_1(DX^{-1})X_0^{-1} &= \frac{A(I - A^2)^{1/2}}{I} \\
X_1((I - D^{1/2})X^{-1})X_0^{-1} &= \frac{2A^2}{-3I} \\
X_2(2D^2(I - D)^{-1/2}X^{-1}X_0^{-1} &= \frac{(2k - 1)(2k)A^2}{-(4k - 1)I} \frac{-(2k - 1)(2k)A^2}{-(4k + 1)I}
\end{align*}
\]

For \(k \geq 3\), we have

\[
\frac{X_k(2k - 1)(2k)D^2X_{k-2}}{-X_k(4k - 1)X_{k-2}^{-1}} = \frac{(2k - 1)(2k)A^2}{-(4k - 1)I} \frac{-(2k - 1)(2k)A^2}{-(4k + 1)I}
\]

By applying the result of proposition 2.5 to the sequence \((X_n)_{n \geq 1}\), we finish the proof of theorem 3.2.

Before giving continued fraction expansions of \(\arccos(A)\) and \(\arccosh(A)\), we begin with the real case in the following lemma

**Lemma 3.4. (see [7])** Let \(x\) be a real number such that \(0 < x < 1\), Then there holds

\[
\arccos \left( \frac{x}{(1 - x^2)^{1/2}} \right) = \left[ 0; \frac{(2k - 1)(2k)x^2}{-(4k - 1)}, \frac{(2k - 1)(2k)x^2}{-(4k + 1)} \right]_{k=1}^{+\infty}
\]  

(3.4)

Now we establish a main theorem which is a matrix version of the lemma 4.

**Theorem 3.5.** Let \(A \in M_m\) be a positive definite matrix such that \(\|A\| < 1\). Then a continued fraction expansion of \(\arcsin(A)\) is

\[
\arcsin(A) = \left[ 0; \frac{A(I - A^2)^{1/2}}{I}, \frac{(2k - 1)(2k)A^2}{-(4k - 1)I}, \frac{(2k - 1)(2k)A^2}{-(4k + 1)I} \right]_{k=1}^{+\infty}
\]  

(3.5)

since \(\arccosh(A) = i\arccos(A)\), by virtue of proposition 2.6 and theorem 3.5, we have:

**Corollary 3.6.** A continued fraction expansion of \(\arccosh(A)\) is given by

\[
\arccosh(A) = \left[ 0; \frac{A(I - A^2)^{1/2}}{I}, \frac{(2k - 1)(2k)A^2}{(4k - 1)I}, \frac{(2k - 1)(2k)A^2}{(4k + 1)I} \right]_{k=1}^{+\infty}
\]  

(3.6)

With a similar method as in theorem 3.2, we prove the result of this theorem.
References